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Results on r-regular near-rings

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Abstract

The objective of this investigation is to provide some characterizations on r-regular near-rings with IFP as well as with mate functions.

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1 Introduction

The development of new ideas on “Near-rings” is demonstrated by investigating several characteristics such as regularity, primitivity, radicals, and so on. The notion regularity was first defined by Roos [6] in Ring theory. With this idea, this concept is exhibited to near-rings and a lot of research was done by Dheena [1], Gerhard [10], Ramakotaiah and Rao[5]. Recently, Manikantan and Kumar [3] established relations on pseudo symmetric, primary ideals in right near-rings.

2 Preliminaries

Definition 2.1. *[4] A triple* $(\mathfrak{K}, +, .)$ *is referred as Right near-ring, where*

1. $\mathfrak{K}$ *satisfies the properties of a "Group" under addition.*
2. $\mathfrak{K}$ *satisfies the properties of a "Semi-group" under multiplication.*
3. $(t^1 + x^1)\,.s^1 = t^1.s^1 + x^1.s^1, \forall\, t^1, x^1, s^1 \in \mathfrak{K}$ *(right distributive law).*

Moreover, we consider the Right near-ring$(\mathfrak{K}, +, .)$ and we designate a right near-ring as $\mathfrak{K}$ unless otherwise mentioned. We write t^1s^1 to denote $t^1.s^1$, for any two elements t^1 and s^1 in a near-ring $\mathfrak{K}$. For additional definitions and results, we refer the reader to [4]. We recall the following:

Definition 2.2. *[4] Let* $\mathfrak{K}$ *refer to "Zero-symmetric near-ring" if* $k0 = 0\ \forall k \in \mathfrak{K}$ *i.e.,* $\mathfrak{K} = \mathfrak{K}_o$.

Example 2.3. *Let* $(\mathfrak{K}, +)$*, where* $\mathfrak{K} = \{i^1, p^1, q^1, r^1\}$ *be a Klein's four group with addition and product tables mentioned below is an example for a Zero-symmetric near-ring.* $(\mathfrak{K}, +)$ *is a Zero-symmetric near-ring which we denote by* $\mathfrak{K} \in \eta_0$*.*

Table 1: Addition table

$+$	i^1	p^1	q^1	r^1
i^1	i^1	p^1	q^1	r^1
p^1	p^1	i^1	r^1	q^1
q^1	q^1	r^1	i^1	p^1
r^1	r^1	q^1	p^1	i^1

Table 2: Product table

$.$	i^1	p^1	q^1	r^1
i^1	i^1	i^1	i^1	i^1
p^1	i^1	p^1	q^1	r^1
q^1	i^1	i^1	i^1	i^1
r^1	i^1	p^1	q^1	r^1

Definition 2.4. *[4] A subgroup* $\mathfrak{D}$ *of* $\mathfrak{K}$ *is said to be* $\mathfrak{K}$*-subgroup (*$\mathfrak{K}$*-SG) if* $\mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}$*.*

Notation 2.5. *If* $S, T \subseteq \mathfrak{K}$*, then we define* $ST = \{st/s \in S, t \in T\}$*.*

We now designate a normal subgroup as NSG.

Definition 2.6. *[4] A NSG* $\mathfrak{I}$ *of* $(\mathfrak{K}, +)$ *is referred as left ideal of* $\mathfrak{K}$*, if* $\forall\, t, p \in \mathfrak{K}$*,* $\forall s \in \mathfrak{I}$*,* $t(p + s) - tp \in \mathfrak{I}$*.*

Definition 2.7. *[4] A NSG* $\mathfrak{I}$ *of* $(\mathfrak{K}, +)$ *is referred as right ideal of* $\mathfrak{K}$*, if* $\mathfrak{I}\mathfrak{K} \subseteq \mathfrak{I}$*.*

Definition 2.8. *[4] A NSG* $\mathfrak{I}$ *of* $(\mathfrak{K}, +)$ *is referred as ideal (two-sided ideal)if it satisfies both the definitions of left ideal and a right ideal of* $\mathfrak{K}$*.*

Theorem 2.9. *[4] For a near-ring* $\mathfrak{K} \in \eta_0$*, every ideal is a* $\mathfrak{K}$*-SG of* $\mathfrak{K}$*.*

Definition 2.10. *[4] Assume that* F *is a non-void subset in* $\mathfrak{K}$*. Then* $\{\mathrm{L_s}/s \in I\}$ *is the family of all left ideals which contain* F*.* $L = \cap_{s\in I} L_s$ *is* *the smallest left ideal containing* F *is referred to as "left ideal generated by* F*".*

Definition 2.11. *[4] An ideal* $\mathfrak{A}$ *of* $\mathfrak{K}$ *is called a "principal ideal" if* $\mathfrak{A}$ *is generated by one component.*
If an ideal $\mathfrak{A}$ *is generated by an element 'a', then* $\mathfrak{A}$ *is symbolized by* $\langle a \rangle$.
If a left ideal $\mathfrak{A}$ *is generated by a single component 'a', then* $\mathfrak{A}$ *is symbolized by* $\langle a|$.

Definition 2.12. *[4] The center of a near-ring* $\mathfrak{K}$ *is defined as*
$\mathfrak{C} = \{x \in \mathfrak{K}/nx = xn, \forall\ n \in \mathfrak{K}\}$. *Elements in* $\mathfrak{C}$ *are said to be central.*

Definition 2.13. *[4] A component ' p ' is called an idempotent element of* $\mathfrak{K}$ *if* $p^2 = p$, *for* $p \in \mathfrak{K}$.

Definition 2.14. *[1] [4] Let* $\mathfrak{K}$ *be identified as Insertion of Factors Property(IFP), if* $ts = 0 \implies tps = 0, \forall t, s, p \in \mathfrak{K}$.

The above mentioned near-ring Example 2.3 is an example of an IFP near-ring.

Definition 2.15. *[4] For each individual component* $k \in \mathfrak{K}$, *if* $k^2 = 0 \Rightarrow k = 0$, *then* $\mathfrak{K}$ *is known as a reduced near-ring.*

Lemma 2.16. *[4] For each individual* d, l *in* $\mathfrak{K} \in \eta_0$, *which is a reduced near-ring,* $dlt = dtl$, *where* $t^2 = t$, t *is in* $\mathfrak{K}$.

Theorem 2.17. *[4] If* $\mathfrak{K} \in \eta_0$ *has no non-zero nilpotent components, then* $\mathfrak{K}$ *satisfies the IFP.*

Definition 2.18. *[4] If for each individual component $c \in \mathfrak{K}$, $\mathfrak{K}c = \mathfrak{K}c^2$, then $\mathfrak{K}$ is known as "left bipotent".*

Definition 2.19. *[4] If for each individual component $k \in \mathfrak{K}$ there is a component l in $\mathfrak{K}$ such that $k = klk$, then $\mathfrak{K}$ is known as "regular near-ring (RN)".*

Definition 2.20. *[4] If for each individual component $p \in \mathfrak{K}$ there is a component l in $\mathfrak{K}$ such that $p = lp^2$, then $\mathfrak{K}$ is known as "left strongly regular near-ring (left SRN)".*

Definition 2.21. *[7] If for each individual component $q \in \mathfrak{K}$ there is a component l which is an idempotent in $\mathfrak{K}$ such that $q = ql, l \in \langle q|$, then $\mathfrak{K}$ is known as "r-regular near-ring (r-RN)".*

Theorem 2.22. *[7] If $\mathfrak{K}$ is r-RN with 1 and has IFP, then $a = al$ implies $a = la$, where l is an idempotent in $\mathfrak{K}$, $l \in \langle a|$.*

3 Characterization of "r-regular near-rings".

The principal object "m -regular near-ring" was cited by KrishnaMoorthy, Veega, and Geetha [2] who proved some results. In this section, we introduce "m-regular near-ring with r-regular near-ring" and give some characterization.

Definition 3.1. *[2] If for each individual component $k \in \mathfrak{K}$ there is a component l in $\mathfrak{K}$ such that $k = kl^m k$ where $m \geq 1$ is a fixed integer, then $\mathfrak{K}$ is known as "m-regular near-ring (m-RN)".*

Example 3.2. *Let* $(\mathfrak{K}, +)$, *where* $\mathfrak{K} = \{i^1, p^1, q^1, r^1\}$ *be a Klein's four group with addition and product tables mentioned below.*

Table 3: Addition table

$+$	i^1	p^1	q^1	r^1
i^1	i^1	p^1	q^1	r^1
p^1	p^1	i^1	r^1	q^1
q^1	q^1	r^1	i^1	p^1
r^1	r^1	q^1	p^1	i^1

Table 4: Product table

$.$	i^1	p^1	q^1	r^1
i^1	i^1	i^1	i^1	i^1
p^1	i^1	q^1	r^1	p^1
q^1	i^1	r^1	p^1	q^1
r^1	i^1	p^1	q^1	r^1

Then $(\mathfrak{K}, +, .)$ is an example for m-RN.

Lemma 3.3. *[2] Let* $\mathfrak{K}$ *be a m-RN,* $a \in \mathfrak{K}$ *and* $a = ab^m a$. *Then*

- *The idempotents are* ab^m *and* $b^m a$.
- $ab^m\mathfrak{K} = a\mathfrak{K}$ & $\mathfrak{K}b^m a = \mathfrak{K}a$.

Definition 3.4. *Let* $\mathfrak{D}$ *be an ideal of* $\mathfrak{K}$ *is known as semi-prime ideal(S-PI)supposing that for all ideals* $\mathfrak{I}$ *of* $\mathfrak{K}$, $\mathfrak{I}^2 \subseteq \mathfrak{D}$ *implies* $\mathfrak{I} \subseteq \mathfrak{D}$.

Definition 3.5. *[4] For all ideals* $\mathfrak{D}$ *of* $\mathfrak{K}$, $xy \in \mathfrak{D}$ *implies that* $yx \in \mathfrak{D}$ *where* $x, y \in \mathfrak{K}$, *then we say that* $\mathfrak{K}$ *is said to be satisfies the property* $\mathfrak{P}_4$*(*$\mathfrak{P}_4$*).*

Definition 3.6. *[4] For any element* $p \in \mathfrak{K}, p^2 \in \mathfrak{D}$ *implies* $p \in \mathfrak{D}$ *then the ideal* $\mathfrak{D}$ *is said to be "completely semi-prime ideal" of* $\mathfrak{K}$*(C-Semi-prime ideal).*

Theorem 3.7. *Let* $\mathfrak{K} \in \eta_0$ *be a m-RN, r-RN in which all the idempotents are central with unity then*

1. *Any ideal* $\mathfrak{D}$ *of* $\mathfrak{K}$ *is C-Semi-prime ideal.*

2. *$\mathfrak{K}$ satisfies the $\mathfrak{P}_4$.*

Proof. 1. Suppose $\mathfrak{D}$ be any ideal of $\mathfrak{K}$ such that $p^2 \in \mathfrak{D}$. Let $p \in \mathfrak{K}$, by definition m-RN, for all $p \in \mathfrak{K}$, there exists $l \in \mathfrak{K}$ such that $p = pl^m p$ where $m \geq 1$, a fixed integer. By using the lemma 3.3 and by the theorem 2.22, we have $p = pl^m p = l^m p^2 \in \mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}$.(Since $\mathfrak{K} \in \eta_0$, we have $\mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}$). Therefore $p^2 \in \mathfrak{D}$ implies $p \in \mathfrak{D}$. Hence $\mathfrak{D}$ is C-semi-prime ideal.

2. Let $t, p \in \mathfrak{K}$ such that $tp \in \mathfrak{D}$. Then $(pt)^2$=$(pt)(pt) = p(tp)t \in \mathfrak{K}\mathfrak{D}\mathfrak{K} \subset \mathfrak{D} \Rightarrow (pt)^2 \in \mathfrak{D}$.Since by the above 1, $\mathfrak{D}$ is C-semi-prime ideal, $pt \in \mathfrak{D}$.Hence $\mathfrak{K}$ satisfies $\mathfrak{P}_4$.

□

4 r-Regular near-rings with mate function

In [8] and [9] to study regularity structure in a significant way, the notion " mate function " in $\mathfrak{K}$ has been initiated.

Definition 4.1. *[8], [9] For each individual component $p \in \mathfrak{K}$ then $p = p\psi(p)p$, where ψ is a mapping from $\mathfrak{K}$ into $\mathfrak{K}$ then $\psi(p)$ is said to be a mate function of p in $\mathfrak{K}$.*

Example 4.2. *Let $(\mathfrak{K}, +)$ where $\mathfrak{K} = \{i^1, p^1, q^1, r^1\}$ be a Klein's four group with addition and product tables mentioned below.*

Then $(\mathfrak{K}, +, .)$ is an example for r-RN. Here the identity function serves as a mate function.

Table 5: Addition table

$+$	i^1	p^1	q^1	r^1
i^1	i^1	p^1	q^1	r^1
p^1	p^1	i^1	r^1	q^1
q^1	q^1	r^1	i^1	p^1
r^1	r^1	q^1	p^1	i^1

Table 6: Product table

$.$	i^1	p^1	q^1	r^1
i^1	i^1	i^1	i^1	i^1
p^1	i^1	p^1	i^1	p^1
q^1	i^1	i^1	q^1	q^1
r^1	i^1	p^1	q^1	r^1

It is an important point that the mentioned Example2.3 is not a r-RN. It does not admit mate functions.

Lemma 4.3. *[8], [9] Let $\mathfrak{E}$ represents set of all idempotents of $\mathfrak{K}$. If ψ is a mate function of $\mathfrak{K}$ and for each component p in $\mathfrak{K}$ then*

- $p\psi(p), \psi(p)p \in \mathfrak{E}$.
- $\mathfrak{K}p = \mathfrak{K}\psi(p)p$ *and* $p\mathfrak{K} = p\psi(p)\mathfrak{K}$

Theorem 4.4. *Let $\mathfrak{K} \in \eta_0$ be a r-RN satisfies IFP with unity and $\mathfrak{K}$ admits mate function. Then $\mathfrak{K}$ fulfils the below mentioned conditions.*

1. *$\mathfrak{K}$ is left bi potent.*
2. *$\mathfrak{K}$ is left SRN.*

Proof. 1. Let $\mathfrak{K} \in \eta_0$ be r-RN with unity and satisfies IFP. Let ψ be a mate function in $\mathfrak{K}$. Let $p \neq 0$ and $p \in \mathfrak{K}p$. Now $p = pe, e^2 = e \in \mathfrak{K}, e \in \langle p| \subseteq \langle p\rangle$. By the theorem 2.22 and using the lemma 4.3, we have $p = pe = ep = ep\psi(p)p = e\psi(p)p^2 \subseteq \mathfrak{K}p^2$ which implies $\mathfrak{K}p \subseteq \mathfrak{K}p^2$. Since $\mathfrak{K}p^2 \subseteq \mathfrak{K}p$.Therefore $\mathfrak{K}p = \mathfrak{K}p^2$ for all $p \in \mathfrak{K}$.Hence $\mathfrak{K}$ is a left bi potent near-ring.

2. $p = pk, k^2 = k, k \in \langle p| \subseteq \langle p\rangle$. Now $p = pk = p\psi(p)pk = p\psi(p)p = \psi(p)p^2 \in \mathfrak{K}p^2 = yp^2$ for some $y = \psi(p) \in \mathfrak{K}$.Thus, $\mathfrak{K}$ is a left SRN.

□

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