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Domination and Inverse Domination in Wrapped Butterfly Networks

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Abstract

In a graph G a set S of vertices is called a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. Let S be a minimum dominating set G. If $V(G) - S$ contains a dominating set say $S^{'}$ of G, then $S^{'}$ is called an inverse dominating set with respect to S. A dominating set S of a graph G is non-split dominating set if the induced subgraph $V(G) - S$ is connected. A Wrapped butterfly network $WBF(n), n \geq 3$, is obtained by merging the first and last levels of a butterfly network $BF(n), n \geq 3$. In this paper we determine the domination, inverse domination and non-split domination numbers of $WBF(n)$.

1 Introduction

In the past few decades the problem concerning domination of graphs plays a major role in research branch of graph theory. Historically, the domination type problems mainly arise from chess game to obtain minimum number of queens needed to attack or dominate every square on the chessboard. Domination problems used to find the sets of representatives, in monitoring communication or electrical networks, and in land surveying where it is

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necessary to minimize the number of places a surveyor must stand in order to take height measurements for an entire region. It also plays a vital role in parallel processing and supercomputing, which continues to exert great influence in the development of modern science and engineering. Similarly inverse domination plays a major role in communication and electrical network which was introduced by Kulli and Sigarkanti. The inverse domination number which is the minimum cardinality of a dominating set whose complement contains a minimum dominating set, motivated to find the two disjoint minimum dominating set for any network gives the full back up at any critical situation.

It is a natural question why to devote special attention to the case of two disjoint dominating sets rather than k disjoint dominating sets for a general k. The reason is that, by Ores observation, the trivial necessary minimum degree condition is also sufficient for the existence of two disjoint dominating sets. For all fixed $k \geq 3$, it is NP-complete to decide the existence of k disjoint dominating sets and no minimum degree condition is sufficient for the existence of three disjoint dominating sets. One can find applications for two disjoint dominating sets in networks. In any network (or graphs), dominating sets are central sets and they play a vital in routing problems in parallel computing. Also finding efficient dominating sets is always concern in finding optimal central sets in networks.

Suppose S is a dominating set in a graph (or network) G, when the network fails in some nodes in S, the inverse dominating set in VS will take care of the role of S. In this aspect, it is worthwhile to concentrate on dominating and inverse dominating sets in graphs.The network of processors and interconnections play a vital role in facilitating the communication between processors in parallel computers. Some of the popular interconnection networks are rings, toroids, hypercube, Butterfly Graphs and wrapped Butterfly networks. The domination problem has been proved to be NP-complete [4]. In this paper we consider the domination, inverse domination, split and non-split domination problems for the wrapped butterfly network $WBF(n), n \geq 3$.

2 Basic Concepts

Definition 2.1. [1] *A dominating set S of a graph G is a subset of vertices of G with the condition that every vertex in $V \setminus S$ is adjacent to some vertex in S. Such a set with the minimum cardinality yields the domination number denoted by $\gamma(G)$.*

Definition 2.2. [**5**] *Let S be a minimum dominating set of G. If VD contains a dominating set say $S^{'}$, then $S^{'}$ is called an inverse dominating set with respect to S. The inverse domination number $\gamma^{'}(G)$ of G is the order of a smallest inverse dominating set in G.*

Definition 2.3. [**6**] *A dominating set of a graph $G = (V, E)$ is a non-split dominating set. If the induced subgraph $V - D$ is connected. The non-split domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a non-split dominating set.*

Following results on domination number and inverse domination number for connected graphs exist in the literature already.

Theorem 2.4. [**2**] *For any graph G of order p and maximum degree Δ, we have $\gamma(G) \geq p/(\Delta + 1)$.*

Theorem 2.5. [**6**] *If G is a connected graph with p vertices and q edges, then we have $\gamma_{ns}(G) \geq 2p - q - 1/2$.*

Theorem 2.6. [**3**] *For any graph G, $\gamma(G) \leq \gamma_{ns}(G)$.*

Theorem 2.7. [**5**] *Let S be a minimum dominating set of G. If for every vertex $v \in S$, the induced subgraph $< N[v] >$ is a complete graph of order at least two, then $\gamma(G) = \gamma^{'}(G)$.*

3 Main results

In this section we determine a lower bound for the domination number, inverse domination number, and non-split domination number of wrapped butterfly networks.

Definition 3.1. [**8**] *The n-dimensional butterfly network $BF(n)$ has vertex set $V = \{(x; i)/x = (x_1, x_2, x_n), x_i = 0$ or $1, 1 \leq i \leq n\}$. Two vertices $(x; i)$ and $(y; j)$ are linked by an edge in $BF(n)$ if and only if $j = i + 1$ and either (i) $x = y$, or (ii) x differs from y in precisely the j^{th} bit.*
Wrapped butterfly, denoted by $WBF(n)$ is an n- level graph with $n.2^n$ vertices and each vertex of degree 4.

Lemma 3.2. Let G be the connected undirected graph $WBF(n), n \geq 3$. Then $\gamma(G) \geq n.2^{n-2}$
Proof.

Let S be a dominating set in G. In general $WBF(n)$ has n rows, each containing 2^n vertices representing the columns. We divide the columns into two halves H_1 and H_2 as the columns represented by the first 2^{n-1} vertices and the columns represented by the next 2^{n-1} vertices respectively. Also let $L_1, L_2, L_3, ..., L_n$ denotes the levels in $WBF(n)$.

Claim 1: For H_1, when n is even, we need atleast $2^{n-2} + 2^{n-3}$ vertices from any of the three consecutive levels except top and bottom level in H_1.

Proof of Claim 1: Assume the three consecutive levels as $i - 1$, i,$i + 1$. To dominate level i we had atleast 2^{n-2} vertices from level $i-1$ or level $i+1$. Now we have three cases, i.e.) Let T be the dominating set in level i or level $i - 1$, or level $i + 1$.

Case(i): If T be the dominating set in level i

a) Suppose we choose the 2^{n-2} vertices from level i consecutively then it dominates only 2^{n-2} vertices in level $i-1$ and level $i+1$ but we have remaining consecutive 2^{n-2} vertices in all three levels i, $i-1$, $i+1$ are still not dominated. To dominate those remaining vertices we need again 2^{n-2} vertices . Thus we have atleast 2.2^{n-2} vertices to dominate all the three levels.

b) Divide the first row of H_1 into 4 sets of 2^{n-3} vertices from the 4 quarters of the consecutive columns in H_1. Suppose we choose the first quarter and the third quarter vertices among 2^{n-1} vertices in level i, now it dominates all the 4 quarters in level $i+1$, also first and third quarter in level $i-1$, here we have remaining 2^{n-2} vertices in level $i - 1$. For dominating these vertices we need to choose atleast 2^{n-3} vertices from level i or level $i - 1$ then we have atleast $2^{n-2} + 2^{n-3}$ vertices to dominate all the three levels in H_1.

Case(ii): If T be the dominating set in level $i + 1$.

Now we choose the 2^{n-2} vertices from level $i+1$ then it dominates only 2^{n-2} vertices in level i and not dominate the vertices in level $i-1$. For dominating this level we again choose 2^{n-2} vertices from level i. Thus we have atleast 2.2^{n-2} vertices to dominate all the three levels.

Case(iii): If T be the dominating set in level $i - 1$.

Now we choose the 2^{n-2} vertices from level $i-1$ then it dominates only 2^{n-2} vertices in level i and not dominate the vertices in level $i+1$. For dominating this level we again choose 2^{n-2} vertices from level i. Thus we have atleast 2.2^{n-2} vertices to dominate all the three levels. Among all the above cases, case(i)b is the best case. Hence the claim for H_1. Similarly it is true in the Claim for H_2. So we have in G for $(\frac{n-2}{3})$ consecutive level (leaving top and bottom level), then

$$|S| \geq 2.\Big[\Big(\frac{n-2}{3}\Big)(2^{n-2} + 2^{n-3})\Big]$$

.

Claim 2: To dominate top and bottom level in H_1 we need atleast 2^{n-2} vertices.

Proof of Claim 2: By using previous claim, there are atleast 2.2^{n-3} vertices which are not dominated in top and bottom level. Therefore, we need atleast 2^{n-2} vertices in level 1 or in level n to dominate both top and bottom level in H_1. Claim is also true for H_2. So we need 2.2^{n-2} vertices to dominate both top and bottom level in G. Hence for $WBF(n)$, the cardinality of dominating set

$$|S| \geq 2.\left[\left(\frac{n-2}{3}\right)(2^{n-2} + 2^{n-3}) + 2^{n-2}\right]$$

$$|S| \geq n.2^{n-2}.$$

Hence the proof is true for when n even.

Claim 3: Suppose when n is odd, we need atleast $2^{n-2} + 2^{n-3}$ vertices from any of the three consecutive levels except top or bottom level in H_1. By using Claim 1 we prove claim 3 in similar manner. Also it holds for H_2. So we have in G for $(\frac{n-1}{3})$ consecutive level (leaving top or bottom level), then

$$|S| \geq 2.\left[\left(\frac{n-1}{3}\right)(2^{n-2} + 2^{n-3})\right].$$

Claim 4: To dominate top or bottom level in H_1 we need atleast 2^{n-2} vertices.

Proof of Claim 4: By using previous claim, there are atleast 2.2^{n-3} vertices which are not dominated in top or bottom level. Therefore, we need atleast 2^{n-3} vertices in level 1 or in level n to dominate both top and bottom level in H_1. Claim is also true for H_2. So we need 2.2^{n-3} vertices to dominate both top or bottom level in G. Hence for $WBF(n)$, the cardinality of dominating set

$$|S| \geq 2.\left[\left(\frac{n-1}{3}\right)(2^{n-2} + 2^{n-3})\right] + 2^{n-2}.$$

$$|S| \geq n.2^{n-2}.$$

Hence the proof is true for when n is odd.

Therefore, the proof holds for both even and odd cases.

Claim 5: When n is the multiple of 3.

Proof of Claim 5: Similarly using by claim1, case(i)b, we have

$$|S| \geq 2.\left[\left(\frac{n}{3}\right)(2^{n-2} + 2^{n-3})\right].$$

$$|S| \geq n.2^{n-2}.$$

Hence For any G connected undirected graph $WBF(n), n \geq 3$. Then we have $\gamma(G) \geq n.2^{n-2}$.
Lemma 3.3. [7] Let G be the connected undirected graph $WBF(n), n \geq 3$.Then $\gamma(G) \leq n.2^{n-2}$.
By Lemma 3.2 and Lemma 3.3, we have the following theorem.
Theorem 3.4. Let G be the connected undirected graph $WBF(n), n \geq 3$. Then $\gamma(G) = n.2^{n-2}$.
Lemma 3.5. Let G be the connected undirected graph $WBF(n), n \geq 3$. Then $\gamma^{'}(G) \leq n.2^{n-2}$.
Proof.
$WBF(n)$ has n rows, each containing 2^n vertices representing the columns. We divide the columns into two halves H_1 and H_2 as the columns represented by the first 2^{n-1} vertices and the columns represented by the next 2^{n-1} vertices respectively. Let D be the minimum dominating set in H_1. We select vertices from H_1 in a set $D^{'} = V \setminus D$ as follows:
1. Divide the first row of H_1 into 4 sets of 2^{n-3} vertices from the 4 quarters of the consecutive columns in H_1. Select the second and fourth quarter vertices in $D^{'}$.
2. Divide the i^{th}row of H_1 into 2^{i+1} sets, $S_1, S_2, S_3, ..., S_{2^{i+1}}$, each consisting of 2^{n-i-2} vertices from the 2^{i+1} sets of consecutive columns of size 2^{n-i-2} in Row i, $2 \leq i \leq n-3$. Select 2^{i-1} sets from among $S_1, S_2, S_3, ..., S_{2^{i+1}}$, in $D^{'}$ such that the vertices in the selected set S_j are not dominated by end vertices in Level $(i-1)$, of straight edges and oblique edges incident at them, $2 \leq i \leq n-3$. The number of selected vertices H_i in Level i is $2^{i-1} \times 2^{n-i-2} = 2^{n-3}$.
3. Select 2^{n-3} number of vertices at Level $n-2$ such that they are consecutive pairs satisfying the condition that they are not dominated by end vertices of straight edges and oblique edges in Level $n-3$.
4. Include in $D^{'}$, the mirror images in H_2 of the already selected vertices in H_1. We claim that $D^{'}$ is a dominating set of G. It is clear from the choice of vertices in $D^{'}$, that none of the 4 neighbouring vertices of any vertex in $D^{'}$ belongs to $D^{'}$. Hence every vertex in $D^{'}$ belonging to Level i dominates 2 vertices in Level $(i-1)$ and 2 vertices in Level i, $2 \leq i \leq n-2$. This implies that all vertices in Level i which are not in $D^{'}$ are dominated by the vertices in $D^{'}$ belonging to Level $(i+1)$, $2 \leq i \leq n-2$. Vertices in $D^{'}$ from Level 1 dominate vertices in Level n. Thus $D^{'}$ is an inverse dominating set of $WBF(n), n \geq 3$. The cardinality of $D^{'}$ is $2(2^{n-1} + (n-4)2^{n-3}) = n.2^{n-2}$. Hence $\gamma^{'}(G) \leq n.2^{n-2}$.
Theorem 3.6. Let G be the connected undirected graph $WBF(n), n \geq 3$.

Then we have $\gamma^{'}(G) = n.2^{n-2}$.

Proof.

By Theorem 3.4 $\gamma(G) = n.2^{n-2}$ and by the virtue of the definition of inverse domination we know that $\gamma(G) \leq \gamma^{'}(G)$. Thus $\gamma^{'}(G) \geq n.2^{n-2}$. By the Lemma 3.5 $\gamma^{'}(G) \leq n.2^{n-2}$. Therefore $\gamma^{'}(G) = n.2^{n-2}$.

Theorem 3.7. [**3**] A dominating set D of a graph G is minimal if and only if for each vertex $v \in D$, one of the following conditions is satisfied:

i) there exists a vertex $u \in V - D$ such that $N(u) \cup D = \{v\}$; and

ii) v is an isolated vertex in $\langle D \rangle$.

Theorem 3.8. [**6**] A nonsplit dominating set D of a graph G is minimal if and only if for each vertex $v \in D$, one of the following conditions is satisfied:

i) there exists a vertex $u \in V - D$ such that $N(u) \cup D = \{v\}$; and

ii) v is an isolated vertex in $\langle D \rangle$; and

iii)$N(v) \cup (v - D) = \emptyset$.

Lemma 3.9. Let G be the connected undirected graph $WBF(n), n \geq 3$. Then $\gamma_{ns}(G) \leq n.2^{n-2}$.

Proof.

$WBF(n)$ has n rows, each containing 2^n vertices representing the columns. We divide the columns into two halves H_1 and H_2 as the columns represented by the first 2^{n-1} vertices and the columns represented by the next 2^{n-1} vertices respectively. Proof follows from Lemma 3.5.

Theorem 3.10. Let G be the connected undirected graph $WBF(n), n \geq 3$. Then $\gamma_{ns}(G) = n.2^{n-2}$.

Proof.

By Theorem 3.4 $\gamma(G) = n.2^{n-2}$. Since $\gamma_{ns}(G) \geq \gamma(G)$, we have $\gamma_{ns}(G) \geq n.2^{n-2}$. By Lemma 3.9 $\gamma_{ns}(G) \leq n.2^{n-2}$. These two results imply that $\gamma_{ns}(G) = n.2^{n-2}$. Hence the proof.

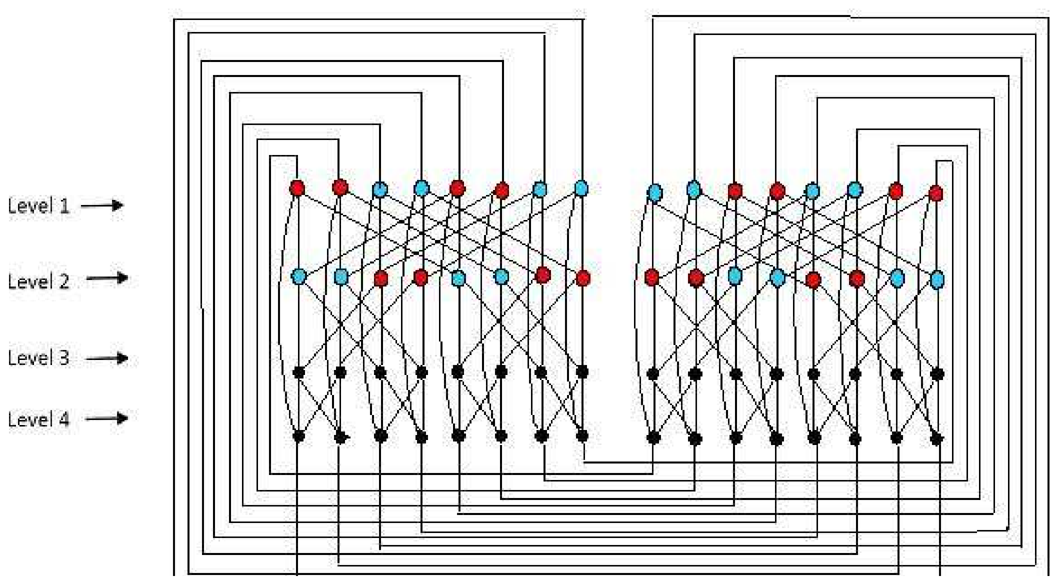

Figure 1: $WB(4)$ with dominating set marked in red and inverse and non-split dominating set marked in blue.

4 Conclusion

In this paper we find the domination number, inverse domination number and non-split domination number for $WBF(n)$, where $n \geq 3$. It is interesting to note that $\gamma(G) = \gamma^{'}(G) = \gamma_{ns}(G)$. Finding Graphs satisfying this property it would be an intersting line of research to identify graph G with $\gamma(G) = \gamma^{'}(G) = \gamma_{ns}(G)$.

References

[1] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, 1998.

[2] C. Berge, Graphs and Hypergraphs, 1973.

[3] E. J. Cockayne, R. M. Dawe, S. T. Hedetniemi, Towards a theory of domination in graphs, Networks, **7,** no. 3, (1977), 247–261.

[4] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, 1979.

[5] V. R. Kulli, S. C. Sigarkanti, Inverse domination in graphs, Nat. Acad. Sci. Letters, **14,** (1991), 473–475.

[6] V. R. Kulli, B. Janakiram, The non-spilt domination number of a graph, Indian J. Pure., Appl. Math., **31,** no. 5, (2000), 545–550.

[7] V. Shalini, Indra Rajasingh, Domination and Total domination in Wrapped Butterfly Networks, Procedia Computer Science-Elsevier, **172,** (2020), 66–70.

[8] J. Xu., Topological Structure and Analysis of Interconnection Networks, 2001.