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A family of modified Chebyshev-Halley iterative methods with two parameters for nonlinear equations

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Abstract

In this paper, a family of modified Chebyshev-Halley iterative methods free from second derivative with two parameters is presented. Per iteration, the new methods require two evaluations of the functions and one evaluation its first derivative. We prove that the new methods have at least third order of convergence. Some numerical examples are given to show the performance and efficiency of the presented method.

1 Introduction

Solving nonlinear equations of the form $f(x) = 0$, where $f : D \subset R \longrightarrow R$ for an open interval D is still a challenging problem for mathematicians. As we know, the most appropriate method for solving nonlinear equations is the iterative methods. The basic iterative method for solving nonlinear equations is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, ... \tag{1}$$

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This method is quadratically convergent [1]. To get higher order of convergence, many modified iterative methods have been introduced. One of these, is a family of third order iterative methods, called Chebyshev-Halley methods [2], defined by

$$x_{n+1} = x_n - \left[1 + \frac{L_f(x_n)}{2\big[1 - \beta L_f(x_n)\big]}\right], n = 0, 1, 2, ... \tag{2}$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}$$

and β is a parameter. If $\beta = 0$, then the family reduces to the classical Chebyshev method; if $\beta = \frac{1}{2}$, then the family becomes the Halley method; if $\beta = 1$, then the family becomes the Super-Halley method.

Although this family of methods is efficient, it has some problems in computing the second derivative of the function per iteration. To overcome this drawback, we modify the Chebyshev-Halley methods by replacing the second derivative of the function f with a finite difference scheme. In this way, we obtain a family of modified iterative methods for solving nonlinear equations free from second derivative. We also prove that the new modified methods have at least third-order convergence. Some examples are given to illustrate the efficiency and advantage of these modified methods.

2 The Method

Consider the Chebyshev-Halley iterative methods

$$x_{n+1} = x_n - \left[1 + \frac{L_f(x_n)}{2\big[1 - \beta L_f(x_n)\big]}\right], n = 0, 1, 2, ...$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}$$

and β is a parameter. We will replace $f''(x_n)$ in $L_f(x_n)$ by a finite difference scheme as follows

$$f''(x_n) \cong \frac{f'(x_n) - f'(y_n)}{x_n - y_n},$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then the term of $L_f(x_n)$ in the Chebyshev-Halley iterative methods becomes

$$L_f(x_n) = \frac{\big[f'(x_n) - f'(y_n)\big] f(x_n)}{(x_n - y_n) f'^2(x_n)} = \frac{\big[f'(x_n) - f'(y_n)\big]}{f'(x_n)} \tag{5}$$

To decrease the number of functions in (2), let us consider an approximation to $f'(y_n)$ which is introduced by Chun and Ham [3],

$$f'(y_n) \approx \frac{f'(x_n)\big[f(x_n) + (\alpha - 2) f(y_n)\big]}{f(x_n) + \alpha f(y_n)} \tag{6}$$

where α is a parameter. Then we obtain

$$f'(x_n) - f'(y_n) = \frac{2f(y_n)}{f(x_n) + \alpha f(y_n)} \tag{7}$$

Substituting (7) into (5), we obtain

$$L_f(x_n) = \left(\frac{2f(y_n)}{f(x_n) + \alpha f(y_n)}\right) \frac{1}{f'(x_n)} \tag{8}$$

Substituting (8) into (2), we obtain a new family of two-step modification of Chebyshev-Halley methods as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left[\frac{f(x_n) + (\alpha - 2\beta + 1) f(y_n)}{f(x_n) + (\alpha - 2\beta) f(y_n)}\right] \frac{f(x_n)}{f'(x_n)} \tag{9}$$

The order of convergence of (9) is analyzed in the following section.

3 Analysis of convergence

Theorem 1. *Let $D \subset R$ be an open interval and the function $f : D \subset R \longrightarrow R$ has a simple root $\gamma \in D$ If f is sufficiently smooth in a neighborhood of the root γ, then the order of convergence of the methods (9) is at least three.*

Proof 1. *Let e_n be the error in x_n; that is, e_n=$x_n - \gamma$. By using Taylor expansion around $x = \gamma$, we obtain*

$$f(x_n) = f'(\gamma)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)], \tag{10}$$

and

$$f'(x_n) = f'(\gamma)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)], \tag{11}$$

where $c_k = \frac{f^{(k)}(\gamma)}{k!f'(\gamma)}, k = 2, 3, 4, ...$
From

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

by using (10) and (11) and $e_n = x_n - \gamma$, we obtain

$$y_n - \gamma = c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O(e_n^5) \tag{12}$$

Expanding $f(y_n)$ around $x = \gamma$, we obtain

$$f(y_n) = f'(\gamma)\big[c_2e_n^2 - 2(c_2^2 + c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O(e_n^5)\big] \tag{13}$$

From (10) and (13), we obtain

$$\begin{aligned} f(x_n) + (\alpha - 2\beta + 1)f(y_n) =& e_n + (2 + \alpha - 2\beta)c_2e_n^2 + \big[(3 + 2\alpha - 4\beta)c_3 \\ &+ (4\beta - 2\alpha - 1)c_2^2\big]e_n^3 + \big[(4 + 3\alpha - 6\beta)c_4 \\ &+ 7(2\beta - \alpha - 1)c_2c_3 + 5(\alpha - 2\beta + 1)c_2^3\big]e_n^4 \\ &+ O(e_n^5) \end{aligned} \tag{14}$$

$$\text{and} \quad f(x_n)+(\alpha-2\beta)f(y_n) = e_n + (1+\alpha-2\beta)c_2e_n^2 + \big[(1+2\alpha-4\beta)c_3 + (4\beta-2\alpha)c_2^2\big]e_n^3 + \big[(1+3\alpha-6\beta)c_4 + 7(2\beta-\alpha)c_2c_3 + 5(\alpha-2\beta)c_2^3\big]e_n^4 + O(e_n^5) \tag{15}$$

From (14) and (15), we get

$$\frac{f(x_n)+(\alpha-2\beta+1)f(y_n)}{f(x_n)+(\alpha-2\beta)f(y_n)} = 1 + c_2e_n + \big[2c_3 - (3+\alpha-2\beta)c_2^2\big]e_n^2 + \big[3c_4 - (10+4\alpha-8\beta)c_2c_3 + (8-12\beta+6\alpha+\alpha^2-4\alpha\beta+4\beta+4\beta^2)c_2^3\big]e_n^3 + O(e_n^4). \tag{16}$$

From (9), we have

$$x_{n+1} = x_n - \left[\frac{f(x_n)+(\alpha-2\beta+1)f(y_n)}{f(x_n)+(\alpha-2\beta)f(y_n)}\right]\frac{f(x_n)}{f'(x_n)}.$$

Substituting (10), (11), (16) and $e_n = x_n - \gamma$ into (9) and simplifying yield

$$e_{n+1} = (2+\alpha-2\beta)c_2^2e_n^3 + \big[(7+4\alpha-8\beta)c_2c_3 - (9-14\beta+7\alpha+\alpha^2-4\alpha\beta + 4\beta^2)c_2^3\big]e_n^4 + O(e_n^5). \tag{17}$$

This shows that the order of convergence of the methods defined by (9) are three but if we choose the value of α and β which make $\alpha - 2\beta = -2$ in (17), the order of convergence will be four. Therefore, the order of convergence of the methods defined by (9) are at least three. If we choose $\alpha = 0$ and $\alpha = 1$ in (9), the new methods become the Ostrowski's method [4].

4 Numerical examples

In this section, we employ the newly obtained method with $\alpha = -2, \beta = 0$ to solve some of nonlinear equations and compare this method with the following fourth-order iterative methods.

(i) The method of Ezzati and Salaki (ESM1) [5] is defined by

$$x_{n+1} = y_n + f(y_n)\left(\frac{1}{f'(x_n)} - \frac{4}{f'(x_n) + f'(y_n)}\right),$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(i) The other method of Ezzati and Salaki (ESM2) [5] is defined by

$$x_{n+1} = y_n + \frac{f(y_n)}{f'(y_n)} + \frac{2f(x_n)f(y_n)}{f'(x_n)(f(x_n) - f(y_n))},$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(ii) The Newton-Steffensen-Potra-Pták method (NSPP) [6] defined by

$$x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))},$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(iii) The method of Chun (CM1) [7] defined by

$$x_{n+1} = y_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n f(y_n) + 2f^2(y_n)} \frac{f(y_n)}{f'(x)},$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We see that all of above methods are free from second derivative and the number of their functional evaluations per iteration is three. If we consider the definition of efficiency index as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method therefore the efficiency index of the method ESM1 and ESM2 are $4^{1/4} \approx 1.414$ while the method NSPP, CM1, and NEW are $4^{1/3} \approx 1.587$.

All computations are carried out with 125 digit floating point arithmetic. Displayed in Table 1 are the number of iterations n to obtain $|x_n - x_{n-1}| \leq 10^{-15}$ and $|f(x_n)| < 10^{-15}$, , the value of function f in the last iteration, the distance between the two last iteration. The test functions and their roots are as follows:

$$f_1(x) = x^3 + 4x^2 - 10, \gamma = 1.365230013414097$$

$$f_2(x) = x^2 - e^x - 3x + 2, \gamma = 0.257530285439861$$

$$f_3(x) = (x-1)^3 - 2, \gamma = 2.259921049894873$$

$$f_4(x) = (x+2)e^x - 1, \gamma = -0.442854401002389$$

$$f_5(x) = x^3 - 10, \gamma = -2.154434690031884$$

$$f_6(x) = x^2 - xe^x + \cos x, \gamma = 0.409992017989137$$

$$f_7(x) = \sin^2(x) - x^2 + 1, \gamma = 1.404491648215341$$

Table 1 : Comparison of various fourth order iterative methods.

	n	x_n	$\|f(x_n)\|$	$\|x_n - x_{n-1}\|$
$f_1(x), x_0 = 3.5$				
NSPP	5	1.365230013414001	0.11e-125	0.101e-46
ESM1	5	1.365230013414001	0.61e-126	0.607e-45
ESM2	5	1.365230013414001	0.61e-126	0.101e-46
CM1	5	1.365230013414001	0.10e-122	0.179e-44
NEW	4	1.365230013414001	0.67e-21	0.51e-21
$f_2(x), x_0 = 0.5$				
NSPP	3	0.257553028543986	0.91e-91	0.54e-22
ESM1	3	0.257553028543986	0.87e-99	0.55e-24
ESM2	3	0.257553028543986	0.91e-91	0.54e-22
CM1	3	0.257553028543986	0.95e-91	0.54e-22
NEW	3	0.257553028543986	0.11e-88	0.16e-21
$f_3(x), x_0 = 1.5$				
NSPP	9	2.259921049894873	0.19e-89	0.23e-22
ESM1	9	2.259921049894873	0.32e-72	0.46e-18
ESM2	9	2.259921049894873	0.19e-89	0.23e-22
CM1	6	2.259921049894873	0.1e-123	0.78e-33
NEW	5	2.259921049894873	0.67e-85	0.12e-33
$f_4(x), x_0 = -0.5$				
NSPP	3	-0.442854401002389	0.30e-80	0.69e-20
ESM1	3	-0.442854401002389	0.42e-78	0.22e-19
ESM2	3	-0.442854401002389	0.30e-80	0.69e-20
CM1	3	-0.442854401002389	0.17e-80	0.59e-20
NEW	3	-0.442854401002389	0.20e-97	0.55e-24
$f_5(x), x_0 = 3.6$				
NSPP	4	-2.154434690031884	0.58e-68	0.63e-17
ESM1	4	-2.154434690031884	0.58e-68	0.63e-17
ESM2	4	-2.154434690031884	0.58e-68	0.63e-17
CM1	4	-2.154434690031884	0.23e-65	0.28e-16
NEW	4	-2.154434690031884	0.25e-97	0.40e-24
$f_6(x), x_0 = -0.3$				
NSPP	6	0.409992017989137	0.65e-98	0.22e-24
ESM1	7	0.409992017989137	0	0.24e-40
ESM2	6	0.409992017989137	0.65e-98	0.22e-24
CM1	5	0.409992017989137	0.71e-124	0.37e-31
NEW	4	0.409992017989137	0.20e-124	0.24e-36
$f_7(x), x_0 = -0.3$				
NSPP	13	1.404491648215341	0.11e-127	0.19e-46
ESM1	13	1.404491648215341	0.21e-126	0.27e-46
ESM2	13	1.404491648215341	0.11e-127	0.19e-46
CM1	7	1.404491648215341	0.23e-60	0.51e-15
NEW	6	1.404491648215341	0.13e-63	0.11e-15

The results in Table 1 show that for most of the function we tested, the newly obtained method requires a lesser number of iterations than the other methods.

5 Conclusions

In this paper we presented a family of modified Chebyshev-Halley iterative methods with two parameters. The new methods removed the second derivative from Chebyshev-Halley iterative methods by using finite difference. Per iteration, these new methods used three functions and one first derivative evaluation. We proved that the order of convergence of the new method is at least three. Numerical results show that these methods are comparable to the some of existing methods in term of the number of iterations.

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