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Even Strongly Multiplicative Labeling in Certain Splitting Graphs

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Abstract

A graph $G = (V(G), E(G))$ with p vertices is said to be even strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, ..., p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct and even. In this paper we prove that the splitting graph of Path, Star and cycle graph are even strongly multiplicative.

1 Introduction

Graph labeling concerns the assigning of values, usually represented by integers, to the edges and/or vertices of a graph [1]. Graph labeling serves as a frontier between number theory and structure of graphs [2]. Splitting Graph was introduced by Sampathkumar and Walikar [4].

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Definition 1.1. *A graph $G = (V(G), E(G))$ with p vertices is said to be even strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, 3, ..., p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct and even.*

Definition 1.2. *A graph $G = (V(G), E(G))$ with p vertices for each vertex v of a graph G, take a new vertex $v^{'}$,then splitting graph of G, is obtained by join $v^{'}$ to all the vertices of G adjacent to v. The splitting graph G, is denoted by $S(G)$ [3].*

2 Main results

2.1 The Splitting Graph of Path Graph P_n

Theorem 2.1. *The Splitting Graph of a Path P_n is even strongly multiplicative for all $n \geq 2$.*

Proof.
Let the vertex set of P_n be $v_1, v_2, ..., v_n$,then the vertex set of splitting graph $S(P_n)$ is $V = \{v_i/1 \leq i \leq 2n\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$ where
$E_1 = \{e_i = (v_i, v_{i+1})\,/1 \leq i \leq n-1\}$,
$E_2 = \left\{e_i = \left(v_i^{'}, v_{i+1}\right)/1 \leq i \leq n-1\right\}$ and
$E_3 = \left\{e_i = \left(v_i^{'}, v_{i-1}\right)/2 \leq i \leq n\right\}$.
The labeling of vertices of splitting graph P_n is defined as $f : V \to N$ such that $f\,(v_i) = 2i,\ \ f\,(v_i^{'}) = 2i-1$ for $1 \leq i \leq n$.

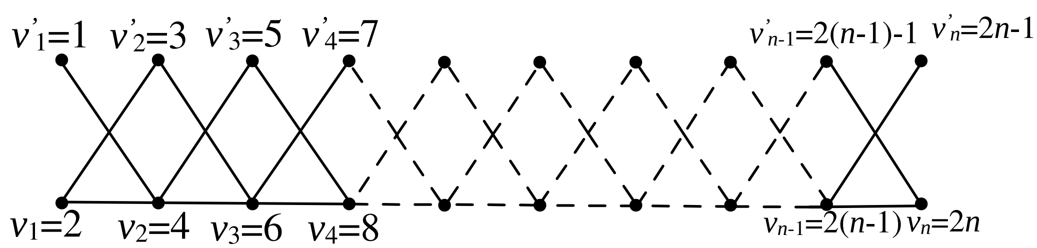

Figure 1: Splitting Graph of Path $S(P_n)$

We claim that all the labels induced on the edges, by the product of the labels of the end vertices in E are distinct and even.
To prove the edge labelings in E_1 are distinct and even:

Define an edge induced function $g : E_1 \rightarrow N$ such that for all $e_i \in E_1$,
$g(e_i) = f(v_i)f(v_{i+1}) = (2i)\ 2[i+1]\ , 1 \leq i \leq n-1$.
Therefore the edge labelings in E_1 are even.
If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 1 \leq i, p \leq n-1$. Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$$
$$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$$
$$(2i)2(i+1) = (2p)2(p+1),$$
$$\Rightarrow i = -1-p, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p)$, $\forall\ 1 \leq i, p \leq n-1$.
Thus all edge labelings in E_1 are distinct and even.
The edge labelings in E_2 and E_3 can be proved similarly to be distinct and even.
To prove that the labelings in edge set E_1 and E_2 are distinct and even:
If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 1 \leq i, p \leq n$. Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_p^{'}, v_{p+1})$$
$$f(v_i)f(v_{i+1}) = f(v_p^{'})f(v_{p+1})$$
$$(2i)2[i+1] = (2p-1)2(p+1)$$
$$i^2 + i = p^2 + \tfrac{p}{2} - \tfrac{1}{2}$$
$$\Longrightarrow i + \tfrac{1}{2} = \sqrt{(p+\tfrac{1}{4})^2 - \tfrac{5}{16}}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p)$, $\forall\ 1 \leq i, p \leq n$.
Thus all edge labelings in E_1 and in E_2 are distinct and even.
To prove that the labelings in edge set E_2 and E_3 are distinct and even:
If $e_i \in E_2$ and $e_p \in E_3$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 1 \leq i \leq n-1$ and $2 \leq p \leq n$. Assume that $g(e_i) = g(e_p)$

$$g(v_i^{'}, v_{i+1}) = g(v_p^{'}, v_{p-1})$$
$$f(v_i^{'})f(v_{i+1}) = f(v_p^{'})f(v_{p-1})$$
$$(2i-1)2[i+1] = (2p-1)2(p-1)$$
$$i^2 + \tfrac{i}{2} - \tfrac{1}{2} = p^2 - \tfrac{3p}{2} + \tfrac{1}{2}$$
$$\Longrightarrow i + \tfrac{1}{4} = \sqrt{p^2 - \tfrac{3}{2}p + \tfrac{17}{16}}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p)$, $\forall\ 1 \leq i \leq n-1$.
Thus all edge labelings in E_2 and in E_3 are distinct and even.
Therefore all the edge labeling of E are even and distinct.
Hence the splitting graph of Path is even strongly multiplicative for all $n \geq 2$.

2.2 Splitting Graph of Star Graph S_n

Theorem 2.2. *The Splitting graph of star graph S_n is even strongly multiplicative for all $n \geq 2$.*

Proof.
Let the vertex set of S_n be $v_1, v_2, ..., v_n$, then the vertex set of splitting graph $S(S_n)$ is $V = \{v_i/1 \leq i \leq 2n\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_i = (v_1, v_i)/2 \leq i \leq n\}, E_2 = \left\{e_i = (v_1^{'}, v_i)/2 \leq i \leq n\right\}$ and $E_3 = \left\{e_i = (v_1^{'}, v_i^{'})/2 \leq i \leq n\right\}$.
The labeling of vertices of splitting graph of S_n is defined as $f : V \to N$ such that

$$f\ (v_i) = \left\{ \begin{array}{c} 2n,\ i = 1 \\ 2(i-1), 2 \leq i \leq n \end{array} \right\},\ f\ (v_i^{'}) = \left\{ \begin{array}{c} 2n-1,\ i = 1 \\ 2(i-1)-1, 2 \leq i \leq n \end{array} \right\}$$

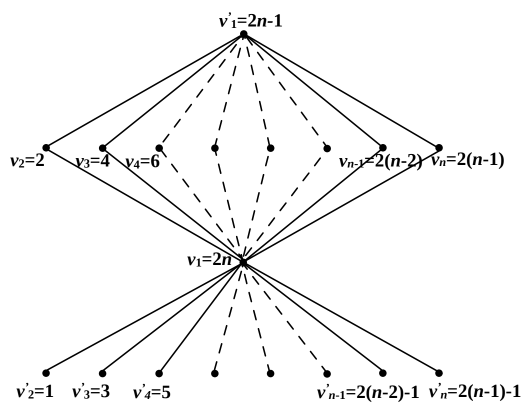

Figure 2: Splitting Graph of Star $S(S_n)$

We claim that all the labels induced on the edges, by the product of the labels of the end vertices in E are distinct and even.
To prove the edge labelings in E_1 are distinct and even:
Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,
$g(e_i) = f(v_1)f(v_i) = (2n)\ 2[i-1]\ , 2 \leq i \leq n.$
Therefore the edge labelings in E_1 is even.
If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 2 \leq i, p \leq n$. Assume that $g(e_i) = g(e_p)$

$$g(v_1, v_i) = g(v_1, v_p)$$
$$f(v_1)f(v_i) = f(v_1)f(v_p)$$
$$2n2(i-1) = 2n2(p-1) \Rightarrow i = p, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 2 \leq i, p \leq n$.
Thus all edge labelings in E_1 are distinct and even.
The edge labelings in E_2 and E_3 can be proved similarly to be distinct and even.
To prove that the labelings in edge set E_1 and E_2 are distinct and even:
If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 2 \leq i, p \leq n$. Assume that $g(e_i) = g(e_p)$

$$g(v_1, v_i) = g(v_1^{'}, v_p)$$
$$f(v_1)f(v_i) = f(v_1^{'})f(v_p)$$
$$2n2[i-1] = (2n-1)2(p-1)$$
$$\Longrightarrow i-1 = \frac{(2n-1)(p-1)}{2n}$$
, a contradiction for i.

Hence $g(e_i) \neq g(e_p), \forall\ 2 \leq i, p \leq n$.
Thus all edge labelings in E_1 and in E_2 are distinct and even.
To prove that the labelings in edge set E_2 and E_3 are distinct and even:
If $e_i \in E_2$ and $e_p \in E_3$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.
For $i \neq p, 2 \leq i, p \leq n$. Assume that $g(e_i) = g(e_p)$

$$g(v_1^{'}, v_i) = g(v_1, v_p^{'})$$
$$f(v_1^{'})f(v_i) = f(v_p^{'})f(v_1)$$
$$(2n-1)2[i-1] = (2p-3)2n$$
$$\Longrightarrow i-1 = \frac{(2p-3)n}{2n-1}$$
, a contradiction for i.

Hence $g(e_i) \neq g(e_p), \forall\ 2 \leq i, p \leq n$.
Thus all edge labelings in E_2 and in E_3 are distinct and even.
Therefore all the edge labelings of E are even and distinct.
Hence the splitting graph of Star S_n is even strongly multiplicative for all $n \geqslant 2$.

2.3 Splitting Graph of Cycle Graph C_n

Theorem 2.3. *The Splitting graph of cycle graph C_n is even strongly multiplicative for all $n \equiv 0 \pmod{16}$.*

Proof.
Let the vertex set of C_n be $v_1, v_2, ..., v_n$,then the vertex set of splitting graph $S(C_n)$ where $n \equiv 0 \pmod{16}$, is $V = \{v_i/1 \leq i \leq 2n\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$,where

$$E_1 = \{e_i = (v_i, v_{i+1}) \cup (v_1, v_n)/1 \leq i \leq n-1\},$$
$$E_2 = \left\{e_i = \left(v_i^{'}, v_{i+1}\right) \cup (v_n^{'}, v_1)/1 \leq i \leq n-1\right\} \text{ and}$$
$$E_3 = \left\{e_i = \left(v_1^{'}, v_{i-1}\right) \cup (v_1^{'}, v_n)/2 \leq i \leq n\right\}.$$

The labeling of vertices of splitting graph of C_n is defined as $f : V \to N$

such that $f\ (v_i) = 2i,\ f\ (v_i^{'}) = 2i - 1,\ i = 1, 2, ..., n.$

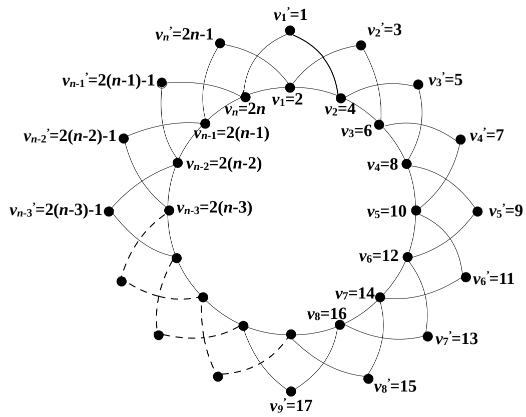

Figure 3: Splitting Graph of Cycle $S(C_n)$

We claim that all the labels induced on the edges, by the product of the labels of the end vertices in E are distinct and even.

To prove the edge labelings in E_1 are distinct and even:

Define an edge induced function $g : E_1 \to N$ such that for all $e_i \in E_1$,

$$g(e_i) = f(v_i)f(v_{i+1}) = 2i[2(i + 1)], 1 \le i \le n - 1.$$

Therefore the edge labelings in E_1 is even.

If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 1 \le i, p \le n - 1$, Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_p, v_{p+1})$$
$$f(v_i)f(v_{i+1}) = f(v_p)f(v_{p+1})$$
$$2i\ 2(i + 1) = 2p\ 2(p + 1)$$
$$i + p = -1 \Rightarrow i = -1 - p, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 1 \le i, p \le n - 1$.

If e_i and e_p are distinct edges in E_1 then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 1 \le i \le n - 1$ and $e_p = (v_n, v_1)$,

Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_n, v_1)$$
$$f(v_i)f(v_{i+1}) = f(v_n)f(v_1)$$
$$2i\ 2(i + 1) = 2n\ 2.1$$
$$i^2 + i - n = 0,\ i = \frac{-1 \pm \sqrt{1+4n}}{2}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 1 \le i \le n - 1$.

Thus all edge labelings in E_1 are distinct and even.

The edge labelings in E_2 and E_3 can be proved similarly to be distinct and even.

To prove that the labelings in edge set E_1 and E_2 are distinct and even:

Case 1: If $e_i \in E_1$ and $e_p \in E_2$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i, p \leq n-1$. Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_p^{'}, v_{p+1})$$
$$f(v_i)f(v_{i+1}) = f(v_p^{'})f(v_{p+1})$$
$$2i.2(i+1) = (2p-1)2(p+1)$$
$$(i+\tfrac{1}{2}) = \sqrt{(p+\tfrac{1}{4})^2 - \tfrac{5}{16}}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i \leq n-1$.

Case 2: If $e_i \in E_1$ and $e_p = (v_n^{'}, v_1)$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i \leq n-1$.

Assume that $g(e_i) = g(e_p)$

$$g(v_i, v_{i+1}) = g(v_n^{'}, v_1)$$
$$f(v_i)f(v_{i+1}) = f(v_n^{'})f(v_1)$$
$$2i.2(i+1) = (2n-1)2 \Rightarrow (i+\tfrac{1}{2}) = \sqrt{(n-\tfrac{1}{4})}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i \leq n-1$.

Case 3: If $e_i = (v_n, v_1)$ and $e_p = (v_n^{'}, v_1)$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. Assume that $g(e_i) = g(e_p)$

$$g(v_n, v_1) = g(v_n^{'}, v_1)$$
$$f(v_n)f(v_1) = f(v_n^{'})f(v_1)$$
$$2n.2 = (2n-1)2$$
$$n = (n-\tfrac{1}{2}), \text{ a contradiction for n.Hence } g(e_i) \neq g(e_p),$$

Thus all edge labelings in E_1 and in E_2 are distinct and even.

To prove that the labelings in edge set E_2 and E_3 are distinct and even:

Case 1: If $e_i \in E_2$ and $e_p \in E_3$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$. For $i \neq p, 1 \leq i \leq n-1, 2 \leq p \leq n$.

Assume that $g(e_i) = g(e_p)$

$$g(v_i^{'}, v_{i+1}) = g(v_p^{'}, v_{p-1})$$
$$f(v_i^{'})f(v_{i+1}) = f(v_p^{'})f(v_{p-1})$$
$$(2i-1).2(i+1) = (2p-1)2(p-1)$$
$$(i+\tfrac{1}{4}) = \sqrt{(p-\tfrac{3}{4})^2 + \tfrac{1}{2}}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p), \forall\ 1 \leq i \leq n-1$.

Case 2: If $e_i \in E_2$ and $e_p = (v_1^{'}, v_n)$ are distinct edges, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 1 \leq i \leq n-1$. Assume that $g(e_i) = g(e_p)$

$$g(v_i^{'}, v_{i+1}) = g(v_1^{'}, v_n)$$

$$f(v_i^{'})f(v_{i+1}) = f(v_1^{'})f(v_n)$$
$$(2i-1).2(i+1) = 1.2n$$
$$(i+\tfrac{1}{4}) = \sqrt{\tfrac{n}{2}+\tfrac{9}{16}}, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p)$, $\forall\ 1 \leq i \leq n-1$.
Case 3: If $e_i = (v_n^{'}, v_1)$ and $e_p = (v_1^{'}, v_n)$, then to prove $g(e_i) \neq g(e_p)$.
Assume that $g(e_i) = g(e_p)$

$$g(v_n^{'}, v_1) = g(v_1^{'}, v_n)$$
$$f(v_n^{'})f(v_1) = f(v_1^{'})f(v_n)$$
$$(2n-1).2 = (1)2n \Rightarrow 2n-1 = 2n, \text{ a contradiction for } i.$$

Hence $g(e_i) \neq g(e_p)$, when $e_i = (v_n^{'}, v_1)$.
Thus all edge labelings in E_2 and in E_3 are distinct and even.
Therefore all the edge labelings of E are even and distinct.
Hence the splitting graph of cycle even strongly multiplicative for all $n \equiv 0$ (mod 16).

3 Conclusion

In this paper we have proved certain splitting graph in path, star and cycle graphs are even strongly multiplicative. Finding even strongly multiplicative labeling for other interconnection networks are quite challenging.

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