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On graded J_{gr}-2-absorbing and graded weakly J_{gr}-2-absorbing submodules of graded modules over graded commutative rings

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Abstract

Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concepts of graded J_{gr}-2-absorbing and graded weakly J_{gr}-2-absorbing submodules of M and give some basic properties of these classes of graded submodules.

1 Introduction and Preliminaries

The concepts of graded 2-absorbing ideal and graded weakly 2-absorbing ideal, generalizations of graded prime ideals and graded weakly prime ideals, respectively, were studied by Al-Zoubi, Abu-Dawwas, Ceken among other authors [3, 13, 18]. For the concepts of graded prime submodule and graded weakly prime submodule, see for example [1, 10-12, 14-15, 22]. The concepts of graded 2-absorbing submodule and graded weakly 2-absorbing submodules, generalizations of graded prime submodules and graded weakly prime submodules, respectively, were introduced by Al-Zoubi and Abu-Dawwas in [2] and studied in [6-7]. Then many generalizations of graded (weakly)

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2-absorbing submodules such as graded primary 2-absorbing [16], graded weakly primary 2-absorbing [5] and graded classical 2-absorbing [4] were studied.

Here, we introduce the concept of graded (weakly) J_{gr}-2-absorbing submodule as a new generalization of a graded (weakly) 2-absorbing submodule. We generalize some basic properties of graded (weakly) 2-absorbing submodules to graded (weakly) J_{gr}-2-absorbing submodules. Throughout this paper, all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer the reader to [17] and [19-21] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group with identity e. By a G-graded ring, we mean a ring R together with direct sum decomposition (as abelian group) $R = \bigoplus_{\alpha\in G} R_\alpha$ with the property that $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all α, $\beta \in G$. The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$; i.e., $h(R) = \cup_{\alpha\in G} R_\alpha$. If $a \in R$, then a can be written uniquely as $\sum_{\alpha\in G} a_\alpha$, where a_α is called a homogeneous component of a in R_α. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal A of R is said to be a graded ideal if $A = \bigoplus_{\alpha\in G}(A \cap R_\alpha) := \bigoplus_{\alpha\in G} A_\alpha$ [21].

Let R be a G-graded ring and M be an R-module. Then M is called *a G-graded R-module* if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Also, if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called homogeneous. Let R be a G-graded ring and M be a graded R-module. A submodule N of M is said to be *a graded submodule of* M if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. In this case, N_α is called the α-component of N [21].

Let R be a G-graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{\alpha\in G}(S^{-1}R)_\alpha$, where $(S^{-1}R)_\alpha = \{r/s : r \in R, s \in S \ \ \textit{and}\ \alpha = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{\alpha\in G}(S^{-1}M)_\alpha$, where $(S^{-1}M)_\alpha = \{m/s : m \in M, s \in S \ \ \textit{and}\ \alpha = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \bigcup_{\alpha\in G}(S^{-1}R)_\alpha$ and $h(S^{-1}M) = \bigcup_{\alpha\in G}(S^{-1}M)_\alpha$ [21].

Let R be a G-graded ring, M a graded R-module and K a graded submodule of M. Then $(K :_R M)$ is defined as $(K :_R M) = \{a \in R : aM \subseteq K\}$. If K is a graded submodule of M, then $(K :_R M)$ is a graded ideal of R [15].

A proper graded submodule K of M is said to be *a graded maximal submodule* if there is no graded submodule L of M such that $K \subsetneq L \subsetneq M$, [21].

The graded Jacobson radical of a graded module M, denoted by $J_{gr}(M)$, is defined to be the intersection of all graded maximal submodules of M (if M has no graded maximal submodule, then we shall take, by definition, $J_{gr}(M) = M$) [21].

A proper graded submodule N of a graded R-module M is said to be *a graded (resp. graded weakly) 2-absorbing submodule of M* if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$ (resp. $0 \neq r_g s_h m_\lambda \in N$), then either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $r_g s_h \in (N :_R M)$ [2].

A proper graded submodule N of a graded R-module M is said to be *a graded J_{gr}-prime submodule* if whenever $r_g \in h(R)$ and $m_\lambda \in h(M)$ with $r_g m_\lambda \in N$, then either $m_\lambda \in N + J_{gr}(M)$ or $r_g \in (N + J_{gr}(M) :_R M)$.

2 Graded J_{gr}-2-absorbing submodules

Definition 2.1. *Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be a graded J_{gr}-2-absorbing submodule of M if whenever r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N + J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$.*

It is clear that every graded 2-absorbing submodule is a graded J_{gr}-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.2. *Let $G = \mathbb{Z}_2$ and let $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{16}$ be a graded R-module with $M_0 = \mathbb{Z}_{16}$ and $M_1 = \{0\}$. Now, consider the graded submodule $N = (\bar{8})$ of M. Then N is not a graded 2-absorbing submodule since $2 \in R_0, \bar{2} \in M_0$ with $2 \cdot 2 \cdot \bar{2} \in (\bar{8})$ but neither $2 \cdot \bar{2} \in (8)$ nor $2 \cdot 2 = 4 \in ((\bar{8}) :_{\mathbb{Z}} \mathbb{Z}_{16}) = 8\mathbb{Z}$. However, an easy computation shows that N is a graded J_{gr}-2-absorbing submodule.*

It is clear that every graded J_{gr}-prime submodule is a graded J_{gr}-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.3. *Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_6$ be a graded R-module with $M_0 = \mathbb{Z}_6$ and $M_1 = \{0\}$. Now, consider the graded submodule $N = (\bar{0})$ of M. Then N is not a graded J_{gr}-prime since $2 \in R_0$ and $\bar{3} \in M_0$ with $2 \cdot \bar{3} \in (\bar{0})$ but neither $\bar{3} \in N + J_{gr}(M) = (\bar{0}) + (\bar{0}) = (\bar{0})$ nor $2 \in (N + J_{gr}(M) :_R M) = ((\bar{0}) + (\bar{0}) :_R \mathbb{Z}_6) = 6\mathbb{Z}$. However an easy computation shows that N is a graded J_{gr}-2-absorbing submodule of M.*

Remark 2.4. *Let R be a G-graded ring and M a graded R-module.*

(i) *If $J_{gr}(M) = 0$, then every graded J_{gr}-2-absorbing submodule of M is a graded 2-absorbing submodule of M.*

(ii) *If N is a graded J_{gr}-2-absorbing submodule of M with $J_{gr}(M) \subseteq N$, then N is a graded 2-absorbing submodule of M.*

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N, L be two graded submodules of M such that $N \subsetneq L$. If N is a graded J_{gr}-2-absorbing submodule of M and $J_{gr}(M) \subseteq J_{gr}(L)$, then N is a graded J_{gr}-2-absorbing submodule of L.*

Proof. Let r_g, $s_h \in h(R)$ and $m_\lambda \in L \cap h(M)$ such that $r_g s_h m_\lambda \in N$. Then either $r_g m_\lambda \in N + J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$ as N is a graded J_{gr}-2-absorbing submodule of M. Since $J_{gr}(M) \subseteq J_{gr}(L)$, we get either $r_g m_\lambda \in N + J_{gr}(L)$ or $s_h m_\lambda \in N + J_{gr}(L)$ or $r_g s_h \in (N + J_{gr}(L) :_R M)$. Therefore, N is a graded J_{gr}-2-absorbing submodule of L. □

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module and N, L be two graded submodules of M with $L \nsubseteq N$ and $J_{gr}(M) = J_{gr}(L)$. If N is a graded J_{gr}-2-absorbing submodule of M, then $N \cap L$ is a graded J_{gr}-2-absorbing submodule of L.*

Proof. Since $L \nsubseteq N$, $N \cap L$ is a proper graded submodule of L. Now, let $r_g, s_h \in h(R)$ and $l_\lambda \in L \cap h(M)$ with $r_g s_h l_\lambda \in N \cap L$, so $r_g s_h l_\lambda \in N$. Then either $r_g l_\lambda \in N + J_{gr}(M) = N + J_{gr}(L)$ or $s_h l_\lambda \in N + J_{gr}(M) = N + J_{gr}(L)$ or $r_g s_h M \subseteq N + J_{gr}(M) = N + J_{gr}(L)$. If $r_g l_\lambda \in N + J_{gr}(L)$, then $r_g l_\lambda \in (N + J_{gr}(L)) \cap L \subseteq (N \cap L) + J_{gr}(L)$ by modular law. If $s_h l_\lambda \in N + J_{gr}(L)$, then $s_h l_\lambda \in (N + J_{gr}(L)) \cap L \subseteq (N \cap L) + J_{gr}(L)$ by modular law. Now, if $r_g s_h M \subseteq N + J_{gr}(L)$, then $r_g s_h L = r_g s_h M \cap r_g s_h L \subseteq (N + J_{gr}(L)) \cap L \subseteq (N \cap L) + J_{gr}(L)$ by modular law. Therefore, $N \cap L$ is a graded J_{gr}-2-absorbing submodule of L. □

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded J_{gr}-2-absorbing submodule of M, then $S^{-1}N$ is a graded J_{gr}-2-absorbing submodule of $S^{-1}M$.*

Proof. Let $\frac{r_{g_1}}{s_{h_1}}, \frac{r_{g_2}}{s_{h_2}} \in h(S^{-1}R)$ and let $\frac{m_{g_3}}{s_{h_3}} \in h(S^{-1}M)$ such that $\frac{r_{g_1}}{s_{h_1}}\frac{r_{g_2}}{s_{h_2}}\frac{m_{g_3}}{s_{h_3}} \in S^{-1}N$. Then there exists $s_{h_4} \in S$ such that $s_{h_4}r_{g_1}r_{g_2}m_{g_3} \in N$, so either $s_{h_4}r_{g_1}m_{g_3} \in N + J_{gr}(M)$ or $s_{h_4}r_{g_2}m_{g_3} \in N + J_{gr}(M)$ or $r_{g_1}r_{g_2} \in (N + J_{gr}(M) :_R M)$. If $s_{h_4}r_{g_1}m_{g_3} \in N + J_{gr}(M)$, then $\frac{s_{h_4}r_{g_1}m_{g_3}}{s_{h_4}s_{h_1}s_{h_3}} = \frac{r_{g_1}}{s_{h_1}}\frac{m_{g_3}}{s_{h_3}} \in S^{-1}(N + J_{gr}(M)) \subseteq S^{-1}N + J_{gr}(S^{-1}M)$. If $s_{h_4}r_{g_2}m_{g_3} \in N + J_{gr}(M)$, then $\frac{s_{h_4}r_{g_2}m_{g_3}}{s_{h_4}s_{h_2}s_{h_3}} = \frac{r_{g_2}}{s_{h_2}}\frac{m_{g_3}}{s_{h_3}} \in S^{-1}(N+J_{gr}(M)) \subseteq S^{-1}N+J_{gr}(S^{-1}M)$. If $r_{g_1}r_{g_2} \in (N+J_{gr}(M) :_R M)$, then $\frac{r_{g_1}r_{g_2}}{s_{h_1}s_{h_2}} = \frac{r_{g_1}}{s_{h_1}}\frac{r_{g_2}}{s_{h_2}} \in S^{-1}(N + J_{gr}(M) :_R M) \subseteq (S^{-1}N + J_{gr}(S^{-1}M) :_{S^{-1}R} S^{-1}M)$. Therefore, $S^{-1}N$ is a graded J_{gr}-2-absorbing submodule of $S^{-1}M$. □

Let R be a G-graded ring and M, M' graded R-modules. Let $f : M \longrightarrow M'$ be an R-module homomorphism. Then f is said to be a graded homomorphism if $f(M_g) \subseteq M'_g$ for all $g \in G$ (see [21].)

Recall that a proper graded submodule S of a graded R-module M is said to be a gr-small submodule of M (for short $S <<_g M$) if for every proper graded submodule K of M, we have $S + K \neq M$ (see [9].)

Theorem 2.8. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ be a graded epimorphism.*

(i) *If N is a graded J_{gr}-2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded J_{gr}-2-absorbing submodule of M'.*

(ii) *If N' is a graded J_{gr}-2-absorbing submodule of M' with $ker(f) <<_g M$, then $f^{-1}(N')$ is a graded J_{gr}-2-absorbing submodule of M.*

Proof. (i) Suppose that N is a graded J_{gr}-2-absorbing submodule of M. It is easy to see that $f(N)$ is a proper graded submodule of M'. Now, let $r_g, s_h \in h(R)$ and $m'_\lambda \in h(M')$ with $r_gs_hm'_\lambda \in f(N)$. Since f is a graded epimorphism, there exists $m_\lambda \in h(M)$ such that $f(m_\lambda) = m'_\lambda$. So $r_gs_hf(m_\lambda) = f(r_gs_hm_\lambda) \in f(N)$, so there exists $n_\alpha \in N \cap h(M)$ such that $f(r_gs_hm_\lambda) = f(n_\alpha)$. Thus $r_gs_hm_\lambda - n_\alpha \in ker(f) \subseteq N$, it follows that $r_gs_hm_\lambda \in N$. Then either $r_gm_\lambda \in N + J_{gr}(M)$ or $s_hm_\lambda \in N + J_{gr}(M)$ or $r_gs_hM \subseteq N+J_{gr}(M)$ as N is a graded J_{gr}-2-absorbing submodule of M. This implies that either $r_gm'_\lambda \in f(N)+f(J_{gr}(M))$ or $s_hm'_\lambda \in f(N)+f(J_{gr}(M))$ or $r_gs_hM' \subseteq f(N)+f(J_{gr}(M))$. By [8, Theorem 2.12 (i)], we have $f(J_{gr}(M)) \subseteq$

$J_{gr}(M')$. Hence either $r_g m'_\lambda \in f(N) + J_{gr}(M')$ or $s_h m'_\lambda \in f(N) + J_{gr}(M')$ or $r_g s_h M' \subseteq f(N) + J_{gr}(M')$. Therefore, $f(N)$ is a graded J_{gr}-2-absorbing submodule of M'.

(ii) Suppose that N' is a graded J_{gr}-2-absorbing submodule of M'. It is easy to see $f^{-1}(N')$ is a proper graded submodule of M. Now, let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in f^{-1}(N')$, so $r_g s_h f(m_\lambda) \in N'$. Hence either $r_g f(m_\lambda) \in N' + J_{gr}(M')$ or $s_h f(m_\lambda) \in N' + J_{gr}(M')$ or $r_g s_h M' \subseteq N' + J_{gr}(M')$ as N' is a graded J_{gr}-2-absorbing submodule of M'. Since $ker(f) <<_g M$, by [8, Theorem 2.12 (ii)], we get $f(J_{gr}(M)) = J_{gr}(M')$. This implies that either $r_g m_\lambda \in f^{-1}(N') + J_{gr}(M)$ or $s_h m_\lambda \in f^{-1}(N') + J_{gr}(M)$ or $r_g s_h M \subseteq f^{-1}(N') + J_{gr}(M)$. Therefore, $f^{-1}(N')$ is a graded J_{gr}-2-absorbing submodule of M. □

3 Graded weakly J_{gr}-2-absorbing submodules

Definition 3.1. *Let R be a G-graded ring and let M a graded R-module. A proper graded submodule N of M is said to be a graded weakly J_{gr}-2-absorbing submodule of M, if whenever r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N + J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$.*

It is clear that every graded weakly 2-absorbing submodule is a graded weakly J_{gr}-2-absorbing submodule. The converse is not true in general. For example, let's take a graded module M over a G-graded ring R as in Example 2.2. Then $N = (\bar{8})$ is a graded weakly J_{gr}-2-absorbing submodule submodule of M. However, N is not a graded weakly 2-absorbing submodule since $0 \neq 2 \cdot 2 \cdot \bar{2} \in (\bar{8})$ but neither $2 \cdot \bar{2} \in (\bar{8})$ nor $2 \cdot 2 = 4 \in ((\bar{8}) :_{\mathbb{Z}} \mathbb{Z}_{16}) = 8\mathbb{Z}$.

Remark 3.2. *Let R be a G-graded ring and M a graded R-module.*

(i) *If $J_{gr}(M) = 0$, then every graded weakly J_{gr}-2-absorbing submodule of M is a graded weakly 2-absorbing submodule of M.*

(ii) *If N is a graded weakly J_{gr}-2-absorbing submodule of M with $J_{gr}(M) \subseteq N$, then N is a graded weakly 2-absorbing submodule of M.*

The following example shows that the intersection of graded weakly J_{gr}-2-absorbing submodules need not be a graded weakly J_{gr}-2-absorbing submodule.

Example 3.3. *Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 6\mathbb{Z}$ and $K = 7\mathbb{Z}$ of M. Clearly N and K are graded weakly J_{gr}-2-absorbing submodules since they are graded weakly 2-absorbing submodule of M. But $N \cap K = 42\mathbb{Z}$ is not a graded weakly J_{gr}-2-absorbing submodule of M since $0 \neq 2 \cdot 3 \cdot 7 \in 42\mathbb{Z}$ and neither $2 \cdot 7 \in 42\mathbb{Z}$ nor $3 \cdot 7 \in 42\mathbb{Z}$ nor $2 \cdot 3 \in 42\mathbb{Z}$.*

Theorem 3.4. *Let R be a G-graded ring, M a graded R-module and N, L be two graded submodules of M such that $N \subseteq J_{gr}(M)$ and $L \subseteq J_{gr}(M)$. If N and L are graded weakly J_{gr}-2-absorbing submodules of M, then $N \cap L$ is a graded weakly J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N \cap L$. Since N is a graded weakly J_{gr}-2-absorbing submodule of M, we get either $r_g m_\lambda \in N + J_{gr}(M) = J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M) = J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M) = (J_{gr}(M) :_R M)$. Similarly, since L is a graded weakly J_{gr}-2-absorbing submodule of M, we get either $r_g m_\lambda \in J_{gr}(M)$ or $s_h m_\lambda \in J_{gr}(M)$ or $r_g s_h \in (J_{gr}(M) :_R M)$. Thus either $r_g m_\lambda \in N \cap L + J_{gr}(M)$ or $s_h m_\lambda \in N \cap L + J_{gr}(M)$ or $r_g s_h \in (N \cap L + J_{gr}(M) :_R M)$. Therefore, $N \cap L$ is a graded weakly J_{gr}-2-absorbing submodule of M. □

Theorem 3.5. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If for each r_g, $s_h \in h(R)$ and graded submodule L of M with $0 \neq r_g s_h L \subseteq N$, implies either $r_g L \subseteq N + J_{gr}(M)$ or $s_h L \subseteq N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$, then N is a graded weakly J_{gr}-2-absorbing submodule of M.*

Proof. Let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$. Let $L = Rm_\lambda$ be a graded submodule of M generated by m_λ. Then $0 \neq r_g s_h L \subseteq N$. By our assumption, we get either $r_g L \subseteq N + J_{gr}(M)$ or $s_h L \subseteq N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$, which yields that either $r_g m_\lambda \in N + J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$. Therefore, N is a graded weakly J_{gr}-2-absorbing submodule of M. □

Theorem 3.6. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded weakly J_{gr}-2-absorbing submodule of M, then $S^{-1}N$ is a graded weakly J_{gr}-2-absorbing submodule of $S^{-1}M$.*

Proof. The proof is similar to that of Theorem 2.7, so we omit it. □

Theorem 3.7. *Let R be a G-graded ring and let M, M' be two graded R-modules and $f : M \to M'$ be a graded homomorphism.*

(i) *If f is a graded epimorphism and N is a graded weakly J_{gr}-2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded weakly J_{gr}-2 -absorbing submodule of M'.*

(ii) *If $f : M \to M'$ is a graded isomorphism and N' is a graded weakly J_{gr}-2-absorbing submodule of M' with $ker(f) <<_g M$, then $f^{-1}(N')$ is a graded weakly J_{gr}-2-absorbing submodule of M.*

Proof. The proof is similar to that of Theorem 2.8, so we omit it. □

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