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Disjoint Union of Two $SSG(2)$ is Odd Graceful

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Abstract

A subdivided shell graph is obtained by subdividing the edges in the path of the shell graph. Let G_1, G_2, G_3, …, G_n be 'n' subdivided shell graphs of any order. The graph $SSG(n)$ is obtained by adding an edge to apexes of G_i and G_{i+1}, i = 1, 2, …, $(n-1)$. The graph $SSG(n)$ is called a path union of $'n'$ subdivided shell graphs of any order. In this paper, the disjoint union of two $SSG(2)$ is proved to be odd graceful.

1 Introduction

Odd graceful labeling is a variation of Rosa's graceful labeling. In 1967, Rosa [9] introduced the labeling method called β-valuation which was renamed as graceful labeling by Golomb [7]. A graph which admits a graceful labeling is called a graceful graph. Odd graceful labeling was introduced by Gnanajothi [6] in the year 1991. A graph G with 'q' edges is said to be odd graceful if there is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ such that when

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each edge xy is assigned the label $\mid f(x) - f(y) \mid$, the resulting edge labels are$\{1, 3, 5, \ldots, (2q-1)\}$. Many graphs are proved to be odd graceful. Gnanajothi [6] proved that the graphs P_n, C_n (n - even), $K_{m,n}$, combs, books, the disjoint union of copies of C_4 are odd-graceful.

A *shell graph* [4] is a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. A *subdivided shell graph* is obtained by subdividing the edges in the path of the shell graph. Let G_1, G_2, G_3, $\ldots$, G_n be 'n' subdivided shell graphs of any order. The graph $SSG(n)$ is obtained by adding an edge to apexes of G_i and $G_{i+1}, i = 1, 2, \ldots, (n-1)$. The graph *SSG(n)* is called a path union of $'n'$ subdivided shell graphs of any order. For a good and an exhaustive survey on different graph labeling methods, applications and open problems, one may refer to the dynamic survey by Gallian [5].

Graceful labeling and its variations have many applications in various fields. These labelings are used in Communication network[8], Coding theory [2], X-ray crystallography [1] and in Network addressing [3], to mention a few.

2 Main Result

In this section, odd gracefulness of a shell related graph is proved.

Theorem 2.1. *The disjoint union of two copies of the graph $SSG(2)$ is odd graceful.*

Proof. Let G be the disjoint union of two copies of the graph $SSG(2)$ with n vertices and q edges. The number of vertices present in the subdivided path is denoted as $'m'$. Let G_{11} denote the right hand side subdivided shell in the first copy. Let t_0 denote the apex of the subdivided shell. The vertices, from bottom to top, in the path of G_{11} are $t_1, t_2, \ldots, t_m$. Let G_{12} denote the left hand side subdivided shell in the first copy. The apex is denoted as w_0. The vertices, from bottom to top, in the the path of G_{12} are $w_1, w_2, \ldots, w_m$.

Let G_{21} denote the right hand side subdivided shell in the second copy. Its apex is denoted by u_0. The vertices, from bottom to top, in the the path of G_{21} are $u_1, u_2, \ldots, u_m$. Let G_{22} denote the left hand side subdivided shell in the second copy. Its apex is denoted by v_0. The vertices, from bottom to top, in the the path of G_{22} are $v_1, v_2, \ldots, v_m$. The graph G has n =$(4m + 4)$

vertices and $q = 6m$ edges.

The vertex labeling of the graph $G, f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ is defined as follows. Define the vertices of G_{11} as $f(t_0) = 0, f(t_{2i-1}) = (3m + 2i - 2), for\ 1 \leq i \leq \frac{(m+1)}{2}, f(t_{2i}) = (6m - 2i), for\ 1 \leq i \leq \frac{(m-1)}{2}$.

The vertices of G_{12} are labeled as follows.$f(w_0) = (2q-1), f(w_{2i-1}) = 2i, for\ 1 \leq i \leq \frac{(m+1)}{2}, f(w_{2i}) = 11m - 2i, for\ 1 \leq i \leq \frac{(m-1)}{2}$.

The vertices of G_{21} are labeled as follows. $f(u_0) = 1, f(u_{2i-1}) = (6m + 2i - 2), for\ 1 \leq i \leq \frac{(m+1)}{2}, f(u_{2i}) = (2m - 2i + 1), for\ 1 \leq i \leq \frac{(m-1)}{2}$.

The vertices of G_{22} are labeled as follows. $f(v_0) = (9m - 1), f(v_{2i-1}) = (9m + 2i - 2), for\ 1 \leq i \leq \frac{(m+1)}{2}, f(v_{2i}) = (6m - 2i + 2), for\ 1 \leq i \leq \frac{(m-1)}{2}$.

From the above equations, it is clear that all vertex labels are distinct. For, suppose that, $f(w_{2i-1}) = f(u_{2i})$, then from the above equations we get $2 \leq 1$, which is absurd. Hence no two vertex labels are equal. Also one can note that $f(v) \in \{0, 1, 2, \ldots (2q-1)\}$ for all $v \in V(G)$.

The edge labels in the first copy of $SSG(2)$ are computed as follows:

$$
\begin{aligned}
&| f(t_0) - f(t_{2i-1}) | = | 3m + 2i - 2 |, \quad for\ \ 1 \leq i \leq \tfrac{(m+1)}{2} \\
&| f(t_{2i-1}) - f(t_{2i}) | = | 3m - 4i + 2 |, \quad for\ \ 1 \leq i \leq \tfrac{(m-1)}{2} \\
&| f(t_{2i}) - f(t_{2i+1}) | = | 3m - 4i |, \quad for\ \ 1 \leq i \leq \tfrac{(m-1)}{2} \\
&| f(t_0) - f(w_0) | = | 2q - 1 | \\
&| f(w_0) - f(w_{2i-1}) | = | 2q - 1 - 2i |, \quad for\ \ 1 \leq i \leq \tfrac{(m+1)}{2} \\
&| f(w_{2i-1}) - f(w_{2i}) | = | 11m - 4i |, \quad for\ \ 1 \leq i \leq \tfrac{(m-1)}{2} \\
&| f(w_{2i}) - f(w_{2i+1}) | = | 11m - 4i - 2 |, \quad for\ \ 1 \leq i \leq \tfrac{(m-1)}{2}
\end{aligned}
\tag{2.1}
$$

The edge labels in the second copy of $SSG(2)$ are computed as follows.

$$\begin{aligned}
&| f(u_0) - f(u_{2i-1}) |=| 6m + 2i - 3 |, \quad for \;\; 1 \le i \le \tfrac{(m+1)}{2} \\
&| f(u_{2i-1}) - f(u_{2i}) |=| 4m + 4i - 3 |, \quad for \;\; 1 \le i \le \tfrac{(m-1)}{2} \\
&| f(u_{2i}) - f(u_{2i+1}) |=| 4m + 4i - 1 |, \quad for \;\; 1 \le i \le \tfrac{(m-1)}{2} \\
&| f(u_0) - f(v_0) |=| 9m - 1 | \\
&| f(v_0) - f(v_{2i-1}) |=| 2i - 1 |, \quad for \;\; 1 \le i \le \tfrac{(m+1)}{2} \\
&| f(v_{2i-1}) - f(v_{2i}) |=| 7m + 4i - 4 |, \quad for \;\; 1 \le i \le \tfrac{(m-1)}{2} \\
&| f(v_{2i}) - f(v_{2i+1}) |=| 7m + 4i - 2 |, \quad for \;\; 1 \le i \le \tfrac{(m-1)}{2}
\end{aligned} \tag{2.2}$$

From the above equations , it is clear that all the edge labels are distinct and odd integers from 1 to $(2q - 1)$. Both vertex labels and edge labels satisfy the required conditions of odd graceful labeling. This proves that the labeling defined in the graph G is odd graceful. Hence the disjoint union of two $SSG(2)$ is odd graceful.

□

An illustration for the above theorem is given in Figure 1, when $m = 9$, $n = 40$, $q = 54$, $2q - 1 = 107$.

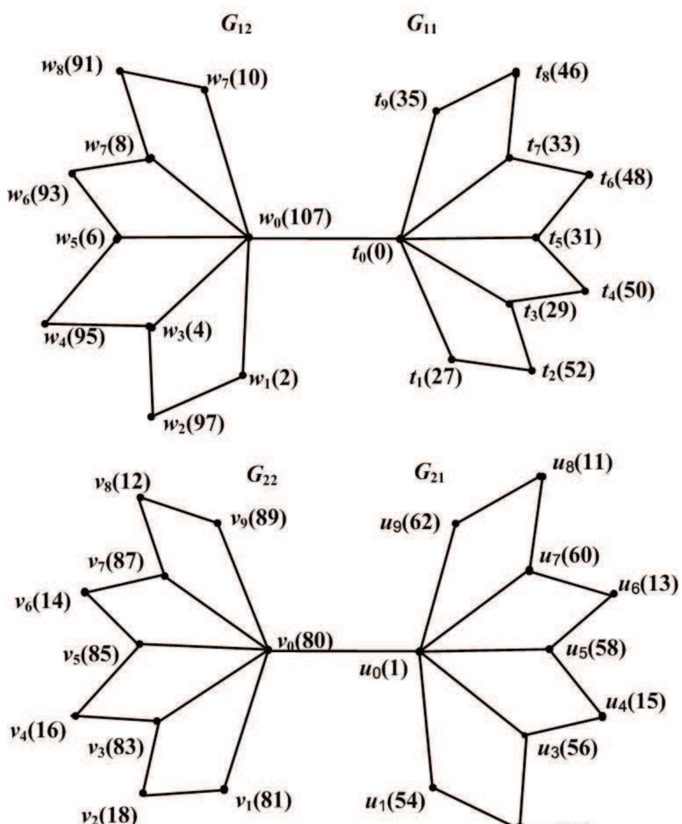

Figure 1: Odd graceful disjoint union of two SSG(2), $n = 40$, $q = 54$

3 Conclusion

In this paper, we have proved that the disjoint union of two $SSG(2)$ is odd graceful. One can try other type of labelings such as one modulo three graceful labeling, k - graceful labeling, harmonious labeling, elegant labeling on this graph. Also, one can find out whether graph labelings can be applied on the disjoint union of two SSG(n) when $n \geq 3$.

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