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The exact solutions for the nonlinear Schrödinger equation forced by multiplicative noise on Itô sense

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Abstract

We consider the stochastic nonlinear Schrödinger equation in Itô sense with multiplicative noise. Moreover, we apply the He's semi-inverse and tanh-coth techniques in order to get new exact solutions which describe some complex phenomena in applied science. Furthermore, we display the influence of multiplicative noise on the solutions for the stochastic nonlinear Schrödinger equation.

1 Introduction

Throughout the field of nonlinear mathematical physics the search for solutions to nonlinear partial differential equations (NPDEs) has drawn much interest [1, 2, 3, 4, 5, 6]. Several useful approaches have been used and improved to construct the exact solutions of nonlinear PDEs; for instance, sine-cosine method [7], $(\frac{G'}{G})$-expansion method [8], perturbation method [9, 10], tanh-sech method [11], Riccati-Bernoulli sub-ODE technique [12, 13], $exp(-(\varphi))$-expansion method [14], Jacobi elliptic function method [15]. One of the basic models of nonlinear waves is the nonlinear Schrödinger equation (NLSE). It has many applications in the theory of solids [16] and crystals [17], in laser beams [18], and in electromechanical systems [19].

Noise, often called fluctuations or randomness, has now been found to be significant in many phenomena. Thus, it has become important to involve random effects when modeling various physical phenomena that occur in biology, physics, engineering, environmental sciences, oceanography, meteorology, to name a few. Equations that consider time-dependent random fluctuations are called stochastic differential equations.

Thermal fluctuations may be modeled using stochastic processes in Schrödinger equations. The existence and uniqueness of stochastic Schrödinger equations of either multiplicative or additive noise have been studied by several authors. The additive noise case is discussed in [20, 21], while the multiplicative noise case is studied in [22, 23, 24]. In addition, both cases are discussed in [25, 26].

We consider the following stochastic NLSE with multiplicative noise in the Itô sense:

$$i\psi_t - \psi_{xx} + 2\left|\psi\right|^2\psi - 2\rho^2\psi + \sigma\psi W_t = 0, \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}, \quad (1.1)$$

where $\psi(t,x)$ is a complex-valued process, ρ and σ are constants, and $W(t)$ is the Wiener process (also called Brownian motion) and $W_t = \frac{dW}{dt}$. The Wiener process $W(t)$ is a stochastic process and satisfies the following properties: (i) $W(0) = 0$, (ii) The process $\{W(t)\}_{t\geq 0}$ has stationary, independent increments, (iii) W has continuous trajectories, (iv) For $s < t$ the stochastic variable $W(t) - W(s)$ has the normal distribution $N(0; t-s)$. The multiplicative noise in Eq. (1.1) describes a process in which the excitation phase is disrupted. This kind of noise in crystals refers to the dispersion of exciton by phonons due to thermal molecular vibrations.

We aim to get the stochastic solutions for Eq. (1.1) by using two distinct approaches, namely He's semi-inverse and tanh-coth techniques. For physicists, the acquired solutions are very beneficial in explaining various interesting physical phenomena. Moreover, we study the effect of multiplicative noise on the obtained solutions of nonlinear Schrödinger Eq. (1.1).

The rest of the article is divided as follows:
In section 2, we obtain the exact solutions for Eq. (1.1) by applying two distinct methods. In section 3, we discuss the influence of multiplicative noise on the solutions for stochastic NLSE (1.1). In section 4, we conclude this article.

2 Results and Discussions

We present the stochastic solutions of stochastic NLSE (1.1). Consider the wave transformation

$$\psi(t,x) = u(\xi)e^{i\chi}, \quad \xi = k(x + 2\alpha t), \text{ and } \chi = \alpha x + \upsilon t + \sigma W(t), \tag{2.2}$$

where $u(\xi)$ is the traveling wave solution , σ is the noise strength, α, ν and k are unknown constants. Using (2.2) and

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= (2\alpha k u' + i\upsilon u + i\sigma u W_t e^{i\chi}, \\ \frac{\partial^2 \psi}{\partial x^2} &= (k^2 u'' + 2i\alpha k u' - \alpha^2 u)e^{i\chi}, \end{aligned}$$

Eq. (1.1) is transformed into the following ODE:

$$-k^2 u'' + 2u^3 + Hu = 0\,, \tag{2.3}$$

where

$$H = \alpha^2 - 2\rho^2 - \nu. \tag{2.4}$$

2.1 He's semi-inverse method

Here, we utilize semi-inverse technique [27, 28] to get the stochastic solution of (1.1). From Eq. (2.3), we formulate the following variation:

$$J(\xi) = \int_0^\infty \{\frac{k^2}{2}(u')^2 - \frac{1}{2}u^4 - \frac{1}{2}Hu^2\}d\xi. \tag{2.5}$$

We presume the solitary wave solution of Eq. (2.3), according to Ref. [28], is

$$u(\xi) = A \sec h(\xi). \tag{2.6}$$

After substituting Eq. (2.6) into Eq. (2.5),

$$\begin{aligned} J &= \int_0^\infty \left[\frac{A^2k^2}{2}\sec h^2(\xi)\tanh^2(\xi) - \frac{A^4}{2}\sec h^4(\xi) - \frac{A^2}{2}H\sec h^2(\xi)\right]d\xi \\ &= \frac{A^2k^2}{6} - \frac{A^4}{3} - \frac{A^2}{6}H. \end{aligned}$$

Setting $\frac{\partial J}{\partial A} = 0$ after differentiating J with respect to A

$$\frac{\partial J}{\partial A} = \frac{1}{3}(k^2 - H)A - \frac{4}{3}A^3 = 0.$$

Solving the above equation, we get

$$A = \sqrt{\frac{(k^2 - H)}{2}} = \sqrt{\rho^2 + \frac{1}{2}\nu}.$$

Hence, Eq. (2.6) takes the form

$$u_1(\xi) = \sqrt{\rho^2 + \frac{1}{2}\nu}\sec h(\xi).$$

Now, the exact stochastic solution of the schrödinger Eq. (1.1) is

$$\begin{aligned} \psi_1(t,x) &= u(\xi)e^{i(\alpha x + vt + \sigma W(t))} \\ &= e^{i(\alpha x + vt + \sigma W(t))}\sqrt{\rho^2 + \frac{1}{2}\nu}\sec h(kx + 2k\alpha t) \end{aligned}$$

Analogously, we can take

$$u(\xi) = A\sec h^2(\xi),\ u(\xi) = A\csc h(\xi) \text{ and } u(\xi) = A\csc h^2(\xi),$$

to get different forms of solitary wave solutions.

2.2 Tanh-Coth Method

We utilize the tanh-coth method to get the stochastic solution of (1.1). This method was defined by Malfliet [29]. We define the solution u in the form

$$u(\xi) = \sum_{r=0}^{m} a_k z^r, \tag{2.7}$$

where $z = \tanh \xi$ or $z = \coth \xi$. First, let us determine the parameter of m by balancing the highest order of nonlinear term u^3 with the linear term u'' of the highest order, yields

$$m = 1.$$

Now we can rewrite Eq. (2.7) as

$$u(\xi) = a_0 + a_1 z. \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.3) gives

$$2a_1k^2(1 - z^2)z + 2(a_0 + a_1z)^3 + H(a_0 + a_1z) = 0.$$

Putting the coefficient of $z^i (i = 0, 1, 2, 3)$ with zero gives a system of algebraic equations. Solving this system yields

$$a_0 = 0,\ a_1 = \pm k \ \text{ and } k = \pm\sqrt{\frac{-H}{2}}.$$

The solutions of Eq. (2.3) are

$$u_2(\xi) = \pm k \tanh \xi \quad \text{or} \quad u_3(\xi) = \pm k \coth \xi.$$

Consequently, the stochastic solutions of NLSE (1.1) are

$$\psi_2(t, x) = u_2(\xi)e^{i[\alpha x+vt+\sigma W(t)]} = \pm k e^{i[\alpha x+vt+\sigma W(t)]} \tanh(kx + 2\alpha kt), \tag{2.9}$$

or

$$\psi_3(t, x) = \pm k e^{i[\alpha x+vt+\sigma W(t)]} \coth(kx + 2\alpha kt). \tag{2.10}$$

In summary, the exact solutions of the nonlinear Schrödinger equation (1.1) have been stated to have been obtained in an explicit manner. These solutions demonstrate the wave pictures of fiber optics, ocean rogue profiles, multiple types of hydrodynamic plasma instability. Indeed, the obtained solutions interpret the space observations and experimental techniques of femtosecond pulse, nuclear physics, telecommunications experiments, spatio-temporal solutions, chaotic pulses laser and transistor [30, 31, 32, 33]. Moreover, in the development of quantum mechanics, especially in the field of quantum hall effect, nuclear medicine and entire computer industry, the analysis of the solutions obtained is of great importance.

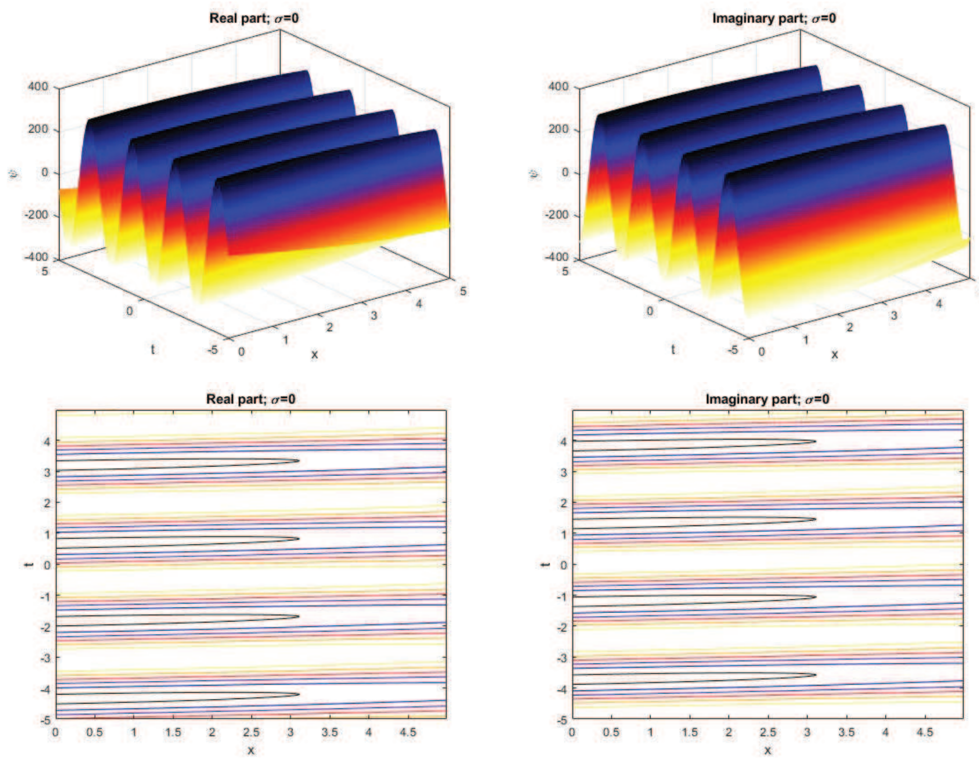

Figure 1: Profile picture to solution of (2.10) for $\sigma = 0$.

3 The Influence of Noise on the NLSE's Solutions

Here, we consider the impact of multiplicative noise on the solution for NLSE (1.1). We implement a variety of numerical simulations conducted by the MATLAB package for the fixed parameters $k = 1$, $\rho = 1$, $\nu = 1$ and $\alpha = 2$ and varying noise intensity σ. One can see that the exact solution of NLSE (1.1) fluctuates and has a pattern if $\sigma = 0$ in Fig. 1. Figures 2–4 show that the pattern starts to destroy when the noise intensity σ increases. Finally, in Figure 5, we give a 2-D graph of the solution of NLSE (1.1) with different values of the noise intensity σ.

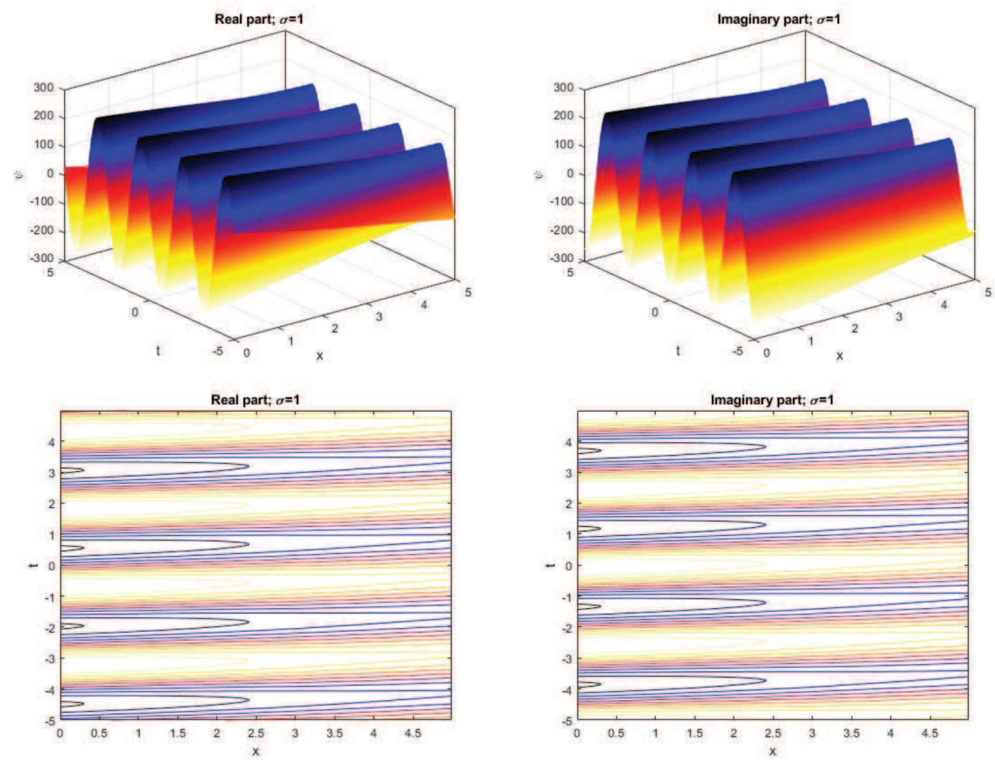

Figure 2: Profile picture to solution of (2.10) for $\sigma = 1$.

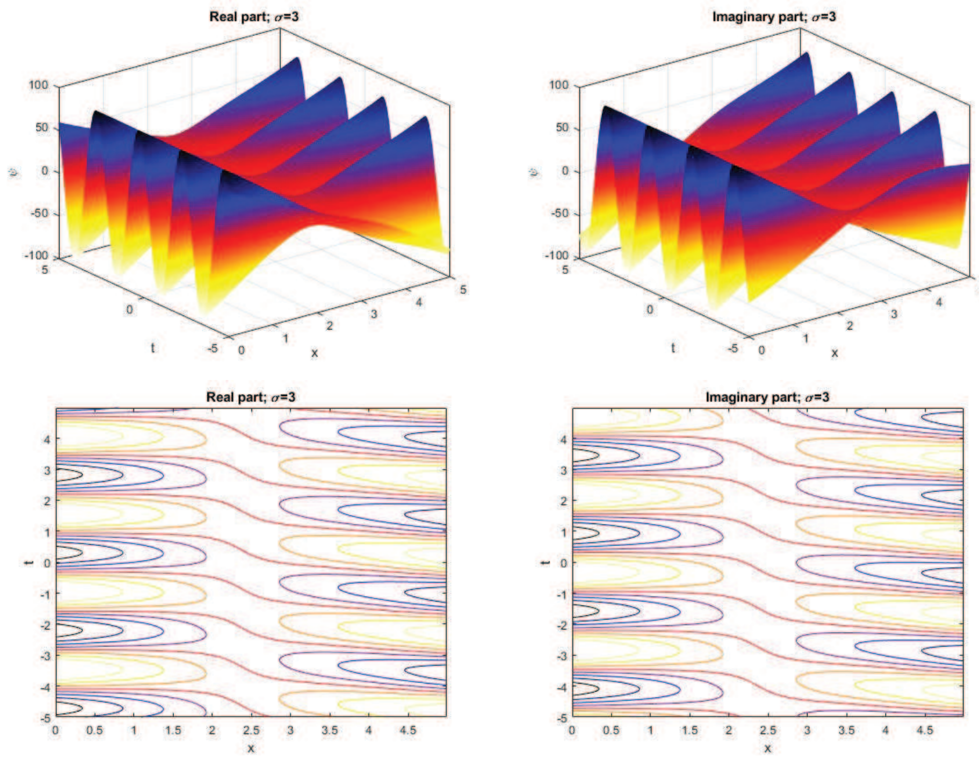

Figure 3: Profile picture to solution of (2.10) for $\sigma = 3$.

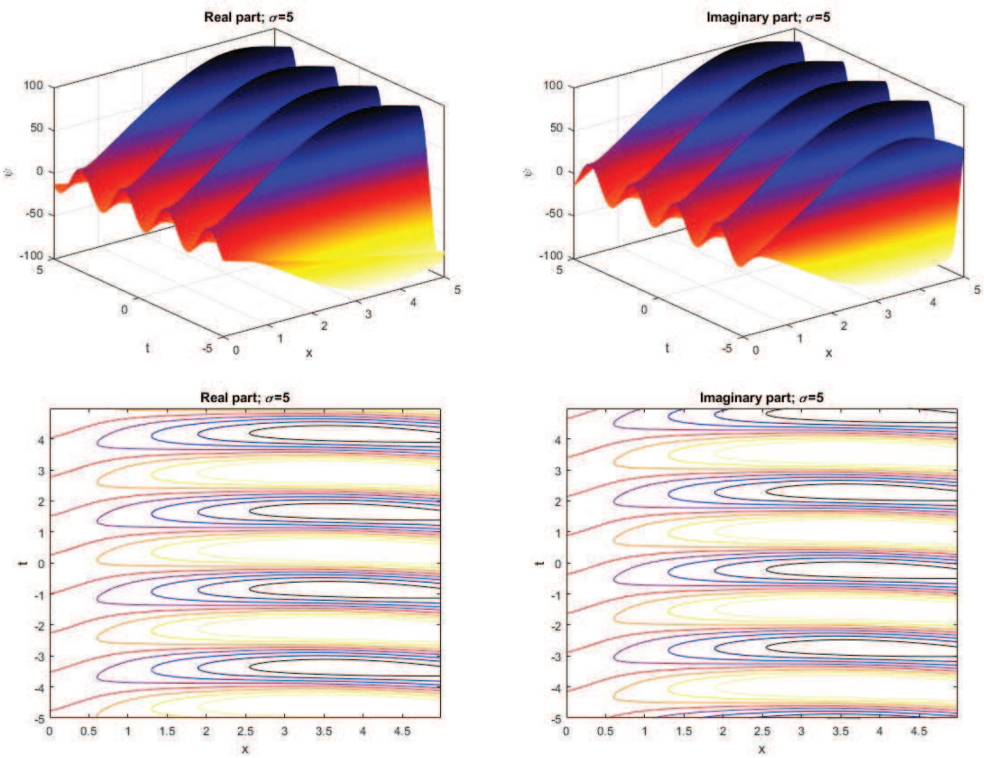

Figure 4: Profile picture to solution of (2.10) for $\sigma = 5$.

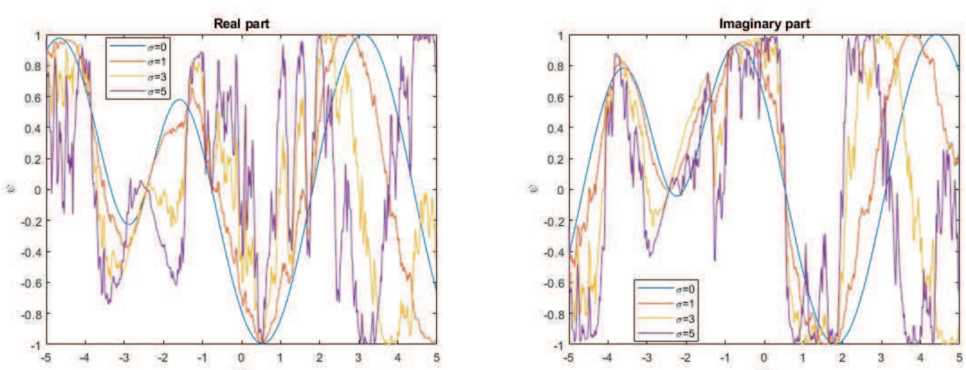

Figure 5: Profile picture to solution of (2.10).

4 Conclusions

We introduced a rich variety of families of wave solutions for stochastic nonlinear Schrödinger equation through Itô sense with multiplicative noise. These solutions are important and helpful for further studies such as the improvement of biomedical, coastal water motions, quasi particle theory, industrial studies, space plasma and fiber applications. We illustrated the influence of multiplicative noise on the exact solutions of stochastic NLSE (1.1). Some graphs were introduced with the aid of Matlab software to illustrate the behavior of these solutions.

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