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Dominion of Some Graphs

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Abstract

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is a dominating set if every vertex in $V \setminus S$ is adjacent to some vertex in S. The dominating set with the least cardinality, γ, is called a γ-set which is commonly known as a minimum dominating set. The dominion of a graph G, denoted by $\zeta(G)$, is the number of its γ-sets. Some relations between these two seemingly distinct parameters are established. In particular, we present the dominions of paths, some cycles and the join of any two graphs.

1 Introduction

A *dominating set* for a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every vertex $v \in V$ is either in S or has a neighbor $u \in S$. The vertex $u \in S$ is said to *cover* the vertex $v \in V$ if either $u = v$ or $uv \in E$. A dominating set S is a

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minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality among all dominating sets of G. Clearly, a graph G can have multiple minimum dominating sets. For simplicity, we refer to those sets as **γ-sets**.

Dominating sets, in particular γ-sets, have been extensively researched. We recommend to the reader Haynes et al.'s book [8] and several other research works [1, 2, 3, 4, 5, 6, 7, 9] on variants and applications of dominating sets. In this literature we found various versions of γ-set such as the perfect dominating set, the double dominating set, the strong dominating set, and the restrained dominating set. In general, each such type of γ-set has some specified properties. Here, we list five of the most commonly studied γ-sets along with their properties. Later in Example 1.1, we illustrate each such γ-set.

1. *Perfect γ-set* requires that no two vertices in a γ-set cover the same vertex (outside the γ-set).

2. *Connected γ-set* requires that the graph induced by the γ-set be connected.

3. *Total γ-set* forbids isolated vertices in the subgraph induced by the γ-set.

4. *Independent γ-set* requires all vertices in a γ-set to be isolated vertices in the subgraph they induce.

5. *clique γ set* requires that the vertices in a γ-set induce a clique.

Because a given graph may have multiple γ-sets, this article addresses a natural but fundamental question: *How many γ-sets does a given graph have?* To answer this question, we introduce the notion of **dominion**.

The *dominion (number)* of a graph $G = (V, E)$, denoted by $\zeta(G)$, is the number of its γ-sets. In other words,

$\zeta(G) := |\{S : S \text{ is a dominating set in } G \text{ and } |S| = \gamma\}|$. For instance, given a complete graph $G = K_n$, it is obvious that each $v \in V(G)$ covers the remaining $n - 1$ vertices of G. Therefore $\zeta(G) = n$ while $\gamma(G) = 1$. Also, for a star graph $G = K_{1,n}$ of order $n + 1 \geq 3$, it is easy to see that $\gamma(G) = 1 = \zeta(G)$ since only the central vertex can cover all of the remaining $n \geq 2$ vertices.

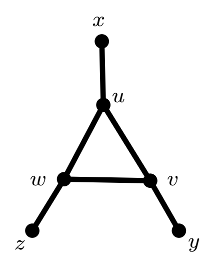

Figure 1: A sun graph G with $\gamma(G) = 3$ and $\zeta(G) = 8$

Example 1.1. *Here, we illustrate the γ-sets of a unicyclic graph G (see Figure 1). Consider the unicyclic graph G as shown in Figure 1. It is easy to see that $\gamma(G) = 3$. We list all the γ-sets with properties which are among the five commonly studied, as previously mentioned.*

- $\{u, v, w\}$ *is a perfect γ-set, a connected γ-set, a total γ-set, and a clique γ-set.*
- $\{x, y, z\}$ *is a perfect γ-set, and an independent γ-set.*
- $\{u, y, z\}$, $\{v, x, z\}$, *and* $\{w, x, y\}$ *are independent γ-sets but not perfect, connected, clique or total.*
- $\{u, v, z\}$, $\{u, w, y\}$, *and* $\{v, w, x\}$ *are γ-sets which have none of the* 5 *special properties that we previously listed.*

We have $\zeta(G) = \sum_{i=0}^{3} \binom{3}{i} = 8$, *the number of γ-sets of G.*

We present a generalization of the previous example for such unicyclic graphs which are commonly known as *sun graphs*.

Proposition 1.2. *If G is a sun graph on $2n \geq 6$ vertices, then* $\zeta(G) = \sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Proof. For each vertex on the cycle, and for each dominating set $S \subseteq V(G)$, either the vertex or its leaf neighbor must be in S. Therefore, we can form all γ-sets S in G from choosing subsets of the set of n leafs, and the cycle neighbors of unchosen leafs. This gives $\sum_{i=0}^{n} \binom{n}{i}$. Conversely, every S assembled by choosing either the vertex on the cycle or the leaf, for each leaf, is a γ-set for G. This gives $\zeta(G) = 2^n$. □

2 Dominion of Some Graphs

Suppose $P_n := v_1 - v_2 - \ldots - v_{n-1} - v_n$, denotes a path on $n \geq 2$ vertices. We establish each of the following claims:

Claim 2.1. *There is no γ-set that includes both vertices v_{n-1}, v_n.*

Proof. Suppose $n = 3k+1$. It is easy to see that $\gamma(P_{3k+1}) = k+1$ and without loss, let's assume that there is a γ-set S, with $|S| = k+1$, that includes both v_{3k}, v_{3k+1}. Since v_{3k-1} is covered by v_{3k} (and vice versa), this leaves exactly $3k-2$ vertices of P_{3k+1} that are to be covered using the remaining $k-1$ vertices of S. Because each vertex of S covers at most 2 other vertices of P_{3k+1}, $k-1$ vertices can cover at most $3(k-1)$ vertices of the path, and $3(k-1) < 3k-2$. A similar argument addresses the case when $n = 3k+2$ and the case when $n = 3k$ follows from a pigeonhole principle argument, since $\gamma(P_{3k}) = k$. □

Claim 2.2. *There is no γ-set that excludes both vertices v_{n-1}, v_n.*

Proof. Suppose S is a γ-set of P_n that excludes both consecutive vertices v_{n-1}, v_n. It follows that $v_n \in V(P_n)$ is left uncovered. Hence, S is not a γ-set of P_n. □

Claim 2.3. *Every γ-set includes either v_{n-1} or v_n but not both.*

Claim 2.4. *When $n = 3k, k \geq 1$, $\zeta(P_n) = 1$ and the unique γ-set in P_n contains v_{n-1}. Moreover, when $n = 3k+2, k \geq 1$, there is only one γ-set in P_n containing v_n.*

Proof. When $n = 3k, k \geq 1$, we have $\zeta(P_n) = k$ and the conclusion follows by the pigeon-hold argument alluded in Claim 2.1. Each of the sets of 3 consecutive vertices $\{v_{3r+1}, v_{3r+2}, v_{3r+3}\}$, $r = 0, \ldots, k-1$, must contain at least one vertex of any γ-set S, and since $\gamma = k$, each of those sets must contain exactly one element of S. Since S is dominating, the element of S in $\{v_1, v_2, v_3\}$ must be v_1 or v_2. If v_1 then the "next" element of S must be v_4, which forces $S = \{v_1, v_4, \ldots, v_{3k-2}\}$, leaving v_{3k} uncovered. Therefore, the first element of S must be v_2, which forces $S = \{v_2, v_5, \ldots, v_{3k-1}, v_{3k+2}\}$.

Suppose that $n = 3k+2$, and let S be a γ-set in $V(P_n)$. Then $|S| = k+1$. Suppose that $v_n \in S$. Then $v_{n-1} \notin S$, so $S - \{v_n\}$ must be a dominating set in $P_n - \{v_n, v_{n-1}\} = P_{3k}$. Since $|S - \{v_n\}| = k = \gamma(P_{3k})$ it follows from the previous argument that $S = \{v_2, v_5, \ldots, v_{3k-1}, v_{3k+2}\}$.

□

Theorem 2.5. *If G is a path on n vertices, then its dominion is*

$$\zeta(G) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3 \qquad (i) \\ \frac{(n+2)(n+11)}{18} - 1, & \text{if } n \equiv 1 \pmod 3 \qquad (ii) \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \pmod 3. \qquad (iii) \end{cases}$$

Proof. Part (i). This follows directly from Claim 2.4.

Part (iii). Suppose that $n = 3k + 2$, $k \geq 1$, and $G = P_n$. We have $\gamma(G) = k + 1$ and every γ-set $S \subseteq V(G)$ contains exactly one of v_n, v_{n-1}. From Claim 2.4 we know that exactly one such S contains v_n, so our task is to show that exactly $\lceil \frac{n}{3} \rceil = k + 1$ such S contain v_{n-1}. Suppose that S is a γ-set for G containing v_{n-1}. By the reasoning in the proof of Claim 2.4, each 3-set $T_r = \{v_{3r+1}, v_{3r+2}, v_{3r+3}\}$, $r = 0, \ldots, k-1$, contains exactly one vertex of S. If $0 \leq r < k-1$ and $S \cap T_r = \{v_{3r+1}\}$ then, because S is dominating in G, $S \cap T_{r+1} = \{v_{3r+4}\}$; consequently, if $S \cap T_r = \{v_{3r+1}\}$ then $S \cap T_s = \{v_{3s+1}\}$, $r \leq s \leq k-1$.

On the other hand, if $S \cap T_r = \{v_{3r+2}\}$ then $S \cap T_{r+1}$ is either $\{v_{3r+4}\}$ or $\{v_{3r+5}\}$. Since $S \cap T_0$ is either $\{v_1\}$ or $\{v_2\}$ (since v_3 does not cover v_1), it follows that the γ-set for $G = P_n$, $n = 3k+2$, containing v_{n-1} are the following:

$$\begin{aligned}
S_0 &= \{v_1, v_4, \ldots, v_{n-4}, v_{n-1}\}, \\
S_1 &= \{v_2, v_4, \ldots, v_{n-4}, v_{n-1}\}, \\
&\vdots \\
S_r &= \{v_2, v_5, \ldots, v_{3r-1}, v_{3r+1}, \ldots, v_{n-4}, v_{n-1}\}, \\
&\vdots \\
S_k &= \{v_2, v_5, \ldots, v_{n-3}, v_{n-1}\},
\end{aligned}$$

and there are exactly $k+1$ of these sets.

Part(ii) We recursively obtain the ζ values of $G_k = P_{3k+1} := v_1 - v_2 - \ldots - v_{3k} - v_{3k+1}$ as we increase the value of $k \geq 1$. For simplicity, let $\zeta(G_k) = \zeta_k$, $k \geq 1$.

When $k = 1$, it is easy to verify that the γ-sets of G_1 are $\{v_1, v_3\}$, $\{v_1, v_4\}$,$\{v_2, v_3\}$, $\{v_2, v_4\}$. Hence $\zeta_1 = 4$.

When $k = 2$, it is clear that $\gamma(G_2) = 3$. Since v_6 covers v_5 and v_7, we add v_6 to each of the γ-sets of G_1 to get $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, $\{v_2, v_3, v_6\}$, $\{v_2, v_4, v_6\}$. Further, we add v_7 to each γ-set of G_1 that includes v_4 since v_4 covers v_5, and v_7 covers v_6. This produces two additional γ-sets for G_2, namely $\{v_1, v_4, v_7\}$, $\{v_2, v_4, v_7\}$. Finally, with the new and unique set $\{v_2, v_5\}$ which covers all other vertices except v_7, we generate two more γ-sets of G_2 using v_6 for one and v_7 for the other. It is clear that v_6 covers v_7 (and vice-versa), giving $\{v_2, v_5, v_6\}$, $\{v_2, v_5, v_7\}$. Hence $\zeta_2 = 8$.

When $k = 3$, it is also clear that $\gamma(G_3) = 4$. We use a very similar construction as in the case when $k = 2$. Since v_9 covers v_8 and v_{10}, we add v_9 to each of the 8 γ-sets of G_2 to get $\{v_1, v_3, v_6, v_9\}$, $\{v_1, v_4, v_6, v_9\}$, $\{v_2, v_3, v_6, v_9\}$, $\{v_2, v_4, v_6, v_9\}$, $\{v_1, v_4, v_7, v_9\}$, $\{v_2, v_4, v_7, v_9\}$, $\{v_2, v_5, v_6, v_9\}$, $\{v_2, v_5, v_7, v_9\}$. Further, we add v_{10} to each γ-set of G_2 that includes v_7 since v_7 covers v_8, and v_{10} covers v_9. This produces three additional γ-sets for G_3, namely $\{v_1, v_4, v_7, v_{10}\}$, $\{v_2, v_4, v_7, v_{10}\}$, $\{v_2, v_5, v_7, v_{10}\}$. Finally, with the unique set $\{v_2, v_5, v_8\}$ which covers all other vertices except v_{10}, we generate two more γ-sets of G_3 using v_9 for one and v_{10} for the other. Thus, v_9 covers v_{10} (and vice-versa) giving $\{v_2, v_5, v_8, v_9\}$, $\{v_2, v_5, v_8, v_{10}\}$. Hence $\zeta_3 = 13$. Now, we generalize the previous constructions for all $k \geq 2$.

It is obvious that $\gamma(G_k) = k + 1$. So, for $k \geq 2$, we repeatedly form the γ-sets of G_k by extending those of G_{k-1} as follows:

(a) Since v_{3k} covers both v_{3k-1} and v_{3k+1}, we add v_{3k} to each γ-set of G_{k-1}.

(b) We add v_{3k+1} to each γ-set of G_{k-1} that includes v_{3k-2} since v_{3k-2} covers v_{3k-1}, and v_{3k+1} covers v_{3k}.

(c) Given the unique set $\{v_2, v_5, \ldots, v_{3k-1}\}$ which covers all other vertices of G_k except v_{3k+1}, we generate two more γ-sets of G_k using v_{3k} for one and v_{3k+1} for the other. Clearly v_{3k} covers v_{3k+1} (and vice-versa) and we have $\{v_2, v_5, \ldots, v_{3k-1}, v_{3k}\}$ and $\{v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}\}$.

From steps (a)-(c), we generate a sequence of ζ_k values, with $k \geq 1$, giving

$$4, 8, 13, 19, 26, 34, 43, \ldots$$

For $k \geq 1$ let $A_k = \{S \subseteq V(G_k) \mid S \text{ is a } \gamma-\text{set in } G_k \text{ and } v_{3k+1} \in S\}$, $B_k = \{S \subseteq V(G_k) \mid S \text{ is a } \gamma-\text{set in } G_k \text{ and } v_{3k+1} \notin S\}$, and $a_k = |A_k|, b_k = |B_k|$. Then $\zeta(G_k) = a_k + b_k$. Notice that $S \in B_k \implies v_{3k} \in S$. If $S \in B_k$, there are 3 possibilities, if $k \geq 2$:

1. $v_{3k-1}, v_{3k-2} \notin S$, in which case $S \setminus \{v_{3k}\} \in B_{k-1}$.

2. $v_{3k-1} \in S$.

3. $v_{3k-2} \in S$.

It is not possible that $\{v_{3k-2}, v_{3k-1}, v_{3k} \subseteq S \in B_k\}$, because this inclusion would imply that $S \setminus \{v_{3k-1}, v_{3k}\}$ is a dominating set in G_{k-1}, which is impossible, since $|S \setminus \{v_{3k-1}, v_{3k}\}| = k - 1 < \gamma(G_{k-1}) = k$. Therefore, for $S \in B_k$, 2 and 3 are disjoint possibilities.

In case 2 we see that $S' = S \setminus \{v_{3k-1}, v_{3k}\}$ is a dominating set in $P_{3(k-1)}$, with $k - 1 = \gamma(P_{3(k-1)})$ vertices, and we know that there is only one such set, $S' = \{v_j \mid j \equiv 2 \pmod 3), 2 \leq j \leq 3k - 4\}$.

In case 3, $S \setminus \{v_{3k}\} \in A_{k-1}$. Furthermore, if $S' \in A_{k-1}$ then $S' \cup \{v_{3k}\} \in B_k$. Similar converses apply to cases 1 and 2: in case 2, if S' is the unique γ-set in $P_{3(k-1)}$ then $S' \cup \{v_{3k-1}, v_{3k}\} \in B_k$, and in case 1, if $S' \in B_{k-1}$ then $S \cup \{v_{3k}\} \in B_k$. Putting these observations together, we have that $b_k = a_{k-1} + b_{k-1} + 1 \quad (*)$.

If $S \in A_k$ then $|S \setminus \{v_{3k+1}\}| = k$; since each set T_r, $r = 0, \ldots, k-1$ must contain at least one vertex of $S \setminus \{v_{3k+1}\}$, it follows that each T_r contains exactly one vertex of S. It follows that $v_{3r+3} \notin S$, $r = o \ldots k - 1$, for if $v_{3r+3} \in S$ then, because there cannot be more than 2 vertices between consecutive elements of S, it would follow that $v_{3s+3} \in S$, $0 \leq s \leq r$. But then v_3 is the sole representative of S in T_0, which contradicts the premise that S is a γ-set in G_k.

Thus, $S \in A_k$ implies $v_{3k} \notin S$, so the sole vertex in $S \cap T_{k-1}$ is either v_{3k-1} or $v_{3k-2} = v_{3(k-1)+1}$. If $v_{3k-1} \in S$, then $S \setminus \{v_{3k+1}\}$ is a γ-set in P_{3k}, so $S = \{v_t \mid t = 2 \pmod 3),\ 2 \leq t \leq 3k - 1\} \cup \{v_{3k+1}\}$. If $v_{3k-2} \in S$, then $S \setminus \{v_{3k+1}\} \in A_{k-1}$; conversely, if $S' \in A_{k-1}$ then $v_{3k-2} \in S'$ and $S' \cup \{v_{3k-2}\} \in A_k$.

From the preceding paragraph we conclude that, for $k \geq 2$, $a_k = 1 + a_{k-1}$. Since $a_1 = 2$, it follows that $a_k = k + 1$, $k = 1, 2, \ldots$

From $(*)$, we have that, for $k \geq 2$,

$$\begin{aligned} b_k &= a_{k-1} + b_{k-1} + 1 = b_{k-1} + k + 1 \\ &= b_1 + [(k+1) + k + \ldots + 3] \\ &= 2 + \frac{(k+2)(k+1)}{2} - 3 \\ &= \frac{k^2 + 3k}{2}. \end{aligned}$$

Therefore, for $k \geq 2$,

$$\begin{aligned} \zeta(G_k) &= a_k + b_k = k + 1 + \frac{k^2 + 3k}{2} \\ &= \frac{k^2 + 5k + 2}{2} = \frac{(n+2)(n+11)}{18} - 1 \end{aligned}$$

if $n = 3k + 1$.

□

Lemma 2.6. *If $n = 3k + 2$, then no γ-set of P_n includes the vertex v_3 when $n = 3k + 2$.*

Proof. Suppose that $k \geq 1$ and $n = 3k+2$. As with any path or cycle, if $S \subseteq V(P_n)$ is dominating, then $V(P_n) \setminus S$ cannot contain 3 "consecutive" vertices, v_{i-1}, v_i, v_{i+1}. Therefore, for each such S, each set $T_r = \{v_{3r+1}, v_{3r+2}, v_{3r+3}\}$, $r = 0, \ldots, k-1$, contains at least one vertex of S.

Now suppose that S is a γ-set in P_n. Then, one of v_{3k+1}, v_{3k+2} is in S; otherwise, v_{3k+2} would not be covered. Since $|S| = 3k + 1$, it must be that each T_r contains exactly one element of S. Therefore, if $v_3 \in S$, then v_3 is the sole representative of S in T_r. But then v_1 is not covered by S. □

Corollary 2.7. *If G is a cycle on $n = 3k$ vertices, then its dominion is $\zeta(G) = 3$.*

Proof. Throughout we obtain $G := C_n$ from a path $P_n := v_1 - v_2 - \ldots - v_{n-1} - v_n$ by connecting v_1 to v_n, i.e., $C_n := v_1 - v_2 - \ldots - v_{n-1} - v_n - v_1$.

Claim 2.8. *Every γ-set of P_n is a γ-set of C_n, if $n = 3k$.*

Proof. The claim is straightforward from the fact that $\gamma(P_{3k}) = \gamma(C_{3k}) = k$, and the fact that any dominating set in P_n will be dominating in C_n. □

Claim 2.9. $\zeta(C_{3k}) = \zeta(P_{3k}) + 2 = 3$ *for all $k \geq 1$.*

Proof. The unique γ-set in P_{3k}, $S = \{v_2, v_5, \ldots, v_{3k-1}\}$, is also a γ-set of C_{3k}. Therefore, the 2 rotations, say, clockwise, of this set by 1 and 2 places, are also γ-sets in C_{3k}, and are distinct subsets of $\{v_1, \ldots, v_{3k}\}$. Further rotations will just duplicate the 3 sets already discovered. Thus, $\gamma(C_{3k}) \geq 3$ and to show equality it suffices to show that any γ-set S of vertices of C_{3k} must be one of these 3:

If S is a γ-set in C_{3k} then $|S| = k$. Then at least two consecutive vertices $u, w \in V(C_{3k}) \setminus S$. Let us rename the vertices around the cycle so that $u = v_{3k}$ and $w = v_1$. Since neither v_1, v_{3k} is in S, yet S is a dominating set in C_{3k}, it must be that S is a dominating set in the path $P_{3k} := v_1 - \ldots - v_{3k}$. Therefore $S = \{v_2, v_5, \ldots, v_{3k-1}\}$, which, with respect to any other naming of the consecutive vertices round C_{3k}, is one of those 3 γ-sets in C_{3k} already discovered.

□

The result follows from Theorem 2.5 and Claim 2.9.

□

Remark 2.10.

The cases when $n = 3k + 1$ and $n = 3k + 2$ turn out to be more difficult to establish and perhaps require a novel technique. However, using Lagrange interpolation on some early recurring ζ-values, we present a conjecture.

Conjecture 2.1. *If G is a cycle on $n \geq 3$ vertices, then its dominion is*

$$\zeta(G) = \begin{cases} \frac{n^2+5n}{6}, & \text{if } n \equiv 1 \pmod 3 \\ n, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

3 Dominion of Join of Graphs

There is no obvious comparative bound between $\gamma(G)$ and $\zeta(G)$ because, while the former determines the size of the minimum dominating set, the latter counts such sets. Yet, with the few upcoming results, we establish some relationships between these two parameters, in particular using the *join* operation. Recall, the join of G_1 and G_2, commonly denoted by $G = G_1 \vee G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

Proposition 3.1. *For any connected graph G on n vertices,* $1 \leq \zeta(G) \leq \binom{n}{\gamma(G)}$.

These trivial bounds follow from the fact that any γ-set is some selection from $V(G)$ and ζ counts such sets of size $\gamma(G)$. The lower bound is tight for stars while the upper bound is tight for complete graphs.

Theorem 3.2. *Suppose* G_1 *and* G_2 *are two connected graphs with* $2 \leq \gamma(G_1) \leq \gamma(G_2)$. *Then* $\zeta(G_1 \vee G_2) \geq |V(G_1)| \cdot |V(G_2)|$.

Proof. Suppose $G = G_1 \vee G_2$ with $2 \leq \gamma(G_1) \leq \gamma(G_2)$. Consider any pair $\{u, v\}$ with $u \in V(G_1)$, $v \in V(G_2)$. It follows that it is a γ-set of G, since u covers $V(G_2)$ and v covers $V(G_1)$, giving the bound. □

The previous argument also establishes that $1 \leq \gamma(G_1 \vee G_2) \leq 2$, given any two graphs G_1, G_2.

Theorem 3.3. *Suppose* G_1 *and* G_2 *are two connected graphs with* $\gamma(G_1) \leq \gamma(G_2)$. *Then*

$$\gamma(G_1 \vee G_2) = \begin{cases} 1 & \text{if } \gamma(G_1) = 1 \\ 2 & \text{if } \gamma(G_1) \geq 2. \end{cases}$$

Moreover,

$$\zeta(G_1 \vee G_2) = \begin{cases} \zeta(G_1) + \zeta(G_2) + \begin{cases} 0 & \text{if } \gamma(G_1) = 1 = \gamma(G_2) \\ |V(G_1)| \cdot |V(G_2)|, & \text{if } \gamma(G_1) = 2 = \gamma(G_2) \end{cases} \\ \\ \zeta(G_1) \qquad + \begin{cases} 0 & \text{if } \gamma(G_1) = 1 < \gamma(G_2) \\ |V(G_1)| \cdot |V(G_2)|, & \text{if } \gamma(G_1) = 2 < \gamma(G_2) \end{cases} \\ \\ |V(G_1)| \cdot |V(G_2)|, & \text{if } 2 < \gamma(G_1) \leq \gamma(G_2). \end{cases}$$

Proof. The first part of the statement regarding the values of $\gamma(G_1 \vee G_2)$ is trivial.

So, we proceed to establish the results in the second part of the statement. Throughout, we denote by S_1 and S_2 any γ-sets of G_1 and G_2, respectively.
(a)
Suppose $\gamma(G_1) = 1 = \gamma(G_2)$. Given the join operation, any vertex $v_1 \in S_1$ also covers any other vertex $v_2 \in V(G_2)$. Likewise, any vertex $v_2 \in S_2$ also covers any other vertex $v_1 \in V(G_1)$. Since S_1 and S_2 are any γ-sets of G_1 and G_2, respectively, the result follows.
(b)
Suppose $\gamma(G_1) = 2 = \gamma(G_2)$. It is clear that each γ-set S_i, $i = 1, 2$, is also a γ- set of $G_1 \vee G_2$, following from (a). The result follows the argument

presented in the proof of Theorem 3.2.
(c)
Suppose $\gamma(G_1) < \gamma(G_2)$ with $\gamma(G_1) \in \{1, 2\}$. It follows that $\gamma(G_1 \vee G_2) = \gamma(G_1)$. Further, if $\gamma(G_1) = 1$, then S_1 is also a γ-set of $G_1 \vee G_2$ and S_2 is not, since its size is greater than 1. Similarly, if $\gamma(G_1) = 2$, S_1 is a γ-set of $G_1 \vee G_2$ but not S_2. The result follows from the argument presented in the proof of Theorem 3.2.
(d)
Finally, suppose $2 < \gamma(G_1) \leq \gamma(G_2)$. It must be that $\gamma(G_1 \vee G_2) = 2$. In which case, neither S_1 nor S_2 is a γ-set of $G_1 \vee G_2$. The result follows from the argument presented in the proof of Theorem 3.2.

□

Corollary 3.4. *For any connected graph G with $\gamma(G) = 1$, $\zeta(\underbrace{G \vee \ldots \vee G}_{r}) = r\zeta(G)$, $r \geq 1$.*

Proof. From Part (a) of Theorem 3.3 (when $\gamma(G_1) = 1 = \gamma(G_2)$), the result follows by induction on $r \geq 1$. □

We note that in the case when $G = K_1$, it is clear that $\underbrace{G \vee \ldots \vee G}_{r} := K_r$ and $\zeta(K_r) = r = r\zeta(K_1)$. Here, we present a more general result for the special case when $G = \overline{K_m}$, the complement of a complete graph on $m \geq 1$ vertices. Note that $\overline{K_{m_1}} \vee \ldots \vee \overline{K_{m_k}}$ is a complete k-partite graph often written as $K(m_1, m_2, \ldots, m_k)$, with $m_i \geq 1$ and $k \geq 2$.

Theorem 3.5. *Suppose $G = K(m_1, m_2, \ldots, m_k)$ is a complete k-partite graph with $m_1 \leq m_2 \leq \ldots \leq m_k$. For some $r \geq 1$, we have*

$$\zeta(G) = \begin{cases} r & \text{if } m_i = 1,\ i = 1, \ldots, r \text{ and } m_i > 1,\ r < i \leq k, \\ \displaystyle\sum_{\substack{1 \leq i,j \leq r \\ i \neq j}} m_i m_j + & \begin{cases} r & \text{if } m_i = 2,\ i = 1, \ldots, r \text{ and } m_i > r,\ r < i \leq k, \\ 0 & \text{if } m_i > 2\ i = 1, \ldots, r = k. \end{cases} \end{cases}$$

Proof. Let V_i denote each disjoint partite (vertex) set of G, with $|V_i| = m_i$, $i = 1, \ldots, k$. If, for some $j = 1, \ldots, r$, $m_j = 1$ and $m_i > 1$ for $i > r$, then it is clear that $\gamma(G) = 1$ and the γ-sets in G are the singleton subsets of $V_1 \cup \ldots \cup V_r$. It follows that $\zeta(G) = r$.

Assume $m_i > 1$, for $i = 1, \ldots, k$. It is clear that $\gamma(G) = 2$. Any vertex $v \in V_i$ covers all vertices $v' \in V(G) \setminus V_i$ except those in V_i. Thus, for each $v \in V_i$ and $v' \in V_j$, $i \neq j$, $\{v, v'\}$ is a γ-set of G. It follows that there are

$\sum_{i \neq j}^{k} m_i m_j$ total such sets. Further, if $|V_i| = 2$ for $i = 1, \ldots, r$, then each such sets, V_i, is an additional γ-set, giving the result. $\square$

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