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Percentage points for testing homogeneity of covariance matrices of several bivariate Gaussian populations

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Abstract

In this article, the exact distribution and exact percentage points for testing equality of covariance matrices of q bivariate Gaussian populations are obtained. The distribution has been derived using the inverse Mellin transformation and the residue theorem. The percentage points have been computed for $q = 2(1)5$.

1 Introduction

In univariate ANOVA models, tests of mean differences in a response y according to one or more factors is of main interest. The validity of the classical F test requires the assumption of homogeneity of variances; that is, all groups have the same variance.

In MANOVA, for testing differences among mean vectors, the standard test statistics (Wilks Lambda, Hotelling-Lawley trace, Pillai-Bartlett trace, Roys maximum root) require an analogous assumption that the within-group covariance matrices for all groups are equal. Moreover, canonical discriminant analysis also makes the assumption that the group covariance matrices are equal. The assumption of equal covariance matrices is also needed in

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profile analysis (GLM repeated models). If the assumption of equality of covariance matrices is not made a priori, then it needs to be tested.

This article focuses on testing equality of covariance matrices. We compute exact percentage points of the likelihood ratio test statistic for testing equality of covariance matrices of several independent bivariate normal populations.

Let $\boldsymbol{X}_{i1},\dots,\boldsymbol{X}_{iN_i}$ be a random sample from a bivariate normal population with mean vector $\boldsymbol{\mu}_i$ and covariance matrix Σ_i, $i=1,\dots,q$. The likelihood ratio test statistic for testing $H:\Sigma_1=\cdots=\Sigma_q=\Sigma$ against general alternatives can be expressed as

$$\Lambda=\frac{N_0^{N_0}\prod_{i=1}^{q}\det(A_i)^{N_i/2}}{\prod_{i=1}^{q}N_i^{N_i}\det(A)^{N_0/2}}, \tag{1.1}$$

where $A_i=\sum_{j=1}^{N_i}(\boldsymbol{X}_{ij}-\bar{\boldsymbol{X}}_i)(\boldsymbol{X}_{ij}-\bar{\boldsymbol{X}}_i)'$, $N_i\bar{\boldsymbol{X}}_i=\sum_{j=1}^{N_i}\boldsymbol{X}_{ij}$, $i=1,\dots,q$, $A=A_1+\cdots+A_q$ and $N_0=N_1+\cdots+N_q$.

This test statistic was first derived by [8]. They also derived null moments of Λ and its distribution in special cases. By replacing N_i by $n_i=N_i-1$ in (1.1), the modified likelihood ratio statistic is obtained as

$$\Lambda^*=\frac{n_0^{n_0}\prod_{i=1}^{q}\det(A_i)^{n_i/2}}{\prod_{i=1}^{q}n_i^{n_i}\det(A)^{n_0/2}}, \tag{1.2}$$

where $n_0=\sum_{i=1}^{q}n_i=N_0-q$. It has been shown by Perlman ([9]) that the test given by Λ^* is unbiased. The h^{th} null moment of the test statistic Λ^* is given by

$$E(\Lambda^{*h})=\frac{n_0^{n_0h}}{\prod_{i=1}^{q}n_i^{n_ih}}\frac{\Gamma(n_0-1)}{\Gamma[n_0(1+h)-1]}\prod_{i=1}^{q}\frac{\Gamma[n_i(1+h)-1]}{\Gamma(n_i-1)}, \tag{1.3}$$

where $\operatorname{Re}(n_ih)>-(n_i-1)$, $i=1,\dots,q$ and $\operatorname{Re}(\cdot)$ denotes the real part of $(\cdot)$. The asymptotic expansion of a constant multiple of $-2\ln\Lambda^*$ is available in [4].

In this article, we give the exact percentage points of $U_1=\Lambda^{*1/n}$ for $n_1=\cdots=n_q=n$. The exact distribution of U_1 has been derived using the inverse Mellin transform and the residue theorem. We compute percentage points using the distributional results given in this article. The distribution and percentage points of the likelihood ratio statistic for testing homogeneity of several univariate and bivariate Gaussian populations have been obtained by [5] and [6].

2 The Null Distribution

Substituting $n_1 = \cdots = n_q = n$, in (1.3) and using the Gauss-Legendre multiplication formula for the gamma function, the h-th moment of $U_1 = (\Lambda^*)^{1/n}$ is simplified as

$$E(U_1^h) = \frac{\Gamma^q(n-1+h)}{\Gamma^q(n-1)} \prod_{k=0}^{q-1} \frac{\Gamma[n+(k-1)/q]}{\Gamma[n+h+(k-1)/q]}.$$

Now, using the inverse Mellin transform and the above moment expression, the density of U_1 is obtained as

$$f(u_1) = K(n,q)(2\pi\iota)^{-1} \int_C \frac{\Gamma^q(n-1+h)}{\prod_{k=0}^{q-1} \Gamma[n+h+(k-1)/q]} u_1^{-1-h}\, dh, \tag{2.4}$$

where $0 < u_1 < 1$, $\iota = \sqrt{-1}$, C is a suitable contour and

$$K(n,q) = \frac{\prod_{k=0}^{q-1} \Gamma[n+(k-1)/q]}{\Gamma^q(n-1)}.$$

Substituting $n-1+h = t$ and simplifying, the density (2.4) is restated as

$$\begin{aligned} f(u_1) = {} & K(n,q)(2\pi\iota)^{-1} u_1^{n-2} \\ & \times \int_{C_1} \frac{\Gamma^{q-1}(t)}{t \prod_{k=0(\neq 1)}^{q-1} \Gamma[t+1+(k-1)/q]} u_1^{-t}\, dt, \end{aligned} \tag{2.5}$$

where $0 < u_1 < 1$ and C_1 is the changed contour. The poles of the integrand in (2.5) are given by $t = -j$, $j = 0, 1, 2, \ldots$, and each pole is of order $q-1$ except $t = 0$ which is of order q. Hence by the residue theorem

$$f(u_1) = K(n,q) u_1^{n-2} \sum_{j=0}^{\infty} R_j,\ 0 < u_1 < 1, \tag{2.6}$$

where R_j is the residue at $t = -j$. Now computing R_j using the residue theorem (see [1], [2], [5, 6], [7], [3]), the density in (2.6) is given by

$$\begin{aligned} f(u_1) = K(n,q) u_1^{n-2} \Bigg[& \frac{1}{(q-1)!} \sum_{r=0}^{q-1} \binom{q-1}{r} A_{00}^{(r)} (-\ln u_1)^{q-1-r} \\ & + \frac{1}{(q-2)!} \sum_{j=1}^{\infty} u_1^j \sum_{r=0}^{q-2} \binom{q-2}{r} A_{j0}^{(r)} (-\ln u_1)^{q-2-r} \Bigg],\ 0 < u_1 < 1, \end{aligned} \tag{2.7}$$

where

$$A_{j0}^{(r)} = \sum_{m=0}^{r-1} \binom{r-1}{m} A_{j0}^{(r-1-m)} B_{j0}^{(m)}, \tag{2.8}$$

with

$$A_{00}^{(0)} = \frac{1}{\prod_{k=0(\neq 1)}^{q-1} \Gamma[1+(k-1)/q]},$$

$$A_{j0}^{(0)} = \frac{(-1)^{(q-1)j+1}}{j(j!)^{q-1} \prod_{k=0(\neq 1)}^{q-1} \Gamma[1-j+(k-1)/q]}, \; j=1,2,\ldots,$$

$$B_{00}^{(m)} = (q-1)\psi^{(m)}(1) - \sum_{\substack{k=0\\ \neq 1}}^{q-1} \psi^{(m)}\left(1+\frac{k-1}{q}\right)$$

and for $j = 1, 2, \ldots$,

$$B_{j0}^{(m)} = (q-1)\psi^{(m)}(1) + \frac{m!}{j^{m+1}} + \sum_{i=1}^{j} \frac{m!(q-1)}{i^{m+1}} - \sum_{\substack{k=0\\ \neq 1}}^{q-1} \psi^{(m)}\left(1-j+\frac{k-1}{q}\right).$$

3 Computation

The computation of the percentage points has been carried out by using $F(u,q) = \int_0^u f(t)\,dt$, where $f(t)$ is given by (2.7). The computation is carried out by using series representation given in (2.7). First, $F(u,q)$ is computed for various values of u. It is checked for monotonicity and for conditions $F(u,q) \to 0$ as $u \to 0$ and $F(u,q) \to 1$ as $u \to 1$. Then u is computed for various values of q, n and $F(u,q) = \alpha$. These are given in Table 1-4. The tables are given for values of q from 2 to 5. We have used MATHEMATICA 8.0 to carry out these computations. To compute u for given value of $\alpha = F(u,q)$, we have used `FindRoot` which searches for a numerical solution to the given equation using Newton's method or a variant of the secant method. An eight place accuracy is kept throughout. We note here that, for $q \geq 6$, the exact density of U can also be derived by applying the technique used in this article. But, the series expansions so obtained are extremely cumbersome and thereby it is difficult to compute the percentage points. Hence the tables are given for $q = 2, 3, 4$ and $q = 5$.

Table 1: Table of the percentage points of U_1 for $q = 2$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	0.00237263	0.00677602	0.0152273	0.0348845
3	0.0526341	0.0882073	0.131221	0.196812
4	0.143573	0.20193	0.262457	0.342905
5	0.235329	0.303507	0.36904	0.450445
6	0.315692	0.386674	0.451877	0.529682
7	0.383537	0.453971	0.516752	0.589699
8	0.44049	0.508843	0.568477	0.636468
9	0.488528	0.554154	0.610499	0.673837
10	0.529378	0.592069	0.645228	0.704332
11	0.564434	0.624195	0.674368	0.729666
12	0.594785	0.651727	0.699146	0.751036
13	0.621285	0.675563	0.720458	0.769296
14	0.644602	0.696387	0.738978	0.785074
15	0.665263	0.714729	0.755214	0.798843
16	0.683688	0.731002	0.769561	0.810961
17	0.700217	0.745534	0.782329	0.821706
18	0.715122	0.758588	0.793764	0.8313
19	0.72863	0.770376	0.804062	0.839916
20	0.740926	0.781073	0.813385	0.847698
21	0.752165	0.790823	0.821864	0.854759
22	0.762476	0.799745	0.829608	0.861196
23	0.771968	0.807941	0.836709	0.867088
24	0.780736	0.815494	0.843243	0.872501
25	0.788857	0.822478	0.849276	0.87749
26	0.796402	0.828954	0.854862	0.882105
27	0.803428	0.834976	0.86005	0.886385
28	0.809987	0.840589	0.864881	0.890365
29	0.816124	0.845834	0.86939	0.894076
30	0.821879	0.850746	0.873608	0.897545

Table 2: Table of the percentage points of U_1 for $q = 3$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	0.000242836	0.000787813	0.00198607	0.00522888
3	0.0155223	0.0279307	0.0443152	0.0718588
4	0.0619774	0.0916889	0.124737	0.172194
5	0.123916	0.166242	0.209439	0.266779
6	0.187871	0.237684	0.285955	0.34708
7	0.248011	0.301734	0.352023	0.413733
8	0.30248	0.357859	0.408431	0.469109
9	0.351086	0.406745	0.456637	0.515497
10	0.394258	0.449369	0.49806	0.554755
11	0.432608	0.486684	0.533911	0.588325
12	0.466761	0.519525	0.56517	0.617312
13	0.497285	0.548591	0.592625	0.642567
14	0.524676	0.574462	0.616905	0.664751
15	0.54936	0.597613	0.638513	0.684381
16	0.571697	0.618436	0.657857	0.701868
17	0.591989	0.637256	0.675268	0.717538
18	0.610496	0.65434	0.691016	0.731659
19	0.627435	0.669913	0.705326	0.744447
20	0.642992	0.684164	0.718382	0.75608
21	0.657325	0.697251	0.730342	0.766707
22	0.67057	0.709308	0.741335	0.776452
23	0.682844	0.720452	0.751475	0.78542
24	0.694249	0.730782	0.760854	0.793699
25	0.704871	0.740381	0.769557	0.801366
26	0.714788	0.749325	0.777651	0.808486
27	0.724067	0.757678	0.785199	0.815115
28	0.732767	0.765495	0.792255	0.821302
29	0.740939	0.772827	0.798863	0.82709
30	0.748631	0.779717	0.805066	0.832516

Table 3: Table of the percentage points of U_1 for $q = 4$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	0.0000318902	0.000114968	0.000318242	0.000936192
3	0.00525564	0.0100146	0.0167181	0.0287958
4	0.0294396	0.0453351	0.0639067	0.092014
5	0.070197	0.0971411	0.125786	0.165508
6	0.118611	0.15391	0.189357	0.235989
7	0.16853	0.209475	0.249052	0.299314
8	0.216775	0.261269	0.303114	0.354932
9	0.261956	0.308499	0.351378	0.403477
10	0.303608	0.35116	0.394271	0.445884
11	0.34172	0.389573	0.4324	0.483069
12	0.376489	0.42416	0.466377	0.515842
13	0.408192	0.455359	0.496762	0.544883
14	0.437125	0.483575	0.524045	0.57076
15	0.463577	0.509172	0.548644	0.593939
16	0.487814	0.53247	0.570914	0.614806
17	0.510075	0.553744	0.591156	0.633679
18	0.530572	0.573233	0.609624	0.650823
19	0.549493	0.591142	0.626533	0.666461
20	0.567003	0.607648	0.642069	0.680779
21	0.583246	0.622904	0.656386	0.693934
22	0.598349	0.637044	0.669621	0.706061
23	0.612424	0.650181	0.681889	0.717274
24	0.625569	0.662417	0.69329	0.72767
25	0.63787	0.673839	0.703912	0.737336
26	0.649403	0.684525	0.713831	0.746345
27	0.660238	0.694542	0.723114	0.754761
28	0.670434	0.70395	0.731819	0.76264
29	0.680045	0.712802	0.739998	0.770032
30	0.689119	0.721146	0.747697	0.77698

Table 4: Table of the percentage points of U_1 for $q = 5$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	$4.716293811350448\times10^{-6}$	0.0000186505	0.0000560038	0.000181321
3	0.00190021	0.00380531	0.0066365	0.0120349
4	0.0146379	0.0233338	0.0339068	0.050594
5	0.0411863	0.0585318	0.0775828	0.104921
6	0.0770493	0.102169	0.128111	0.16326
7	0.1173	0.14849	0.179398	0.219693
8	0.158605	0.194209	0.22846	0.271897
9	0.199047	0.237707	0.274075	0.319249
10	0.237635	0.278306	0.315904	0.361857
11	0.273922	0.315824	0.354023	0.400107
12	0.30777	0.350331	0.388686	0.43447
13	0.339209	0.382008	0.420211	0.465408
14	0.368352	0.411085	0.448919	0.493346
15	0.395353	0.437799	0.475119	0.518659
16	0.420379	0.462381	0.499087	0.541672
17	0.443597	0.485043	0.521073	0.562667
18	0.465165	0.505979	0.541293	0.581885
19	0.485232	0.525362	0.55994	0.599532
20	0.503933	0.543347	0.57718	0.615787
21	0.521391	0.56007	0.593161	0.630804
22	0.537717	0.575652	0.608009	0.644713
23	0.55301	0.590202	0.621837	0.657632
24	0.56736	0.603815	0.634744	0.669659
25	0.580847	0.616574	0.646816	0.680882
26	0.593543	0.628556	0.65813	0.691377
27	0.605513	0.639827	0.668754	0.701213
28	0.616816	0.650447	0.678747	0.710449
29	0.627505	0.66047	0.688163	0.719136
30	0.637625	0.669944	0.697049	0.727323

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