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$\mu\mathscr{I}^\star$-closed sets in ideal strong generalized topological spaces

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Abstract

This paper deals with the concept of $\mu\mathscr{I}^\star$-closed sets in ideal strong generalized topological spaces. Some properties of $\mu\mathscr{I}^\star$-closed sets and $\mu\mathscr{I}^\star$-open sets are investigated.

1 Introduction

General topology plays an important role in both pure and applied sciences such as computer science and mathematical sciences. The theory of generalized topological spaces, which was founded by Császár [1], is one of the most important development of general topology in recent years. In particular, the author defined some basic operators on generalized topological spaces. A large number of papers were devoted to the study of generalized open like sets of a topological space containing the class of open sets and possessing properties more or less similar to those of open sets. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [8] and Vaidyanathaswamy [12]. Janković and Hamlett [7] further studied ideal topological spaces and their applications to various fields. In 1970, Levine

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[9] introduced the concept of generalized closed sets. This concept has been studied extensively in recent years by many topologists. Dontchev, et al. [6] introduced the notion of $\mathscr{I}$-g-closed sets. In [10], the authors called $\mathscr{I}$-g-closed sets $\mathscr{I}_g$-closed and investigated further properties of $\mathscr{I}$-g-closed sets. Noiri and Popa [11] introduced the notion of $\mathscr{I}$-mg-closed sets as generalizations of $\mathscr{I}$-g-closed sets in an ideal topological space. In this paper, we introduce the notion of $\mu\mathscr{I}^\star$-closed sets in ideal strong generalized topological spaces. Moreover, some properties of $\mu\mathscr{I}^\star$-closed sets and $\mu\mathscr{I}^\star$-open sets are investigated.

2 Preliminaries

Let X be a nonempty set and denote by $\mathscr{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathscr{P}(X)$ a *generalized topology* (briefly, GT) on X if $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ [2]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called *μ-open* sets and the complements of μ-open sets are called *μ-closed* sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A [4]. Then $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(A) = X - c_\mu(X - A)$. According to [5], for $A \subseteq X$ and $x \in X$, $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$. Let μ be a GT on a set $X \neq \emptyset$. A GT μ is called *strong* [3] if $X \in \mu$. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $A \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$.

3 Properties of $\mu\mathscr{I}^\star$-closed sets

In this section, we introduce the concept of ideal strong generalized topological spaces. Moreover, we introduce the notions of $\mu\mathscr{I}^\star$-closed sets and $\mu\mathscr{I}^\star$-open sets in ideal strong generalized topological spaces. Furthermore, some properties of $\mu\mathscr{I}^\star$-closed sets and $\mu\mathscr{I}^\star$-open sets are discussed.

Let (X, μ) be a strong generalized topological space and

$$\mu(x) = \{U \mid x \in U, U \in \mu\}$$

be the family of μ-open sets containing x.

Definition 3.1. Let (X,μ) be a strong generalized topological space with an ideal $\mathscr{I}$ on X and $\mathscr{P}(X)$ be the set of all subsets of X. A set operator

$$\mu(.)^\star : \mathscr{P}(X) \to \mathscr{P}(X),$$

called a *strong generalized local function* of A with respect to μ and $\mathscr{I}$ is defined as follows: for $A \subseteq X$, $\mu A^\star(\mathscr{I},\mu) = \{x \in X \mid U \cap A \notin \mathscr{I}$ for every $U \in \mu(x)\}$. In case there is no chance for confusion, $\mu A^\star(\mathscr{I},\mu)$ is simply written as $\mu A^\star$.

Proposition 3.2. *Let (X,μ) be a strong generalized topological space with $\mathscr{I},\mathscr{I}'$ ideals on X and A, B be subsets of X. Then the following properties hold:*

(1) If $A \subseteq B$, then $\mu A^\star \subseteq \mu B^\star$.

(2) If $\mathscr{I} \subseteq \mathscr{I}'$, then $\mu A^\star(\mathscr{I}') \subseteq \mu A^\star(\mathscr{I})$.

(3) $\mu A^\star = c_\mu(\mu A^\star) \subseteq c_\mu(A)$.

(4) $\mu A^\star \cup \mu B^\star \subseteq \mu(A \cup B)^\star$.

(5) $\mu(A \cap B)^\star \subseteq \mu A^\star \cap \mu B^\star$.

(6) $\mu(\mu A^\star)^\star \subseteq \mu A^\star$.

Proof. (1) Let $A \subseteq B$ and $x \in \mu A^\star$. Suppose that $x \notin \mu B^\star$. We have $U \cap B \in \mathscr{I}$ for some $U \in \mu(x)$. Since $U \cap A \subseteq U \cap B$ and $U \cap B \in \mathscr{I}$, $U \cap A \in \mathscr{I}$. Thus, $x \notin \mu A^\star$. This is a contradiction. Therefore, $\mu A^\star \subseteq \mu B^\star$.

(2) Let $\mathscr{I} \subseteq \mathscr{I}'$ and $x \in \mu A^\star(\mathscr{I}')$. Then $U \cap A \notin \mathscr{I}'$ for every $U \in \mu(x)$. By the hypothesis, we have $U \cap A \notin \mathscr{I}$ and hence $x \in \mu A^\star(\mathscr{I})$.

The other proofs are obvious. □

Definition 3.3. Let (X,μ) be a strong generalized topological space with an ideal $\mathscr{I}$ on X. The set operator $c_{\mu(\star)}$ is called the $\mu(\star)$-*closure* and is defined as $c_{\mu(\star)}(A) = A \cup \mu A^\star$ for $A \subseteq X$. We will denote by $\mu(\star)(\mathscr{I},\mu)$ the strong generalized topology generated by $c_{\mu(\star)}$, that is, $\mu(\star)(\mathscr{I},\mu) = \{U \subseteq X \mid c_{\mu(\star)}(X - U) = X - U\}$. $\mu(\star)(\mathscr{I},\mu)$ is called $\mu(\star)$ *strong generalized topology* which is finer than μ.

The elements of $\mu(\star)(\mathscr{I},\mu)$ are called $\mu(\star)$-*open* and the complement of a $\mu(\star)$-open set is called $\mu(\star)$-*closed*. We will simply write $\mu(\star)$ for $\mu(\star)(\mathscr{I},\mu)$. Also, $i_{\mu(\star)}(A)$ denotes the interior of A in $\mu(\star)$.

Definition 3.4. A strong generalized topological space (X, μ) with an ideal $\mathscr{I}$ on X is called an *ideal strong generalized topological space* and is denoted by $(X, \mu, \mathscr{I})$.

Proposition 3.5. *Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space and $A, B \subseteq X$. Then the following properties hold:*

(1) $A \subseteq c_{\mu(\star)}(A)$.

(2) If $A \subseteq B$, then $c_{\mu(\star)}(A) \subseteq c_{\mu(\star)}(B)$.

(3) $c_{\mu(\star)}(A) \cup c_{\mu(\star)}(B) \subseteq c_{\mu(\star)}(A \cup B)$.

Definition 3.6. Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space. A subset A of X is said to be *$\mu\mathscr{I}^\star$-closed* if $\mu A^\star \subseteq U$ whenever $A \subseteq U$ and U is μ-open.

Proposition 3.7. *Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space and $A, B \subseteq X$. If A is $\mu\mathscr{I}^\star$-closed and B is μ-closed, then $A \cap B$ is $\mu\mathscr{I}^\star$-closed.*

Proof. Let $A \cap B \subseteq U$ and $U \in \mu$. Then $A \subseteq U \cup (X - B)$. Since $U \cup (X - B)$ is μ-open, $\mu A^\star \subseteq U \cup (X - B)$ and hence $\mu A^\star \cap B \subseteq U \cap B \subseteq U$. By Proposition 3.2, $\mu(A \cap B)^\star \subseteq \mu A^\star \cap \mu B^\star$. Since $\mu \subseteq \mu(\star)$, B is $\mu(\star)$-closed and $\mu B^\star \subseteq B$. Therefore, we obtain $\mu(A \cap B)^\star \subseteq A^\star_\mu \cap B \subseteq U$. This shows that $A \cap B$ is $\mu\mathscr{I}^\star$-closed. □

Proposition 3.8. *Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space and $A, B \subseteq X$. If A is $\mu\mathscr{I}^\star$-closed and $A \subseteq B \subseteq c_{\mu(\star)}(A)$, then B is $\mu\mathscr{I}^\star$-closed.*

Proof. Let $B \subseteq U$ and $U \in \mu$. Then $A \subseteq U$ and A is $\mu\mathscr{I}^\star$-closed. Hence $\mu A^\star \subseteq U$ and $\mu B^\star \subseteq c_{\mu(\star)}(B) \subseteq c_{\mu(\star)}(c_{\mu(\star)}(A)) = c_{\mu(\star)}(A) = A \cup \mu A^\star \subseteq U$. Thus, B is $\mu\mathscr{I}^\star$-closed. □

Definition 3.9. Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space. A subset A of X is said to be *$\mu\mathscr{I}^\star$-open* if $X - A$ is $\mu\mathscr{I}^\star$-closed.

Theorem 3.10. *A subset A of an ideal strong generalized topological space $(X, \tau, \mathscr{I})$ is $\mu\mathscr{I}^\star$-open if and only if $F \subseteq i_{\mu(\star)}(A)$ whenever $F \subseteq A$ and F is μ-closed.*

Proof. Suppose that A is $\mu\mathscr{I}^\star$-open. Let $F \subseteq A$ and F be μ-closed. Then $X - A \subseteq X - F \in \mu$. Since $X - F \in \mu$ and $X - A$ is $\mu\mathscr{I}^\star$-closed, we have $\mu(X-A)^\star \subseteq X-F$ and $X-i_{\mu(\star)}(A) = c_{\mu(\star)}(X-A) = \mu(X-A)^\star \cup (X-A) \subseteq X - F$. Consequently, $F \subseteq i_{\mu(\star)}(A)$.

Conversely, let $X - A \subseteq U$ and $U \in \mu$. Then $X - U \subseteq A$ and $X - U$ is μ-closed. By hypothesis, we have $X - U \subseteq i_{\mu(\star)}(A)$ and hence $\mu(X - A)^\star \subseteq c_{\mu(\star)}(X - A) = X - i_{\mu(\star)}(A) \subseteq U$. Thus, $X - A$ is $\mu\mathscr{I}^\star$-closed and hence A is $\mu\mathscr{I}^\star$-open. □

Corollary 3.11. *Let $(X, \mu, \mathscr{I})$ be an ideal strong generalized topological space and $A, B \subseteq X$. Then the following properties hold:*

(1) Every μ-open set is $\mu\mathscr{I}^\star$-open.

(2) If A is $\mu\mathscr{I}^\star$-open and B is μ-open, then $A \cup B$ is $\mu\mathscr{I}^\star$-open.

(3) If A is $\mu\mathscr{I}^\star$-open and $i_{\mu(\star)}(A) \subseteq B \subseteq A$, then B is $\mu\mathscr{I}^\star$-open.

Proof. This follows from Propositions 3.7 and 3.8. □

Theorem 3.12. *For a subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mu\mathscr{I}^\star$-closed;

(2) $c_{\mu(\star)}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$;

(3) $c_{\mu(\star)}(A) \cap F = \emptyset$ whenever $A \cap F = \emptyset$ and F is μ-closed.

Proof. (1) ⇒ (2): Let $A \subseteq U$ and $U \in \mu$. By (1), we have $\mu A^\star \subseteq U$ and hence $c_{\mu(\star)}(A) = \mu A^\star \cup A \subseteq U$.

(2) ⇒ (3): Let $A \cap F = \emptyset$ and F be μ-closed. Then $A \subseteq X - F \in \mu$ and by (2), we have $c_{\mu(\star)}(A) \subseteq X - F$. Thus, $c_{\mu(\star)}(A) \cap F = \emptyset$.

(3) ⇒ (1): Let $A \subseteq U$ and $U \in \mu$. Then $A \cap (X - U) = \emptyset$ and $X - U$ is μ-closed. By (3), $c_{\mu(\star)}(A) \cap (X - U) = \emptyset$ and hence $\mu A^\star \subseteq c_{\mu(\star)}(A) \subseteq U$. Therefore, A is $\mu\mathscr{I}^\star$-closed. □

Definition 3.13. Let (X, μ) be a strong generalized topological space and A be a subset of X. The subset $\varsigma_\mu(A)$ is defined as follows: $\varsigma_\mu(A) = \cap\{U \mid A \subseteq U, U \in \mu\}$.

Theorem 3.14. *A subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$ is $\mu\mathscr{I}^\star$-closed if and only if $c_{\mu(\star)}(A) \subseteq \varsigma_\mu(A)$.*

Proof. Suppose that A is $\mu\mathscr{I}^\star$-closed. If $x \notin \varsigma_\mu(A)$, then there exists $U \in \mu$ such that $A \subset U$ and $x \notin U$. Since A is $\mu\mathscr{I}^\star$-closed, by Theorem 3.12, $c_{\mu(\star)}(A) \subseteq U$ and hence $x \notin c_{\mu(\star)}(A)$. Consequently, $c_{\mu(\star)}(A) \subseteq \varsigma_\mu(A)$.

Conversely, suppose that $c_{\mu(\star)}(A) \subseteq \varsigma_\mu(A)$. Let $A \subseteq U$ and $U \in \mu$. Then $c_{\mu(\star)}(A) \subseteq \varsigma_\mu(A) \subseteq U$. Thus, by Theorem 3.12, A is $\mu\mathscr{I}^\star$-closed. □

Theorem 3.15. *A subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$ is $\mu\mathscr{I}^\star$-closed if and only if $c_\mu(\{x\}) \cap A \neq \emptyset$ for each $x \in c_{\mu(\star)}(A)$.*

Proof. Suppose that $\mu\mathscr{I}^\star$-closed and $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in c_{\mu(\star)}(A)$. Then $A \subseteq X - c_\mu(\{x\})$. Since $X - c_\mu(\{x\})$ is μ-open and A is $\mu\mathscr{I}^\star$-closed, by Theorem 3.12, $c_{\mu(\star)}(A) \subseteq X - c_\mu(\{x\}) \subseteq X - \{x\}$. This contradicts the fact that $x \in c_{\mu(\star)}(A)$. Hence, $c_\mu(\{x\}) \cap A \neq \emptyset$ for each $x \in c_{\mu(\star)}(A)$.

Conversely, suppose that A is not $\mu\mathscr{I}^\star$-closed. By Theorem 3.12, we have $\emptyset \neq c_{\mu(\star)}(A) - U$ for some $U \in \mu$ containing A. There exists $x \in c_{\mu(\star)}(A) - U$. Since $x \notin U$, $c_\mu(\{x\}) \cap U = \emptyset$ and hence $c_\mu(\{x\}) \cap A \subseteq c_\mu(\{x\}) \cap U = \emptyset$. This shows that $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in c_{\mu(\star)}(A)$. □

Theorem 3.16. *For a subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mu\mathscr{I}^\star$-closed;

(2) $\mu A^\star - A$ does not contain any nonempty μ-closed set;

(3) $\mu A^\star - A$ is $\mu\mathscr{I}^\star$-open;

(4) $A \cup (X - \mu A^\star)$ is $\mu\mathscr{I}^\star$-closed;

(5) $c_{\mu(\star)}(A) - A$ does not contain any nonempty μ-closed set.

Proof. (1) ⇒ (2): Suppose that A is $\mu\mathscr{I}^\star$-closed. Let $F \subseteq \mu A^\star - A$ and F be μ-closed. Then $A \subseteq X - F \in \mu$ and hence $\mu A^\star \subseteq X - F$. Therefore, $F \subseteq X - \mu A^\star$ and hence $F \subseteq \mu A^\star \cap (X - \mu A^\star) = \emptyset$.

(2) ⇒ (3): Let $F \subseteq \mu A^\star - A$ and F be μ-closed. By (2), we have $F = \emptyset$ and $F \subseteq i_{\mu(\star)}(\mu A^\star - A)$. It follows from Theorem 3.10 that $\mu A^\star - A$ is $\mu\mathscr{I}^\star$-open.

(3) ⇒ (1): Let $A \subseteq U$ and $U \in \mu$. Then $\mu A^\star \cap (X - U) \subseteq \mu A^\star - A$ and by (3), $\mu A^\star - A$ is $\mu\mathscr{I}^\star$-open. By Proposition 3.2, $\mu A^\star$ is μ-closed and hence $\mu A^\star \cap (X - U)$ is μ-closed. Since $\mu A^\star - A$ is $\mu\mathscr{I}^\star$-open, by Theorem 3.10,

$$\mu A^\star \cap (X - U) \subseteq i_{\mu(\star)}(\mu A^\star - A).$$

Now, we have $i_{\mu(\star)}(\mu A^\star - A) = i_{\mu(\star)}(\mu A^\star \cap (X - A)) \subseteq \mu A^\star \cap (X - c_{\mu(\star)}(A)) = \emptyset$. Therefore, $\mu A^\star \cap (X - U) = \emptyset$ and hence $\mu A^\star \subseteq U$. This shows that A is $\mu\mathscr{I}^\star$-closed.

(3) $\Leftrightarrow$ (4): This follows from the fact that $X - (\mu A^\star - A) = (X - \mu A^\star) \cup A$.

(2) $\Leftrightarrow$ (5): This is obvious by the fact that $c_{\mu(\star)}(A) - A = (\mu A^\star \cup A) \cap (X - A) = (\mu A^\star \cap (X - A)) \cup (A \cap (X - A)) = \mu A^\star - A$. □

Theorem 3.17. *A subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$ is $\mu\mathscr{I}^\star$-closed if and only if $A = F - N$, where F is $\mu(\star)$-closed and N contains no nonempty μ-closed set.*

Proof. Suppose that A is $\mu\mathscr{I}^\star$-closed. Let $F = c_{\mu(\star)}(A)$ and $N = \mu A^\star - A$. Then F is $\mu(\star)$-closed and, by Theorem 3.16, N contains no nonempty μ-closed set. Moreover, we have $F - N = (\mu A^\star \cup A) - (\mu A^\star - A) = (\mu A^\star \cup A) \cap (X - (\mu A^\star - A)) = (\mu A^\star \cup A) \cap ((X - \mu A^\star) \cup A) = (\mu A^\star \cap (X - \mu A^\star)) \cup A = A$.

Conversely, suppose that $A = F - N$, where F is $\mu(\star)$-closed and N contains no nonempty μ-closed set. Let $A \subseteq U$ and $U \in \mu$. Then $F - N \subseteq U$ and $F \cap (X - U) \subseteq F \cap (X - (F - N)) = F \cap ((X - F) \cup N) = F \cap N \subseteq N$. By Proposition 3.2, $\mu A^\star$ is μ-closed and hence $\mu A^\star \cap (X - U)$ is μ-closed. Since $A \subseteq F$ and $\mu F^\star \subseteq c_{\mu(\star)}(F) = F$, $\mu A^\star \cap (X - U) \subseteq \mu F^\star \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. Therefore, $\mu A^\star \cap (X - U) = \emptyset$ and hence $\mu A^\star \subseteq U$. Thus, A is $\mu\mathscr{I}^\star$-closed. □

Corollary 3.18. *For a subset A of an ideal strong generalized topological space $(X, \mu, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mu\mathscr{I}^\star$-closed;

(2) $A - i_{\mu(\star)}(A)$ does not contain any nonempty μ-closed set;

(3) $c_\mu(\{x\}) \cap (X - A) \neq \emptyset$ for each $x \in X - i_{\mu(\star)}(A)$.

Proof. This follows from Theorems 3.15 and 3.16. □

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