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Radius Estimates for Some Subclasses of Analytic Functions

Yao Liang Chung, Maisarah Haji Mohd, Shamani Supramaniam

School of Mathematical Sciences
Universiti Sains Malaysia
11800 Pulau Pinang, Malaysia

email: chungyaoliang@gmail.com, maisarah_hjmohd@usm.my,
shamani@usm.my

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Abstract

In this work, we define three subclasses of analytic functions. We then obtain the radius estimates of these functions to belong to various subclasses of starlike functions.

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0 = f'(0) - 1$. By $\mathcal{S}$, we mean the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$. Let $\mathcal{P}$ be the class of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\operatorname{Re}\big(p(z)\big) > 0$. A function $f \in \mathcal{A}$ is starlike of order $\alpha, 0 \leq \alpha < 1$, if $\operatorname{Re}\big(zf'(z)/f(z)\big) > \alpha$ for all $z \in \mathbb{D}$. The class of all starlike functions of order α is denoted by $\mathcal{S}^*(\alpha)$. For $\alpha = 0$, we have $\mathcal{S}^*(0) := \mathcal{S}^*$ which is the starlike functions. For $0 < \gamma \leq 1$, a function $f \in \mathcal{A}$ is said to be strongly starlike of order γ if it satisfies $|\arg(zf'(z)/f(z))| \leq \pi\gamma/2$. We denote the set of all such functions by $\mathcal{S}^*_\gamma$.

Suppose f and g are analytic on $\mathbb{D}$. We say that f is subordinate to g, denoted $f \prec g$, if there exists an analytic function ω in $\mathbb{D}$ with $\omega(0) = 0$

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and $|\omega| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{D}$. Ma and Minda [9] defined the class $\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{D} \right\}$, where $\phi \in \mathcal{P}$ such that $\phi(\mathbb{D})$ is symmetrical with respect to the real axis and starlike with respect to 1. Setting $\phi(z) = (1+z)/(1-z) \in \mathcal{P}$, we have $\mathcal{S}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{D} \right\}$. Several subclasses of starlike functions are defined in terms of subordination as follows:

- $\mathcal{S}^*_L := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_p := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_e := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_c := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_{\sin} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_{\leftmoon} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_R := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{k} \left(\frac{k+z}{k-z} \right),\ k = \sqrt{2}+1,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}^*_{RL} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}},\ z \in \mathbb{D} \right\}$.

For more information on the subclasses, we refer the reader to [1, 3, 4, 10, 11, 12, 15, 18, 19].

Let F and $\mathcal{G}$ be two subfamilies of $\mathcal{A}$. If for every $f \in F$, $r^{-1}f(rz) \in \mathcal{G}$ for $r \le r_0$, and r_0 is the largest number for which this holds, then r_0 is the $\mathcal{G}$ radius (or the radius of the property connected to $\mathcal{G}$) in F. There are various other studies on the radius problem (see, for instance, [2, 6, 8, 13, 14, 20]). Recently, for $\phi(z) = (1+z)/z$, Sebastian and Ravichandran [16] estimated several radii for classes of functions satisfying either (i) $\operatorname{Re}(f(z)/g(z)) > 0$, where $\operatorname{Re} \phi(z)g(z) > 0$; (ii) $|f(z)/g(z) - 1| < 1$, where $\operatorname{Re} \phi(z)g(z) > 0$; (iii) $\operatorname{Re} \phi(z)f(z) > 0$. Motivated by their work and with $\phi(z) = (1-z+z^2)/z$, we introduce the following classes

- $\mathcal{M}_1 := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{g(z)} > 0, \text{ for some } g \in \mathcal{A} \text{ with } \phi(z)g(z) \in \mathcal{P} \right\},$

- $\mathcal{M}_2 := \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1, \text{ for some } g \in \mathcal{A} \text{ with } \phi(z)g(z) \in \mathcal{P} \right\},$

- $\mathcal{M}_3 := \left\{ f \in \mathcal{A} : \phi(z)f(z) \in \mathcal{P} \right\}.$

The main purpose of this paper is to obtain the radius for functions in these three classes $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ to belong to the classes $\mathcal{S}^*_L, \mathcal{S}^*_p, \mathcal{S}^*_e, \mathcal{S}^*_c$, $\mathcal{S}^*_{\sin}, \mathcal{S}^*_{\leftmoon}, \mathcal{S}^*_R$ and $\mathcal{S}^*_{RL}$.

2 Main results

The following theorem gives various starlikeness radii for the class $\mathcal{M}_1$.

Theorem 2.1. *For the class $\mathcal{M}_1$, we have the following radius results:*
(i) The $\mathcal{S}^*_\alpha$ radius is the root of $(1+\alpha)r^2 - 6r + 1 - \alpha = 0$; *i.e.,* $R_{\mathcal{S}^*(\alpha)} = (1-\alpha)/(3+\sqrt{9-\alpha(1-\alpha)}), 0 \leq \alpha < 1$.
(ii) The $\mathcal{S}^*_L$ radius is the root of $2r^2 + (\sqrt{2}-1)r^2 + 6r + 1 - \sqrt{2} = 0$; *i.e.,* $R_{\mathcal{S}^*_L} = 1/((1+\sqrt{2})(3+\sqrt{10})) \approx 0.0672$.
(iii) The $\mathcal{S}^*_p$ radius is the root of $3r^2 - 12r + 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_p} = (6-\sqrt{33})/3 \approx 0.08515$.
(iv) The $\mathcal{S}^*_e$ radius is the root of $(1+e)r^2 - 6er + e - 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_e} = (6e - \sqrt{36e^2 - 4(1+e)(e-1)})/(2(1+e)) \approx 0.1080$.
(v) The $\mathcal{S}^*_C$ radius is the root of $2r^2 - 9r + 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_C} = (9-\sqrt{73})/4 \approx 0.1140$.
(vi) The $\mathcal{S}^*_{\sin}$ radius is the root of $(2+\sin 1)r^2 + 6r - \sin 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_{\sin}} = \left(-6 + \sqrt{36 + 4\sin(1)(2+\sin 1)}\right)/(2(2+\sin 1)) \approx 0.1340$.
(vii) The $\mathcal{S}^*_{\leftmoon}$ radius is the root of $\sqrt{2}r^2 - 6r + 2 - \sqrt{2} = 0$; *i.e.,* $R_{\mathcal{S}^*_{\leftmoon}} = (6 - \sqrt{36 - 4\sqrt{2}(2-\sqrt{2})})/(2\sqrt{2}) \approx 0.09998$.
(viii) The $\mathcal{S}^*_R$ radius is the root of $(2\sqrt{2}-1)r^2 - 6r + 3 - 2\sqrt{2} = 0$; *i.e.,* $R_{\mathcal{S}^*_R} \approx 0.0288$.
(ix) The $\mathcal{S}^*_{RL}$ radius $R_{\mathcal{S}^*_{RL}}$ is the root of $36r^2/((1-r^2)^2) = ((1-(\sqrt{2}-((1+r^2)/(1-r^2)))^2)^{1/2} - (1-(\sqrt{2}-((1+r^2)/(1-r^2)))^2) \approx 0.04706$.
(x) The $\mathcal{S}^*_\gamma$ radius is the root of $r^2\sin(\pi\gamma/2) - 6r + \sin(\pi\gamma/2) = 0$; *i.e.,* $R_{\mathcal{S}^*_\gamma} = \left(3 - \sqrt{9 - \sin^2(\pi\gamma/2)}\right)/\sin(\pi\gamma/2), 0 < \gamma \leq 1$.

Proof. For $f \in \mathcal{M}_1$, let the function $g : \mathbb{D} \to \mathbb{C}$ be such that

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \operatorname{Re}\left(\frac{1-z+z^2}{z}g(z)\right) > 0. \tag{2.1}$$

Define the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ as

$$p_1(z) = \frac{f(z)}{g(z)} \text{ and } p_2(z) = \frac{1-z+z^2}{z}g(z). \tag{2.2}$$

By using (2.1) and (2.2), the functions p_1 and p_2 belong to the class $\mathcal{P}$. It follows from (2.2) that $f(z) = zp_1(z)p_2(z)/(1-z+z^2)$. Then

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{1-z^2}{1-z+z^2}. \tag{2.3}$$

The function $(1-z^2)/(1-z+z^2)$ maps the disk $|z| \leq r$ onto the disk

$$\left|\frac{1-z^2}{1-z+z^2} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}. \tag{2.4}$$

For $p \in \mathcal{P}(\alpha) = \{p \in \mathcal{P} : \operatorname{Re} p > \alpha\}$, the inequality

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)}. \tag{2.5}$$

holds. Since $p_1, p_2 \in \mathcal{P}$, by (2.5), it follows that

$$\left|\frac{zp_1'(z)}{p_1(z)}\right| \leq \frac{2r}{1-r^2} \quad \text{and} \quad \left|\frac{zp_2'(z)}{p_2(z)}\right| \leq \frac{2r}{1-r^2}. \tag{2.6}$$

Using (2.4) and (2.6), it follows from (2.3)

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| &\leq \left|\frac{zp_1'(z)}{p_1(z)}\right| + \left|\frac{zp_2'(z)}{p_2(z)}\right| + \left|\frac{1-z^2}{1-z+z^2} - \frac{1+r^2}{1-r^2}\right| \\ &\leq \frac{6r}{1-r^2}. \end{aligned} \tag{2.7}$$

By (2.7), we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1+r^2-6r}{1-r^2} \geq 0.$$

This implies that the function $f \in \mathcal{M}_1$ is starlike in $|z| \leq 3 - 2\sqrt{2} \approx 0.1716$.

(i) The number $r = \mathcal{R}_{\mathcal{S}^*(\alpha)}$ is the root of $(1+\alpha)r^2 - 6r + 1 - \alpha = 0$. For $0 < r \leq \mathcal{R}_{\mathcal{S}^*(\alpha)}$, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1 + r^2 - 6r}{1 - r^2} \geq \alpha.$$

(ii) From (2.7), we obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zf'(z)}{f(z)} - \frac{1 + r^2}{1 - r^2}\right| + \frac{2r^2}{1 - r^2} \leq \frac{2r^2 + 6r}{1 - r^2}.$$

Let $\Omega_l = \{w \in \mathbb{C} : |w^2 - 1| < 1\}$. By [1, Lemma 2.2], for $2\sqrt{2}/3 \leq b < \sqrt{2}$, the relation

$$\{w \in \mathbb{C} : |w - b| < \sqrt{2} - b\} \subset \Omega_l \tag{2.8}$$

holds. The number $R_{\mathcal{S}^*_L}$ is the root of $2r^2 + (\sqrt{2} - 1)r^2 + 6r + 1 - \sqrt{2} = 0$ If $0 < r \leq R_{\mathcal{S}^*_L}$, then $(2r^2 + 6r)/(1 - r^2) \leq \sqrt{2} - 1$. So, the disk in (2.7) is inside the region Ω_l by (2.8).

(iii) Let $\Omega_p = \{w : \operatorname{Re} w > |w - 1|\}$. By [17, Lemma 1], for $1/2 < b \leq 3/2$, the relation

$$\{w \in \mathbb{C} : |w - b| < b - 1/2\} \subset \Omega_p \tag{2.9}$$

holds. The number $R_{\mathcal{S}^*_p}$ is the root of $3r^2 - 12r + 1 = 0$. If $0 < r \leq R_{\mathcal{S}^*_p}$, then $b = (1 + r^2)/(1 - r^2) \leq 3/2$ and

$$\frac{6r}{1 - r^2} \leq \frac{1 + r^2}{1 - r^2} - \frac{1}{2}.$$

So, the disk in (2.7) is inside region Ω_p by (2.9).

(iv) Let $\Omega_e = \{w \in \mathbb{C} : |\log w| < 1\}$. For $e^{-1} \leq b \leq (e + e^{-1}/2)$, by [11, Lemma 2.2], the relation

$$\{w \in \mathbb{C} : |w - b| < b - 1/e\} \subset \Omega_e \tag{2.10}$$

holds. The number $R_{\mathcal{S}^*_e}$ is the root of $(1+e)r^2 - 6er + e - 1 = 0$. If $0 < r < R_{\mathcal{S}^*_e}$, then $1/e \leq b = (1 + r^2)/(1 - r^2) \leq (e + e^{-1}/2)$ and

$$\frac{6r}{1 - r^2} \leq \frac{1 + r^2}{1 - r^2} - \frac{1}{e}.$$

So, the disk in (2.7) is inside the region Ω_e by (2.10).

(v) Let Ω_c be the domain bounded by the cardioid $\{x+iy : (9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1)=0$. For $1/3 < b \le 5/3$, by [18, Lemma 2.5], the relation

$$\{w \in \mathbb{C} : |w-b| < (3b-1)/3\} \subset \Omega_c \tag{2.11}$$

holds. The number $R_{\mathcal{S}^*_C}$ is the root of $2r^2-9r+1=0$. If $0 < r \le R_{\mathcal{S}^*_c}$, then

$$\frac{6r}{1-r^2} \le \frac{1+r^2}{1-r^2} - \frac{1}{3}.$$

So, the disk in (2.7) is inside the region Ω_c by (2.11).

(vi) Let $q(z) = 1+\sin z$ and $q(\mathbb{D}) := \Omega_s$. For $|b-1| \le \sin 1$, by [3, Lemma 3.3], the relation

$$\{w \in \mathbb{C} : |w-b| < \sin 1 - |b-1|\} \subset \Omega_s \tag{2.12}$$

holds.The number $R_{\mathcal{S}^*_{\sin}}$ is the root in $[0,1]$ of $(2+\sin 1)r^2+6r-\sin 1=0$. If $0 < r \le R_{\mathcal{S}^*_{\sin}}$, then

$$\frac{6r}{1-r^2} \le \sin 1 - \frac{2r^2}{1-r^2}.$$

So, the disk in (2.7) is inside the region Ω_s by (2.12)

(vii) Let $\Omega_m = \{w \in \mathbb{C} : |w^2-1| < 2|w|\}$. By [4, Lemma 2.1], the relation

$$\{w \in \mathbb{C} : |w-b| < 1-|\sqrt{2}-b|\} \subset \Omega_m \tag{2.13}$$

holds. The number $R_{\mathcal{S}^*_{\leftmoon}}$ is the root of $\sqrt{2}r^2-6r+2-\sqrt{2}=0$. If $0 < r < R_{\mathcal{S}^*_{\leftmoon}}$, then

$$\frac{6r-1-r^2}{1-r^2} \le 1-\sqrt{2}.$$

So, the disk in (2.7) is inside the region Ω_m by (2.13).

(viii) Let $\phi(z) = 1+(z^2k+z^2/(k^2-kz))$, $k=\sqrt{2}+1$ and Ω_R is the image of $\mathbb{D}$ under the function ϕ. For $2(\sqrt{2}-1) < b \le \sqrt{2}$, by [7, Lemma 2.2], the relation

$$\{w \in \mathbb{C} : |w-b| < b-2(\sqrt{2}-1)\} \subset \Omega_R. \tag{2.14}$$

holds. The number $R_{\mathcal{S}^*_R}$ is the root of $(2\sqrt{2}-1)r^2-6r+3-2\sqrt{2}=0$. If $0 < r \le R_{\mathcal{S}^*_R}$, then

$$\frac{6r-1-r^2}{1-r^2} \le 2-2\sqrt{2}.$$

So, the disk in (2.7) is inside the region Ω_R by (2.14).

(ix) Let $\Omega_{RL}=\{w\in\mathbb{C}:|(w-\sqrt{2})^2-1|<1\}$. For $\sqrt{2}/3\leq b<\sqrt{2}$, by [10, Lemma 3.2], the relation

$$\{w\in\mathbb{C}:|w-b|<r_{RL}\}\subset\Omega_{RL} \tag{2.15}$$

holds, provided $r_{RL}=((1-(\sqrt{2}-a)^2)^{1/2}-(1-(\sqrt{2}-a)^2))^{1/2}$. If $0<r\leq R_{\mathcal{S}^*_{RL}}$, then $b=(1+r^2)/(1-r^2)<\sqrt{2}$ and

$$\frac{36r^2}{(1-r^2)^2}=((1-(\sqrt{2}-a)^2))^{1/2}-(1-(\sqrt{2}-a)^2).$$

So, the disk in (2.7) is inside the region Ω_{RL} by (2.15).

(x) By [5, Lemma 3.1], we have

$$\{w\in\mathbb{C}:|w-b|<b\sin(\pi\gamma/2)\}\subset\{w\in\mathbb{C}:|\arg w|\leq(\pi\gamma)/2\}. \tag{2.16}$$

The number $R_{\mathcal{S}^*_\gamma}$ is the root of $r^2\sin(\pi\gamma/2)-6r+\sin(\pi\gamma/2)=0$. If $0<r\leq R_{\mathcal{S}^*_\gamma}$, then

$$\frac{6r}{1-r^2}\leq\frac{1+r^2}{1-r^2}\sin(\pi\gamma/2).$$

So, the disk in (2.7) is inside the sector $|\arg w|\leq(\pi\gamma)/2$ by (2.16).

□

Theorem 2.2. *For the class $\mathcal{M}_2$, we have the following radius results:*
(i) The $\mathcal{S}^*_\alpha$ radius is the root of $\alpha r^2-5r+1-\alpha=0$; i.e.. $R_{\mathcal{S}^*(\alpha)}=(5+\sqrt{25-4\alpha(1-\alpha)})/(2\alpha)$.
(ii) The $\mathcal{S}^*_L$ radius is the root of $(\sqrt{2}+2)r^2+5r+1-\sqrt{2}=0$; i.e., $R_{\mathcal{S}^*_L}=(\sqrt{25+4\sqrt{2}}-5)/2(2+\sqrt{2})\approx 0.078621$.
(iii) The $\mathcal{S}^*_p$ radius is the root of $r^2-10r+1=0$; i.e., $R_{\mathcal{S}^*_p}=5-2\sqrt{6}\approx 0.1010$.
(iv) The $\mathcal{S}^*_e$ radius is the root of $r^2-5er+e-1=0$; i.e., $R_{\mathcal{S}^*_e}\approx 0.1276$.
(v) The $\mathcal{S}^*_C$ radius is the root of $r^2-15r+2=0$; i.e., $R_{\mathcal{S}^*_C}=(15-\sqrt{217})/2\approx 0.1345$.
(vi) The $\mathcal{S}^*_{\sin}$ radius is the root of $(3+\sin 1)r^2+5r-\sin 1=0$; i.e., $R_{\mathcal{S}^*_{\sin}}\approx 0.1508$.
(vii) The $\mathcal{S}^*_{\mathbb{C}}$ radius is the root of $(\sqrt{2}-1)r^2-5r+2-\sqrt{2}=0$; i.e., $R_{\mathcal{S}^*_{\mathbb{C}}}\approx 0.1186$.
(viii) The $\mathcal{S}^*_R$ radius is the root of $(2\sqrt{2}-2)r^2-5r+3-2\sqrt{2}=0$; i.e., $R_{\mathcal{S}^*_R}=6-4\sqrt{2}\left(5+\sqrt{81-40\sqrt{2}}\right)\approx 0.0342$.
(ix) The $\mathcal{S}^*_{RL}$ radius $R_{\mathcal{S}^*_{RL}}$ is the root of $(r^2+5r)^2/((1-r^2)^2)=((1-(\sqrt{2}-$

$((1+r^2)/(1-r^2)))^2)^{1/2} - (1-(\sqrt{2}-((1+r^2)/(1-r^2)))^2) \approx 0.05582$.
(x) The $\mathcal{S}^*_\gamma$ radius is the root of $(1-\sin(\pi\gamma)/2)r^2 + 5r - \sin(\pi\gamma)/2 = 0$ *i.e.*

$$R_{\mathcal{S}^*_\gamma} = \left(-5+\sqrt{25+4\sin(\pi\gamma/2)(1-\sin(\pi\gamma/2))}\right)/2(1-\sin(\pi\gamma/2)),\ 0<\gamma\leq 1.$$

Proof. Using the fact that $\operatorname{Re}(1/w) > 1/2$ is equivalent to $|w-1| < 1$, we have $\operatorname{Re} g(z)/f(z) > 1/2$ if and only if $|f(z)/g(z) - 1| < 1$. Let the function $f \in \mathcal{M}_2$. Consider the function $g : \mathbb{D} \to \mathbb{C}$ such that

$$\operatorname{Re}\left(\frac{g(z)}{f(z)}\right) > \frac{1}{2} \text{ and } \operatorname{Re}\left(\frac{1-z+z^2}{z}g(z)\right) > 0. \tag{2.17}$$

Define $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{1-z+z^2}{z}g(z) \quad \text{and} \quad p_2(z) = \frac{g(z)}{f(z)}. \tag{2.18}$$

By (2.17) and (2.18), we obtain $p_1 \in \mathcal{P}$ and $p_2 \in \mathcal{P}(1/2)$. Further, we have $f(z) = zp_1(z)/(p_2(z)(1-z+z^2))$ and hence

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p(z)} - \frac{zp_2'(z)}{p_2(z)} + \frac{1-z^2}{1-z+z^2}. \tag{2.19}$$

Using (2.4) and (2.5), we get

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| &= \left|\frac{zp_1'(z)}{p(z)} - \frac{zp_2'(z)}{p_2(z)} + \frac{1-z^2}{1-z+z^2} - \frac{1+r^2}{1-r^2}\right| \\ &= \frac{r^2+5r}{1-r^2}. \end{aligned} \tag{2.20}$$

It follows from (2.20)

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-5r}{1-r^2} \geq 0 \quad (r \leq 1/5).$$

Hence, all the radii that we estimate will be less than $1/5 = 0.2$.

(i) The number $r = R_{\mathcal{S}^*(\alpha)}$ is the root of $\alpha r^2 - 5r + 1 - \alpha = 0$. For $0 < r \leq R_{\mathcal{S}^*(\alpha)}$, we obtain

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-5r}{1-r^2} \geq \alpha.$$

(ii) From (2.20), we obatin

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| + \frac{2r^2}{1-r^2} \leq \frac{3r^2+5r}{1-r^2}.$$

By [1, Lemma 2.2], for $2\sqrt{2}/3 \leq b < \sqrt{2}$, the relation (2.8) holds. The number $R_{\mathcal{S}_L^*}$ is the root of $(\sqrt{2}+2)r^2 + 5r + 1 - \sqrt{2} = 0$. If $0 < r \leq R_{\mathcal{S}_L^*}$, then $(3r^2+5r)/(1-r^2) \leq \sqrt{2} - 1$. So, the disk in (2.20) is inside the region Ω_l by (2.8).

(iii) By [17, Lemma 1], for $1/2 < b \leq 3/2$, the relation (2.9) holds. The number $R_{\mathcal{S}_p^*}$ is the root of $r^2 - 10r + 1 = 0$. If $0 < r \leq R_{\mathcal{S}_p^*}$, then $a = (1+r^2)/(1-r^2) \leq 3/2$ and

$$\frac{r^2+5r}{1-r^2} \leq \frac{1+r^2}{1-r^2} - \frac{1}{2}.$$

So, the disk in (2.20) is inside the parabolic region Ω_p by (2.9).

(iv) For $e^{-1} \leq b \leq (e + e^{-1}/2)$, by [11, Lemma 2.2], the relation (2.10) holds. The number $R_{\mathcal{S}_e^*}$ is the root of $r^2 - 5er + e - 1 = 0$. If $0 < r < R_{\mathcal{S}_e^*}$, then $1/e \leq a = (1+r^2)/(1-r^2) \leq (e + e^{-1}/2)$ and

$$\frac{r^2+5r}{1-r^2} \leq \frac{1+r^2}{1-r^2} - \frac{1}{e}.$$

So, the disk in (2.20) is inside the region Ω_e (2.10).

(v) For $1/3 < a \leq 5/3$, by [18, Lemma 2.5], the relation (2.11) holds. The number $R_{\mathcal{S}_C^*}$ is the root of $r^2 - 15r + 2 = 0$. If $0 < r \leq R_{\mathcal{S}_c^*}$, then

$$\frac{r^2+5r}{1-r^2} \leq \frac{1+r^2}{1-r^2} - \frac{1}{3}.$$

So, the disk in (2.20) lies inside Ω_c by (2.11).

(vi) Let $q(z) = 1 + \sin z$ and $q(\mathbb{D}) := \Omega_s$ For $|b-1| \leq \sin 1$, by [3, Lemma 3.3], the relation (2.12) holds. The number $R_{\mathcal{S}_{\sin}^*}$ is the root of $(3+\sin 1)r^2 + 5r - \sin 1 = 0$. If $0 < r \leq R_{\mathcal{S}_{\sin}^*}$, then

$$\frac{r^2+5r}{1-r^2} \leq \sin 1 - \frac{2r^2}{1-r^2}.$$

So, the disk in (2.20) is inside Ω_s by (2.12).

(vii) By [4, Lemma 2.1], the relation (2.13) holds. The number $R_{\mathcal{S}^*_{☾}}$ is the root of $(\sqrt{2}-1)r^2 - 5r + 2 - \sqrt{2} = 0$. If $0 < r < R_{\mathcal{S}^*_{☾}}$, then

$$\frac{5r-1}{1-r^2} \leq 1 - \sqrt{2}.$$

So, the disk in (2.20) is inside Ω_m by (2.13).

(viii) Let $\phi(z) = 1 + (z^2k + z^2/(k^2 - kz))$, $k = \sqrt{2}+1$ and $\phi(\mathbb{D}) := \Omega_R$ For $2(\sqrt{2}-1) < a \leq \sqrt{2}$, by [7, Lemma 2.2], the relation (13) holds. The number $R_{\mathcal{S}^*_R}$ is the root of $(2\sqrt{2}-2)r^2 - 5r + 3 - 2\sqrt{2} = 0$. If $0 < r \leq R_{\mathcal{S}^*_R}$, then

$$\frac{5r-1}{1-r^2} \leq 2 - 2\sqrt{2}.$$

So, the disk in (2.20) is inside Ω_R by (2.14).

(ix) For $\sqrt{2}/3 \leq b < \sqrt{2}$, by [10, Lemma 3.2], the relation (14) holds provided $r_{RL} = ((1-(\sqrt{2}-a)^2)^{1/2} - (1-(\sqrt{2}-a)^2))^{1/2}$. If $0 < r \leq R_{\mathcal{S}^*_{RL}}$, then $b = (1+r^2)/(1-r^2) < \sqrt{2}$ and

$$\frac{(r^2+5r)^2}{(1-r^2)^2} = ((1-(\sqrt{2}-a)^2))^{1/2} - (1-(\sqrt{2}-a)^2).$$

So, the disk in (2.20) is inside the region Ω_{RL} by (2.15).

(x) By [5, Lemma 3.1], the relation (2.16) holds. The number $R_{\mathcal{S}^*_\gamma}$ is the root of $(1-\sin(\pi\gamma)/2)r^2 + 5r - \sin(\pi\gamma)/2 = 0$. If $0 < r \leq R_{\mathcal{S}^*_\gamma}$, then

$$\frac{r^2+5r}{1-r^2} \leq \frac{1+r^2}{1-r^2}\sin(\pi\gamma/2).$$

So, the disk in (2.20) is inside the sector $|\arg w| \leq (\pi\gamma)/2$ by (2.16). □

Theorem 2.3. *For the class $\mathcal{M}_3$, the following radius results hold:*
(i) The $\mathcal{S}^*_\alpha$ radius is the root of $(1+\alpha)r^2 - 4r + 1 - \alpha = 0$; *i.e.,* $R_{\mathcal{S}^*(\alpha)} = (1-\alpha)/(2+\sqrt{3+\alpha^2}), 0 \leq \alpha < 1$
(ii) The $\mathcal{S}^*_L$ radius is the root of $(\sqrt{2}+1)r^2 + 4r + 1 - \sqrt{2} = 0$; *i.e.,* $R_{\mathcal{S}^*_L} = (\sqrt{5}-2)/(1+\sqrt{2}) \approx 0.0978$.
(iii) The $\mathcal{S}^*_p$ radius is the root of $3r^2 - 8r + 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_p} = (4-\sqrt{13})/3 \approx 0.1315$.
(iv) The $\mathcal{S}^*_e$ radius is the root of $(1+e)r^2 + 4er + e - 1 = 0$; *i.e.,* $R_{\mathcal{S}^*_e} = (e-1)/(2e+\sqrt{3e^2+1}) \approx 0.1676$.

(v) The $\mathcal{S}_C^*$ *radius is the root of* $2r^2-6r+1=0$; *i.e.,* $R_{\mathcal{S}_C^*}=(3-\sqrt{7})/2\approx 0.1771$.
(vi) The $\mathcal{S}_{\sin}^*$ *radius is the root of* $(2+\sin 1)r^2+4r-\sin 1=0$; *i.e.,* $R_{\mathcal{S}_{\sin}^*}=\sin 1/(2+\sqrt{4+\sin 1(2+\sin 1)})\approx 0.1858$.
(vii) The $\mathcal{S}_{\leftmoon}^*$ *radius is the root of* $\sqrt{2}r^2-4r+2-\sqrt{2}=0$; *i.e.,* $R_{\mathcal{S}_{\leftmoon}^*}=\sqrt{2}-\sqrt{3-\sqrt{2}}\approx 0.1549$.
(viii) The $\mathcal{S}_R^*$ *radius is the root of* $(2\sqrt{2}-1)r^2-4r+3-2\sqrt{2}=0$; *i.e.,* $R_{\mathcal{S}_R^*}=(3-2\sqrt{2})/(2+\sqrt{15-8\sqrt{2}})\approx 0.0438$.
(ix) The $\mathcal{S}_{RL}^*$ *radius* $R_{\mathcal{S}_{RL}^*}$ *is the root of* $\frac{16r^2}{(1-r^2)^2}=((1-(\sqrt{2}-(1+r^2)/(1-r^2))^2))^{1/2}-(1-(\sqrt{2}-(1+r^2)/(1-r^2))^2)\approx 0.0695$.
(x) The $\mathcal{S}_\gamma^*$ *radius is the root of* $r^2\sin(\pi\gamma)/2-4r+\sin(\pi\gamma)/2=0$; *i.e.,* $R_{\mathcal{S}_\gamma^*}=\sin(\pi\gamma/2)/\left(2+\sqrt{4-\sin^2(\pi\gamma/2)}\right), 0<\gamma\leq 1$.

Proof. The proof is similar to those of the previous two theorems. □

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