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Analytical Approximant for a Damped KdV Equation

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Abstract

In this paper, we give an approximate analytical solution to a damped KdV equation with damping that is a function of time. We apply a generalized traveling wave transformation. We then provide an illustrative example. A comparison between the analytical approximant and the numerical solution is analyzed.

1 Introduction

The study of nonlinear evolution equation has been going on for the past few decades. It is an important area of study in the fields of Physics and Mathematics. The Korteweg–de Vries (KdV) equation is a model for many physical phenomena, such as the propagation of small-amplitude large-wavelength waves in shallow waters and in plasma physics. We study a modification of the original KdV equation with a damping that depends on time, which models ion-sound waves damped by ion-neutral collisions. Let us consider the following damped KdV equation [4]

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma(t)u=0. \tag{1.1}$$

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In the case when $\gamma(t) \equiv 0$, we have the following analytical solution

$$u(x,t) = \frac{12\beta k^2 m}{\alpha}\mathrm{cn}^2\left(kx - 4k^3(2m-1)\beta t \middle| m\right). \tag{1.2}$$

Letting $m \to 1$ gives the soliton solution [1], [6]

$$u(x,t) = \frac{12\beta k^2}{\alpha}\mathrm{sech}^2\left(kx - 4\beta k^3 t\right). \tag{1.3}$$

2 Approximate Analytical solution to the damped KdV equation

Suppose that $v = v(x,t)$ is a solution to the KdV

$$\frac{\partial v}{\partial t} + \alpha v\frac{\partial v}{\partial x} + \beta\frac{\partial^3 v}{\partial x^3} = 0. \tag{2.4}$$

We seek an approximate analytical solution to the damped KdV (1.1) in the form

$$u = u(x,t) = f(t)v(g(t)x, h(t)). \tag{2.5}$$

Plugging (2.5) into (1.1) gives

$$\begin{aligned} &\tfrac{\partial u}{\partial t} + \alpha u\tfrac{\partial u}{\partial x} + \beta\tfrac{\partial^3 u}{\partial x^3} + \gamma(t)u = \\ &v(xg(t),h(t))\left(v^{(1,0)}(xg(t),h(t))\left(\alpha f(t)^2 g(t) - \alpha f(t)h\prime(t)\right) + f\prime(t) + f(t)\gamma(t)\right) + \\ &xf(t)g\prime(t)v^{(1,0)}(xg(t),h(t)) + \beta f(t)\left(g(t)^3 - h\prime(t)\right)v^{(3,0)}(xg(t),h(t)). \end{aligned} \tag{2.6}$$

We will choose the functions f, g and h so that $f(0) = g(0) = 1$ and $h(0) = 0$ and

$$\begin{cases} f\prime(t) + f(t)\gamma(t) = 0. \\ f(t)g(t) - h\prime(t) = 0. \\ g(t)^3 - h\prime(t) = 0. \end{cases} \tag{2.7}$$

Solving the system (2.7) gives

$$\begin{aligned} &f(t) = \exp\left(-\textstyle\int_0^t \gamma(\tau)d\tau\right). \\ &g(t) = \sqrt{f(t)} = \exp\left(-\tfrac{1}{2}\textstyle\int_0^t \gamma(\tau)d\tau\right). \\ &h(t) = \textstyle\int_0^t g^3(\tau)d\tau. \end{aligned} \tag{2.8}$$

From (1.2) and (1.3), we obtain the following approximate cnoidal and soliton solutions, respectively :

$$u(x,t) = \frac{12\beta k^2 m}{\alpha} f(t)\mathrm{cn}^2\left(kxg(t) - 4k^3(2m-1)\beta h(t) \middle| m\right) \tag{2.9}$$

$$u(x,t) = u(x,t) = \frac{12\beta k^2}{\alpha} f(t)\mathrm{sech}^2\left(kxg(t) - 4\beta k^3 h(t)\right). \tag{2.10}$$

3 Analysis and Discussion

We have obtained an approximate analytical solution to the damped KdV starting from any exact solution to its undamped counterpart. Let us examine an example.

Example 1. Let $\gamma(t) = \mu t$. Then

$$f(t) = e^{-\frac{\mu t^2}{2}},\ g(t) = e^{-\frac{\mu t^2}{4}},\ h(t) = \sqrt{\frac{\pi}{3\mu}}\mathrm{erf}\left(\frac{1}{2}\sqrt{3\mu}t\right). \tag{3.11}$$

Two approximate analytical solutions are

$$u_1(x,t) = \frac{12\beta k^2 m e^{-\frac{\mu t^2}{2}}}{\alpha}\mathrm{cn}^2\left(e^{-\frac{t^2\mu}{4}}kx - 4k^3(2m-1)\beta\sqrt{\frac{\pi}{3\mu}}\mathrm{erf}\left(\frac{1}{2}\sqrt{3\mu}t\right)\middle| m\right) \tag{3.12}$$

and

$$u_2(x,t) = \frac{12\beta k^2 e^{-\frac{\mu t^2}{2}}}{\alpha}\mathrm{sech}^2\left(kxe^{-\frac{\mu t^2}{4}} - 4\sqrt{\frac{\pi}{3\mu}}\beta k^3\mathrm{erf}\left(\frac{1}{2}\sqrt{3\mu}t\right)\right). \tag{3.13}$$

Figure 1 shows the solution (3.12) for the parameter values

$$\alpha = 6, \beta = 1, \mu = 1, m = 3/4,\ k = 0.4,\ \ -15 \le x \le 15 \text{ and } -3 \le t \le 3. \tag{3.14}$$

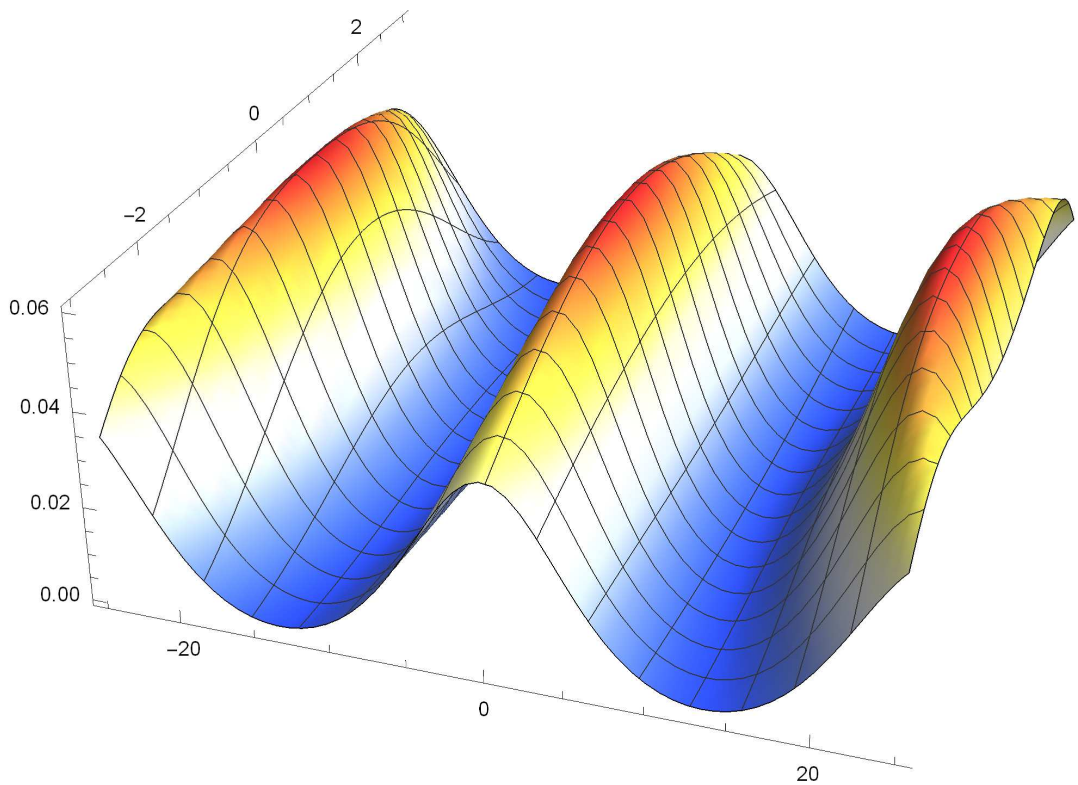

Figure 1. The errors compared with numerical solution using NDSolve Mathematica 12 command with boundary condition $u(x,0)=u_1(x,0)$ are

$$\max_{-15\le x\le 15,-3\le t\le 3}\left|\frac{\partial u_1}{\partial t}+\alpha u_1\frac{\partial u_1}{\partial x}+\beta\frac{\partial^3 u_1}{\partial x^3}+\gamma(t)u_1\right|=0.0173727. \tag{3.15}$$

$$\max_{-15\le x\le 15,-3\le t\le 3}\left|\frac{\partial u_{\text{num}}}{\partial t}+\alpha u_{\text{numl}}\frac{\partial u_{\text{num}}}{\partial x}+\beta\frac{\partial^3 u_{\text{numl}}}{\partial x^3}+\gamma(t)u_{\text{num}}\right|=0.00015. \tag{3.16}$$

is 0.0173727.

Figure 2 shows the solution (3.13) for the same parameter values.

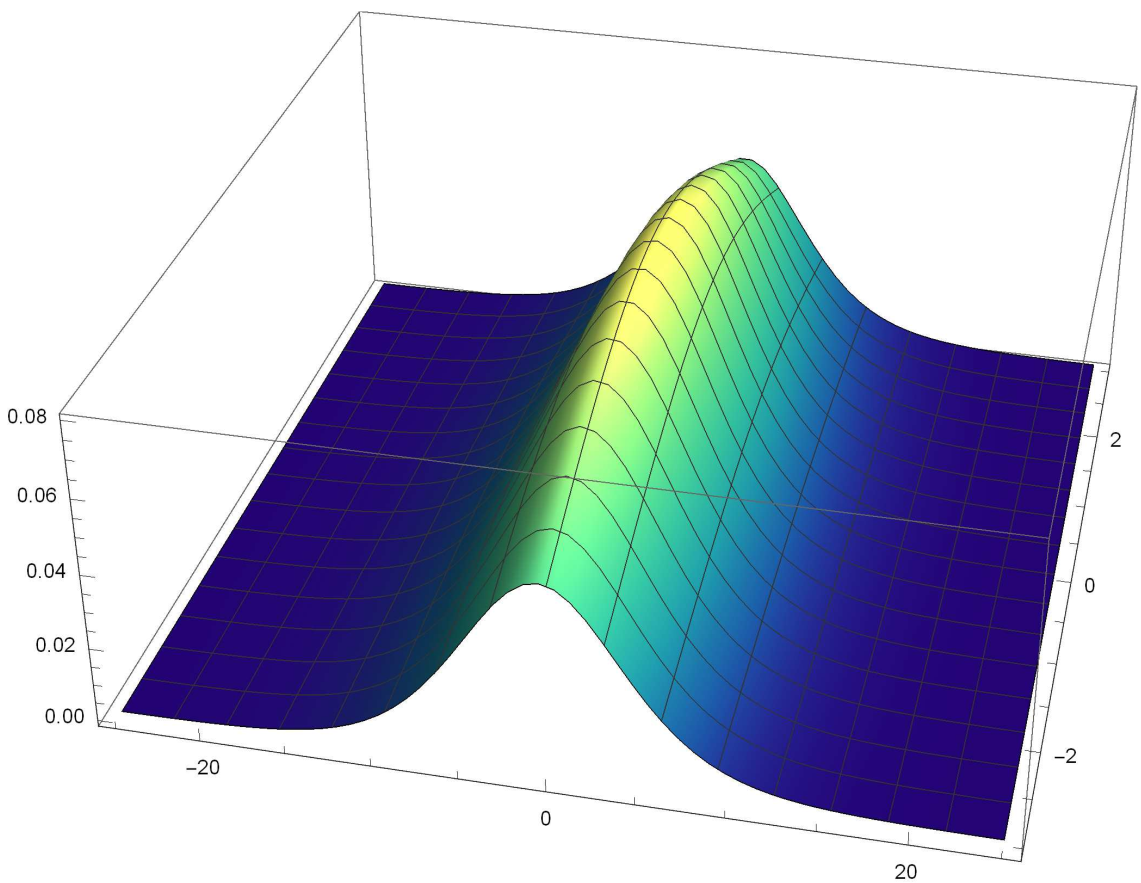

Figure 2.

The errors compared with numerical solution using NDSolve Mathematica 12 command with boundary condition $u(x,0) = u_2(x,0)$ are

$$\max_{-15\le x\le 15, -3\le t\le 3}\left|\frac{\partial u_2}{\partial t} + \alpha u_2\frac{\partial u_2}{\partial x} + \beta\frac{\partial^3 u_2}{\partial x^3} + \gamma(t)u_2\right| = 0.005279. \qquad (3.17)$$

$$\max_{-15\le x\le 15, -3\le t\le 3}\left|\frac{\partial u_{\text{num}}}{\partial t} + \alpha u_{\text{numl}}\frac{\partial u_{\text{num}}}{\partial x} + \beta\frac{\partial^3 u_{\text{numl}}}{\partial x^3} + \gamma(t)u_{\text{num}}\right| = 0.0002. \qquad (3.18)$$

4 Conclusions

We obtained a good approximate analytical solution to the damped KdV equation. We compared the analytical approximant with the solution obtained by means of Wolfram Mathematica. We may use a similar approach to study the damped and driven KDV [3], [1]
$u_t + \alpha u u_x + \beta u_{xxx} + \gamma(t) u = F.$

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