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Hybrid Quasi-associative ideals in BCI-algebras

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Abstract

In this paper, the concept of a hybrid structure is applied to quasi-associative ideals in BCI-algebras. The notion of a hybrid quasi-associative ideal of a BCI-algebra is introduced and some related properties are investigated. Relations between a hybrid ideal and a hybrid quasi-associative ideal in BCI-algebra are provided. Moreover, characterizations of a hybrid quasi-associative ideal are given.

1 Introduction

The BCK/BCI-algebras were initiated in 1966 by Imai and Iséki as a generalization of the concept of propositional calculus and set-theoretic difference [6, 7]. Since then the study on the theory of BCK/BCI-algebras has been considerably developed. The concept of ideal has played an important role in the study of the theory of BCI-algebras. In [22], Yue and Zhang introduced the notion of quasi-associative ideals in BCI-algebras. Then in [9], Jun

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discussed the fuzzification of quasi-associative ideals in BCI-algebras and gave a characterization of fuzzy quasi-associativity in terms of the quasi-associativity of level subsets.

After the introduction of fuzzy sets by Zadeh [23], the theory has been applied in different areas of computer and management sciences and computer engineering such as artificial intelligence,expert systems, operations research, pattern recognition, robotics and control engineering and others. Molodtsov in [14] introduced the concept of soft set to deal with uncertainties with avoiding the difficulties that appears with the usual theoretical approaches. Molodtsov successfully applied the soft set theory in several directions, such as game theory, smoothness of functions, Perron integration, probability, operations research, theory of measurement, etc. (see [14, 15, 16, 17]). Soft set theory is applied to algebraic structures such as BCK/BCI-algebra ([8]), d-algebras ([10]) and decision making ([2, 13]). Muhiuddin et al. studied the fuzzy set theoretical approach to the BCK/BCI-algebras on various aspects, see for example ([19, 20, 18, 21]).

Based on fuzzy sets and soft sets and using a set of parameters over an initial universe set, Jun et al. [11] introduced the notion of hybrid structure and investigated several properties. Using this notion, they introduce the concepts of a hybrid subalgebra, a hybrid field and a hybrid linear space. Recently, the notion of hybrid structure and its applications to semigroups and BCI-algebras have been studied in (see [1, 3, 4, 12] and references there in).

As a follow up, in the present paper, the notion of a hybrid quasi-associative ideal of a BCI-algebra will be introduced and some related properties will be investigated. Relations between a hybrid ideal and a hybrid quasi-associative ideal in BCI-algebra will be provided. Moreover, characterizations of a hybrid quasi-associative ideal will be given.

2 Preliminaries

A *BCI-algebra* is an algebra $(X; *, 0)$ of type $(2, 0)$ that satisfies the following conditions:

(I) $(\forall k, w, z \in X)$ $(((k * w) * (k * z)) * (z * w) = 0)$,

(II) $(\forall k, w \in X)$ $((k * (k * w)) * w = 0)$,

(III) $(\forall k \in X)$ $(k * k = 0)$,

(IV) $(\forall k, w \in X)$ $(k * w = 0, w * k = 0 \Rightarrow k = w)$.

A partial order '$\leq$' is defined on X by $k \leq w$ if and only if $k * w = 0$. The following properties are satisfied in any BCI-algebra X:

(a1) $(\forall k \in X)$ $(k * 0 = k)$.

(a2) $(\forall k, w, z \in X)$ $((k * w) * z = (k * z) * w)$.

(a3) $(\forall k, w, z \in X)$ $(k \leq w \Rightarrow k * z \leq w * z,\ z * w \leq z * k)$.

A nonempty subset B of a BCI-algebra X is said to be an *ideal* of X if it satisfies:

(I1) $0 \in B$,

(I2) $(\forall k, w \in X)$ $(\forall w \in B)$ $(k * w \in B \Rightarrow k \in B)$.

Definition 2.1. *[22] A nonempty subset B of X is said to be a quasi-associative ideal (abbreviated QA-ideal) of X if it satisfies (I1) and*

(I3) $(\forall k, z \in X)$ $(\forall w \in B)$ $(k * (w * z) \in B \Rightarrow k * z \in B)$.

Further, we collect some basic notions and results on hybrid structures due to Jun et al. [11]. Let I, X and $\mathfrak{P}(U)$ be the unit interval, a set of parameters and the power set of an initial universe set U, respectively.

Definition 2.2. *[11] A hybrid structure in X over U is a mapping*

$$\tilde{f}_\mu := (\tilde{f}; \mu) : X \to \mathfrak{P}(U) \times I; k \mapsto (\tilde{f}(k); \mu(k))$$

where $\tilde{f} : X \to \mathfrak{P}(U)$ and $\mu : X \to I$ are mappings.

Definition 2.3. *[11] A hybrid structure $\tilde{f}_\mu = (\tilde{f}, \mu)$ in a BCI-algebra X is called a hybrid subalgebra of X if it satisfies:*

(HS1) $(\forall k, w \in X)$ $(\tilde{f}(k * w) \supseteq \tilde{f}(k) \cap \tilde{f}(w))$,

(HS2) $(\forall k, w \in X)$ $(\mu(k * w) \leq \max\{\mu(k), \mu(w)\})$.

Definition 2.4. *[11] For hybrid structures $\tilde{f}_\mu$ and $\tilde{g}_\nu$ in X over U, the hybrid intersection denoted by $\tilde{f}_\mu \Cap \tilde{g}_\nu$ is a hybrid structure*

$$\tilde{f}_\mu \Cap \tilde{g}_\nu : X \to \mathfrak{P}(U) \times I, k \mapsto ((\tilde{f} \tilde{\cap} \tilde{g})(k), (\mu \vee \nu)(k)),$$

where

$$\tilde{f} \,\tilde{\cap}\, \tilde{g} : X \to \mathfrak{P}(U), k \mapsto \tilde{f}(k) \cap \tilde{g}(k),$$
$$\mu \vee \nu : X \to I, k \mapsto \max\{\mu(k), \nu(k)\}.$$

Definition 2.5. *[11] For hybrid structures $\tilde{f}_\mu$ and $\tilde{g}_\nu$ in X over U, the hybrid union denoted by $\tilde{f}_\mu \Cup \tilde{g}_\nu$ is a hybrid structure*

$$\tilde{f}_\mu \Cup \tilde{g}_\nu : X \to \mathfrak{P}(U) \times I, k \mapsto ((\tilde{f} \tilde{\cup} \tilde{g})(k), (\mu \wedge \nu)(k)),$$

where

$$\tilde{f} \,\tilde{\cup}\, \tilde{g} : X \to \mathfrak{P}(U), k \mapsto \tilde{f}(k) \cup \tilde{g}(k),$$
$$\mu \wedge \nu : X \to I, k \mapsto \min\{\mu(k), \nu(k)\}.$$

Definition 2.6. *[12] A hybrid structure $\tilde{f}_\mu = (\tilde{f}, \mu)$ in a BCI-algebra X is called a hybrid ideal of X if it satisfies:*

(HI1) $(\forall k \in X)$ $(\tilde{f}(0) \supseteq \tilde{f}(k),\ \mu(0) \leq \mu(k))$,

(HI2) $(\forall k, w \in X)$ $(\tilde{f}(k) \supseteq \tilde{f}(k * w) \cap \tilde{f}(w))$,

(HI3) $(\forall k, w \in X)$ $(\mu(k) \leq \max\{\mu(k * w), \mu(w)\})$.

Lemma 2.7. *[12] For a hybrid ideal $\tilde{f}_\mu$ of X over U. If the inequality $k \leq w$ holds in X, then the following are equivalent:*
(1) $(\forall k, w \in X)$ $(\tilde{f}(k) \supseteq \tilde{f}(w), \mu(k) \leq \mu(w))$.
(2) $(\forall k \in X)$ $(\tilde{f}(0) = \tilde{f}(k), \mu(0) = \mu(k))$.

Definition 2.8. *[5] For two BCI-algebras; $(X_1, *_2, 0)$ and $(X_2, *_2, 0)$, the Cartesian product $X_1 \times X_2$ is the set $X_1 \times X_2 = \{(k_1, k_2) |\ k_1 \in X_1, k_2 \in X_2\}$ where*

$$(k_1, k_2) * (w_1, w_2) = (k_1 *_1 w_1, k_2 *_2 w_2), \quad \forall (k_1, k_2), (w_1, w_2) \in X_1 \times X_2.$$

3 Hybrid quasi-associative ideals

In what follows, let X denotes a BCI-algebra unless otherwise specified.

Definition 3.1. *A hybrid structure $\tilde{f}_\mu = (\tilde{f}, \mu)$ in X is called a hybrid quasi-associative ideal (abbreviated HQA-ideal) of X if it satisfies (HI1) and*

(HI4) $(\forall k, w, z \in X)$ $(\tilde{f}(k * z) \supseteq \tilde{f}(k * (w * z)) \cap \tilde{f}(w))$,

(HI5) $(\forall k, w, z \in X)$ $(\mu(k * z) \leq \max\{\mu(k * (w * z)), \mu(w)\})$.

Table 1: Cayley Table of the binary operation $*$

$*$	0	1	a	2
0	0	1	a	2
1	1	0	2	a
a	a	2	0	1
2	2	a	1	0

Example 1. *Let $U = \{e_1, e_2, e_3, e_4\}$ be an initial universe set and $X = \{0, 1, a, 2\}$ be a BCI-algebra with Cayley Table 1.*

Let $\tilde{f}_\mu = (\tilde{f}; \mu)$ be a hybrid structure in X over U which is given by Table 2.

Table 2: Table representation of the hybrid structure $\tilde{f}_\mu$

X	$\tilde{f}$	μ
0	U	0.3
1	$\{e_1, e_2, e_3\}$	0.3
a	$\{e_1, e_2\}$	0.5
2	$\{e_1, e_2\}$	0.5

By direct calculations we verify that $\tilde{f}_\mu$ is a HQA-ideal of X over U.

Theorem 3.2. *Every HQA-ideal is a hybrid ideal in a BCI-algebra.*

Proof. Let $\tilde{f}_\mu = (\tilde{f}, \mu)$ be a HQA-ideal of X. Taking $z = 0$ in (HI4) and (HI5), we have

$$\tilde{f}(k) = \tilde{f}(k * 0) \supseteq \tilde{f}(k * (w * 0)) \cap \tilde{f}(w) = \tilde{f}(k * w) \cap \tilde{f}(w),$$

$$\mu(k) = \mu(k * 0) \leq \max\{\mu(k * (w * 0)), \mu(w)\} = \max\{\mu(k * w), \mu(w)\}$$

for all $k, w \in X$. Hence, $\tilde{f}_\mu = (\tilde{f}, \mu)$ is a hybrid ideal of X. □

The converse of Theorem 3.2 is not true in general as seen in the following example.

Example 2. *Let $U = \{e_1, e_2, e_3, e_4, e_5\}$ be an initial universe set and $X = \{0, 1, 2, 3, 4\}$ be a BCI-algebra with Cayley Table 3.*

Table 3: Cayley Table of the binary operation $*$

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Let $\tilde{f}_\mu = (\tilde{f}; \mu)$ be a hybrid structure in X over U which is given by Table 4.

Table 4: Table representation of the hybrid structure $\tilde{f}_\mu$

X	$\tilde{f}$	μ
0	U	0.2
1	$\{e_1, e_2, e_3, e_4\}$	0.3
2	$\{e_2, e_3\}$	0.4
3	$\{e_2, e_3\}$	0.4
4	$\{e_2, e_3\}$	0.4

By routine calculations we can show that $\tilde{f}_\mu$ is a hybrid ideal of X over U but not HQA-ideal as $\tilde{f}(4*2) = \tilde{f}(3) = \{e_2, e_3\} \not\supseteq U = \tilde{f}(4*(0*2)) \cap \tilde{f}(0) = \tilde{f}(0)$ and/or
$\mu(4 * 2) = \mu(3) = 0.4 \not\leq 0.2 = \mu(0) = \max\{\mu(4 * (0 * 2)), \mu(0)\}$.

Remark 3.3. *From Example 2, we know that a hybrid ideal is not HQA-ideal. So under which condition(s) is every hybrid ideal is HQA-ideal? We give an answer in the following theorem.*

Theorem 3.4. *If a BCI-algebra X is quasi-associative, that is, the inequality $(k * w) * z \leq k * (w * z)$ is valid for all $k, w, z \in X$, then every hybrid ideal is a HQA-ideal.*

Proof. Let $\tilde{f}_\mu = (\tilde{f}, \mu)$ be a hybrid ideal of X. Since X is quasi-associative, we have $(k * w) * z \leq k * (w * z)$ for all $k, w, z \in X$. By Lemma 2.7 (1), for every hybrid ideal $\tilde{f}_\mu = (\tilde{f}, \mu)$ we have, $\tilde{f}$ is order reversing and μ is order preserving. Then, we have

$$\tilde{f}((k*w)*z) \supseteq \tilde{f}(k*(w*z)),\ \mu((k*w)*z) \leq \mu(k*(w*z))$$

so from (a2), (HI2) and (HI3) that

$$\begin{array}{rcl}\tilde{f}(k*z) & \supseteq & \tilde{f}((k*z)*w) \cap \tilde{f}(w) \\ & = & \tilde{f}((k*w)*z) \cap \tilde{f}(w) \\ & \supseteq & \tilde{f}(k*(w*z)) \cap \tilde{f}(w),\end{array}$$

$$\begin{array}{rcl}\mu(k*z) & \leq & \max\{\mu((k*z)*w), \mu(w)\} \\ & = & \max\{\mu((k*w)*z), \mu(w)\} \\ & \leq & \max\{\mu(k*(w*z)), \mu(w)\}.\end{array}$$

Hence, $\tilde{f}_\mu = (\tilde{f}, \mu)$ is a HQA-ideal of X. □

Corollary 3.5. *If a BCI-algebra X satisfies the equality $0*(0*k) = 0*k$ for all $k \in X$, then every hybrid ideal is a HQA-ideal.*

Proof. Straightforward. □

Proposition 3.6. *If $\tilde{f}_\mu = (\tilde{f}, \mu)$ is a HQA-ideal of X, then the following hold:*

(i) $(\forall k, w \in X)\ (k \leq w \Rightarrow \tilde{f}(k) \supseteq \tilde{f}(w), \mu(k) \leq \mu(w))$,

(ii) $(\forall k, w \in X)\ (\tilde{f}(k*w) = \tilde{f}(0) \Rightarrow \tilde{f}(k) \supseteq \tilde{f}(w)$,
$\mu(k*w) = \mu(0) \Rightarrow \mu(k) \leq \mu(w))$,

(iv) $(\forall k, w \in X)\ (\tilde{f}(k*w) \supseteq \tilde{f}(k) \cap \tilde{f}(w), \mu(k*w) \leq \max\{\mu(k), \mu(w)\})$,

(v) $(\forall k, w, z \in X)\ (\tilde{f}(k*w) \supseteq \tilde{f}(k*z) \cap \tilde{f}(z*w)$,
$\mu(k*w) \leq \max\{\mu(k*z), \mu(z*w)\})$,

(vi) $(\forall k \in X)\ (\tilde{f}((0*k)*k) = \tilde{f}(0), \mu((0*k)*k) = \mu(0))$.

Proof. (i) If $k \leq w$, then $k*w = 0$. Hence,

$$\begin{array}{l}\tilde{f}(k) = \tilde{f}(k*0) \supseteq \tilde{f}(k*(w*0)) \cap \tilde{f}(w) \\ \quad = \tilde{f}(k*w) \cap \tilde{f}(w) = \tilde{f}(0) \cap \tilde{f}(w) = \tilde{f}(w),\end{array}$$

$$\begin{array}{l}\mu(k) = \mu(k*0) \leq \max\{\mu(k*(w*0)), \mu(w)\} \\ \quad = \max\{\mu(k*w), \mu(w)\} = \max\{\mu(0), \mu(w)\} = \mu(w).\end{array}$$

(ii) is similar to the case (i).
(iii) For any $k, w \in X$, we have

$$\begin{aligned}\tilde{f}(k*w) &\supseteq \tilde{f}(k*(w*w)) \cap \tilde{f}(w) \\ &= \tilde{f}(k*0) \cap \tilde{f}(w) \\ &= \tilde{f}(k) \cap \tilde{f}(w),\end{aligned}$$

$$\begin{aligned}\mu(k*w) &\leq \max\{\mu(k*(w*w)), \mu(w)\} \\ &= \max\{\mu(k*0), \mu(w)\} \\ &= \max\{\mu(k), \mu(w)\}.\end{aligned}$$

(iv) Using (I) and (i) above, we have

$$\tilde{f}((k*w)*(k*z)) \supseteq \tilde{f}(z*w) \text{ and } \mu((k*w)*(k*z)) \leq \mu(z*w).$$

It follows from Theorem 3.2 that

$$\begin{aligned}\tilde{f}(k*w) &\supseteq \tilde{f}((k*w)*(k*z)) \cap \tilde{f}(k*z) \\ &\supseteq \tilde{f}(z*w) \cap \tilde{f}(k*z),\end{aligned}$$

$$\begin{aligned}\mu(k*w) &\leq \max\{\mu((k*w)*(k*z)), \mu(k*z)\} \\ &\leq \max\{\mu(z*w), \mu(k*z)\}.\end{aligned}$$

(v) In (HI4) and (HI5), if we replace k, w and z by $0*k$, 0 and k, respectively, then

$$\tilde{f}((0*k)*k) \supseteq \tilde{f}((0*k)*(0*k)) \cap \tilde{f}(0) = \tilde{f}(0),$$

$$\mu((0*k)*k) \leq \max\{\mu((0*k)*(0*k)), \mu(0)\} = \mu(0).$$

It follows from (HI1) that $\tilde{f}((0*k)*k) = \tilde{f}(0)$ and $\mu((0*k)*k) = \mu(0)$. The proof is completed. □

Remark 3.7. *Proposition 3.6 (iii) shows that every HQA-ideal is a hybrid subalgebra.*

Proposition 3.8. *Let $\tilde{f}_\mu = (\tilde{f}; \mu)$ be a hybrid ideal of X over U. The following statements are equivalent:*
(1) $\tilde{f}_\mu = (\tilde{f}; \mu)$ is a HQA-ideal of X.
(2) $(\forall k, w \in X)$ $(\tilde{f}(k*w) \supseteq \tilde{f}(k*(0*w)), \mu(k*w) \leq \mu(k*(0*w)))$.
(3) $(\forall k, w, z \in X)$ $(\tilde{f}((k*w)*z) \supseteq \tilde{f}(k*(w*z)), \mu((k*w)*z) \leq \mu(k*(w*z)))$.

Proof. (1) $\Rightarrow$ (2). Assume that $\tilde{f}_\mu = (\tilde{f};\mu)$ is a HQA-ideal. Then by (HI1), (HI4) and (HI5), we have the following for $k, w \in X$

$$\tilde{f}(k*w) \supseteq \tilde{f}(k*(0*w)) \cap \tilde{f}(0) = \tilde{f}(k*(0*w))$$

and

$$\mu(k*w) \leq \max\{\mu(k*(0*w)), \mu(0)\} = \mu(k*(0*w)).$$

Thus, (2) is satisfied.
(2) $\Rightarrow$ (3). Assume that (2) is satisfied. Then by using (a2), (I) and (III), we have

$$\begin{aligned}((k*w)*(0*z))*(k*(w*z)) &= ((k*w)*(k*(w*z)))*(0*z) \\ &\leq ((w*z)*w)*(0*z) \\ &= ((w*w)*z)*(0*z) \\ &= (0*z)*(0*z) \\ &= 0.\end{aligned}$$

That is, $(k*w)*(0*z) \leq k*(w*z)$. Combining the assumption and Lemma 2.7, we get
$\tilde{f}((k*w)*z) \supseteq \tilde{f}(k*(w*z)), \mu((k*w)*z) \leq \mu(k*(w*z))$. Thus (3) is satisfied.
(3) $\Rightarrow$ (1). Assume that (3) is satisfied. By (HI2), (HI3), (a2) and (3), we have for $k, w, z \in X$

$$\begin{aligned}\tilde{f}(k*z) &\supseteq \tilde{f}((k*z)*w) \cap \tilde{f}(w) \\ &= \tilde{f}((k*w)*z) \cap \tilde{f}(w) \\ &\supseteq \tilde{f}(k*(w*z)) \cap \tilde{f}(w)\end{aligned}$$

and

$$\begin{aligned}\mu(k*z) &\leq \max\{\mu((k*z)*w), \mu(w)\} \\ &= \max\{\mu((k*w)*z), \mu(w)\} \\ &\leq \max\{\mu(k*(w*z)), \mu(w)\}\end{aligned}$$

This is (HI4) and (HI5) and so $\tilde{f}_\mu = (\tilde{f};\mu)$ is a HQA-ideal. □

Proposition 3.9. *For two hybrid ideals $\tilde{f}_\mu = (\tilde{f};\mu)$ and $\tilde{g}_\nu = (\tilde{g};\nu)$ of X over U, if the following are satisfied:*
(1) $\tilde{f}(0) = \tilde{g}(0),\ \mu(0) = \nu(0)$,
(2) (for all $k \in X$) $(\tilde{f}(k) \subseteq \tilde{g}(k),\ \mu(k) \geq \nu(k))$
then whenever $\tilde{f}_\mu$ is a HQA-ideal, $\tilde{g}_\nu$ is.

Proof. Starting with (1) and using (III), (a2), Proposition 3.8 we have the following

$$\begin{aligned}\tilde{g}(0)=\tilde{f}(0)&=\tilde{f}((k*(0*w))*(k*(0*w)))\\&=\tilde{f}((k*(k*(0*w)))*(0*w))\\&\subseteq\tilde{f}((k*(k*(0*w)))*w)\\&\subseteq\tilde{g}((k*(k*(0*w)))*w)\\&=\tilde{g}((k*w)*(k*(0*w)))\end{aligned}$$

and

$$\begin{aligned}\nu(0)=\mu(0)&=\mu((k*(0*w))*(k*(0*w)))\\&=\mu((k*(k*(0*w)))*(0*w))\\&\geq\mu((k*(k*(0*w)))*w)\\&\geq\nu((k*(k*(0*w)))*w)\\&=\nu((k*w)*(k*(0*w)))\end{aligned}$$

Using (HI1) it follows that $\tilde{g}(0)=\tilde{g}((k*w)*(k*(0*w)))$ and $\nu(0)=\nu((k*w)*(k*(0*w)))$.
Now as $\tilde{g}_\nu$ is a hybrid ideal then we have

$$\begin{aligned}\tilde{g}(k*w)&\supseteq\tilde{g}((k*w)*(k*(0*w)))\cap\tilde{g}(k*(0*w))\\&\supseteq\tilde{g}(0)\cap\tilde{g}(k*(0*w))\\&=\tilde{g}(k*(0*w))\end{aligned}$$

and

$$\begin{aligned}\nu(k*w)&\leq\max\{\nu((k*w)*(k*(0*w))),\nu(k*(0*w))\}\\&\leq\max\{\nu(0),\nu(k*(0*w))\}\\&=\nu(k*(0*w))\end{aligned}$$

□

Hence, $\tilde{g}_\nu$ is a HQA-ideal, by Proposition 3.8.

Theorem 3.10. *Let $\tilde{f}_\mu=(\tilde{f};\mu)$ be a hybrid structure in X over U. For $\alpha\in\mathfrak{P}(U)$ and $t\in I$, let*

$$\tilde{f}_\mu(\alpha):=\{k\in X\mid\tilde{f}(k)\supseteq\alpha\}\text{ and }\tilde{f}_\mu(t):=\{k\in X\mid\mu(k)\leq t\}.$$

Then, $\tilde{f}_\mu$ is a HQA-ideal of X if and only if the nonempty sets $\tilde{f}_\mu(\alpha)$ and $\tilde{f}_\mu(t)$ are QA-ideals of X.

Proof. Let $\tilde{f}_\mu$ be a HQA-ideal of X. It is obvious form (HI1) that $0 \in \tilde{f}_\mu(\alpha) \cap \tilde{f}_\mu(t)$ and so $\tilde{f}_\mu(\alpha) \neq \emptyset$ and $\tilde{f}_\mu(t) \neq \emptyset$. Let $k*(w*z), w \in \tilde{f}_\mu(\alpha) \cap \tilde{f}_\mu(t)$, then $\tilde{f}(k*(w*z)) \supseteq \alpha, \tilde{f}(w) \supseteq \alpha$ and $\mu(k*(w*z)) \leq t, \mu(w) \leq t$. It follows from $(HI4)$ and $(HI5)$ that

$$\tilde{f}(k*z) \supseteq \tilde{f}(k*(w*z)) \cap \tilde{f}(w) \supseteq \alpha$$

and

$$\mu(k*z) \leq \max\{\mu(k*(w*z)), \mu(w)\} \leq t.$$

Hence, $k*z \in \tilde{f}_\mu(\alpha) \cap \tilde{f}_\mu(t)$, and so $\tilde{f}_\mu(\alpha)$ and $\tilde{f}_\mu(t)$ are QA-ideals of X.

To prove the converse, suppose that $\tilde{f}_\mu(\alpha)$ and $\tilde{f}_\mu(t)$ are QA-ideals of X. Then, $0 \in \tilde{f}_\mu(\alpha) \cap \tilde{f}_\mu(t)$, i.e. $\tilde{f}(0) \supseteq \alpha$ and $\mu(0) \leq t$, for any $\alpha \in \mathfrak{P}(U)$ and $t \in I$. That is, (HI1) is satisfied. let $k, w \in X$ such that $\tilde{f}(k*(w*z)) = \alpha_1$ and $\tilde{f}(w) = \alpha_2$. Taking $\alpha = \alpha_1 \cap \alpha_2$ implies that $k*(w*z), w \in \tilde{f}_\mu(\alpha)$, and so $k*z \in \tilde{f}_\mu(\alpha)$. Hence,

$$\tilde{f}(k*z) \supseteq \alpha = \alpha_1 \cap \alpha_2 = \tilde{f}(k*(w*z)) \cap \tilde{f}(w).$$

Also, taking $t := \max\{\mu(k*(w*z)), \mu(w)\}$ implies $k*(w*z), w \in \tilde{f}_\mu(t)$, and so $k*z \in \tilde{f}_\mu(t)$. It follows that

$$\mu(k*z) \leq t = \max\{\mu(k*(w*z)), \mu(w)\}.$$

Hence, $\tilde{f}_\mu$ is a HQA-ideal of X. □

Theorem 3.11. *Let $\tilde{f}_\mu = (\tilde{f}; \mu)$ be a HQA-ideal of X over U. For $\emptyset \neq \alpha \in \mathfrak{P}(U)$ and $t \in I$, let*

$$\widehat{X} := \{k \in X \mid \tilde{f}(k) \cap \alpha \neq \emptyset, \mu(k) \leq t\}.$$

Then, the nonempty set $\widehat{X}$ is a QA-ideal of X.

Proof. Let $\tilde{f}_\mu$ be a HQA-ideal of X. Then it follows from (HI1) that $0 \in \widehat{X}$ that is (I1) is satisfied. Let $\widehat{X} \neq \alpha$. If $k*(w*z), w \in \widehat{X}$. Then, $\tilde{f}(k*(w*z)) \cap \alpha \neq \emptyset \neq \tilde{f}(w) \cap \alpha, \mu(k*(w*z)) \leq t$ and $\mu(w) \leq t$. From $(HI4)$ and $(HI5)$ it follows that:

$$\begin{aligned} \tilde{f}(k*z) \cap \alpha &\supseteq (\tilde{f}(k*(w*z)) \cap \tilde{f}(w)) \cap \alpha \\ &= (\tilde{f}(k*(w*z)) \cap \alpha) \cap (\tilde{f}(w) \cap \alpha) \neq \emptyset \end{aligned}$$

and

$$\mu(k*z) \leq \max\{\mu(k*(w*z)), \mu(w)\} \leq t.$$

Hence, $k*z \in \widehat{X}$, and so $\widehat{X}$ is a QA-ideal. □

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