International Journal of Mathematics and
Computer Science, **15**(2020), no. 4, 1091–1097

e-Lucky Labeling of Certain Graphs

Tony Augustine, S. Roy

Department of Mathematics
Vellore Institute of Technology
Vellore, India

email: pullikkattiltony@gmail.com

(Received: July 9, 2020, Accepted September 2, 2020)

Abstract

Let G be a connected graph and k be an integer, $k \geq 1$. If $l : V(G) \rightarrow N$ is a vertex labeling of a graph G, then e-lucky sum of a vertex $u \in V(G)$ is $el(u) = \sum_{v \in N(u)} l(v) + e(u)$, where $e(u)$ denotes the eccentricity of u and $N(u)$ denotes the open neighborhood of u. The labeling is e-lucky labeling if $el(u) \neq el(v)$, for every $uv \in E(G)$. The e-lucky number $\eta_{el}(G)$ of G is the smallest integer k for which G has e-lucky labeling.
In this paper, we compute the e-lucky number for book graph, butterfly network and complete graph. Also, we give the sharp bound for any regular bipartite self-centered graphs.

1 Introduction

An assignment of integers to the vertices or edges or both called a graph labeling. The study of graph labeling is an important research area in graph theory. Graph labeling has applications in the Channel assignment process, Social networks, and Database management.
Miller et al. [1] introduced the concept of d-lucky labeling and the d-lucky number for various graphs were obtained in [2] and [3]. Thus, incorporating the concept of d-lucky labeling, we introduce a new labeling called e-lucky labeling.

Key words and phrases: Lucky Labeling, e-Lucky Labeling, Book Graph, Butterfly Network, Complete Graph.
AMS (MOS) Subject Classifications: 05C15, 05C78.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

Definition 1.1. *If $l : V(G) \to N$ is a vertex labeling of a graph G, then the e-lucky sum of a vertex $u \in V(G)$ is $el(u) = \sum_{vN(u)} l(v) + e(u)$, where $e(u)$ denotes the eccentricity of u and $N(u)$ denotes the open neighborhood of u. The labeling is e-lucky labeling if $el(u) \neq el(v)$ for every $uv \in E(G)$. The e-lucky number $\eta_{el}(G)$ of G is the smallest integer k for which G has e-lucky labeling $V(G) \to [k] = \{1, 2, ..., k\}$.*

Also, $\eta_{el}(G) \geq \eta_{el}(H)$, for any subgraph H of G. By neglecting the additive term $e(u)$ in e-lucky labeling, we obtain lucky labeling. In the following section, we determine the e-lucky number for book graph, butterfly network and complete graph.

2 Main results

Definition 2.1. *A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge.*

Theorem 2.2. *If $uv \in E(G)$ and $N[u] = N[v]$, then $l(u) \neq l(v)$ in e-lucky labeling.*

Proof.
On the contrary, assume that $l(u) = l(v)$. Then $N[u] = N[v]$ implies $d(u) = d(v)$ and $e(u) = e(v)$. Therefore, $el(u) = \sum_{x \in N(u)} l(x) + e(u) = \sum_{x \in N(v)} l(x) + e(v) = el(v)$, which is a contradiction.

Theorem 2.3. *For a complete graph $K_n, \eta_{el}(K_n) = n$.*

Proof.
This result follows directly from Theorem 2.2.

Theorem 2.4. *If $G = K_n^*, n \geq 3$ is a connected graph which is obtained from two copies of K_n of the same size by joining any two vertices of K_n, then $\eta_{el}(G) = n$.*

Proof.
Since K_n is a subgraph of G, $\eta_{el}(G) \geq n$.
The graph G consists of two copies of K_n, namely $K_n^{'}$ and $K_n^{''}$ (See Figure 1). The vertices of degree n in $K_n^{'}$ and $K_n^{''}$ are labeled as u_1 and v_1 respectively and the remaining vertices of $K_n^{'}$ and $K_n^{''}$ are labeled as $u_i, i = 2, 3, ..., n$ and $v_i, i = 2, 3, ..., n$ respectively. We define $l : (V(G)) \to \{1, 2, ..., N\}$ by $l(u_i) = i$ for $2 \leq i \leq n$; $l(v_i) = i - 1$ for $2 \leq i \leq n$; $l(u_1) = 1$ and $l(v_1) = n$.

From the labeling defined above, $el(u_1) = \frac{n^2+3n}{2} + 1 \neq \frac{n^2-n}{3} + 3 = el(v_1)$. For $i = 2, 3, ..., n$, u_i and v_i are not adjacent and though u_i $'s$ are adjacent to each other in $K_n^{'}$ and v_i $'s$ are adjacent to each other in $K_n^{''}$ and since the labeling of u_i and v_i are different, we have $\eta_{el}(G) \leq n$. Thus $\eta_{el}(G) = n$.

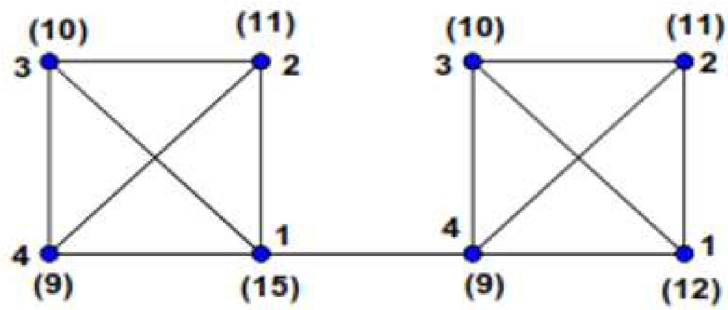

Figure 1: e-lucky labeling of K_4^*

3 Butterfly network

Definition 3.1. *The n-dimensional butterfly network, denoted by $BF(n)$, has a vertex set $V = \{(x, i) : x \in V(Q_n), 0 \leq i \leq n\}$. Two vertices (x, i) and (y, j) are linked by an edge in $BF(n)$ if and only if $j = i + 1$ and either*
(i) $x = y$, or
(ii) x differs from y in precisely the j^{th} bit. For $x = y$, the edge is said to be a straight edge. Otherwise, the edge is a cross edge. For fixed i, the vertex (x, i) is a vertex on level i [4].

Lemma 3.2. *For $BF(1)$,$\eta_{el}(BF(1)) = 2$.*

Proof.
Let $V(BF(1))$ be v_1, v_2, v_3 and v_4. We know that $e(v_i) = 2$, for all i. On the contrary, assume that $\eta_{el}(BF(1)) = 1$. Then, by our assumption, $l(v_i) = 1$, for all $i = 1, ..., 4$ and we get $el(v_i) = el(v_j)$ for every $v_iv_j \in E(BF(1))$. But this is a contradiction to the definition of e-lucky labeling and hence $\eta_{el}(BF(1)) = 2$.

Lemma 3.3. *For $BF(2)$, $\eta_{el}(BF(2)) = 2$.*

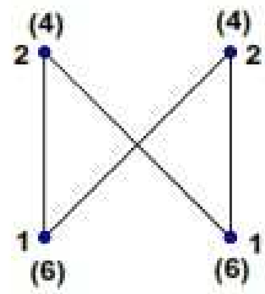

Figure 2: e-lucky labeling of $BF(1)$

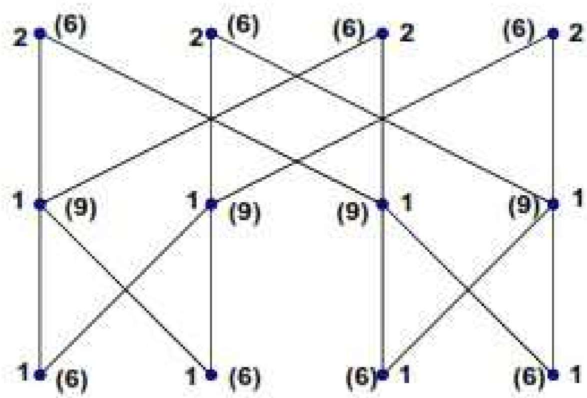

Figure 3: e-lucky labeling of $BF(2)$

Proof.
Since $BF(1)$ is a subgraph of $BF(2)$, $\eta_{el}(BF(2)) \geq 2$. From Figure 3, we get, $\eta_{el}(BF(2)) \leq 2$. Therefore $\eta_{el}(BF(2)) = 2$.

Theorem 3.4. *For $BF(n)$, $n \geq 3$, $\eta_{el}(BF(n)) = 2$.*

Proof.
Since $BF(2)$ is a subgraph of $BF(n)$, $n \geq 3$, $\eta_{el}(BF(n)) \geq 2$.
We give an algorithm to show that the upper bound for $BF(n), n \geq 3$ is 2.
Procedure: Graph $BF(n)$ consists of two copies of $BF(n-1)$, namely $BF'(n-1)$ and $BF''(n-1)$. See Figure 4.
Input: Butterfly network $BF(n), n \geq 3$.
Step 1: Copy the labeling of $BF(n-1)$ to $BF'(n-1)$ and $BF''(n-1)$ in

$BF(n), n \geq 3$.
Strp 2: In level n, label the first 2^{n-1} vertices of $BF(n)$ by 2 and the remaining 2^{n-1} vertices of $BF(n)$ by 1.
Output: $\eta_{el}(BF(n)) = 2, n \geq 3$.
Proof: As we already know that $BF(n-1)$, $2 \leq n \leq 3$ has e-lucky labeling, labeling $BF'(n-1)$ and $BF''(n-1)$ as in $BF(n-1)$, implies that the vertices from level 0 to $(n-1)$ allow e-lucky labeling in $BF(n)$, $n \geq 3$.
Since each vertex in level n has one end in $BF'(n-1)$ and another end in $BF''(n-1)$ for the vertices in level $(n-1)$ and, in particular, the vertices in level n are not adjacent to each other and the degree of each vertex in level $(n-1)$ is different from the vertices in level n, labeling first 2^{n-1} vertices with 2 and the remaining 2^{n-1} vertices with 1 allow e-lucky labeling even though the vertices in level n have the same e-lucky sum and same eccentricity. Hence $\eta_{el}(BF(n)) = 2$.

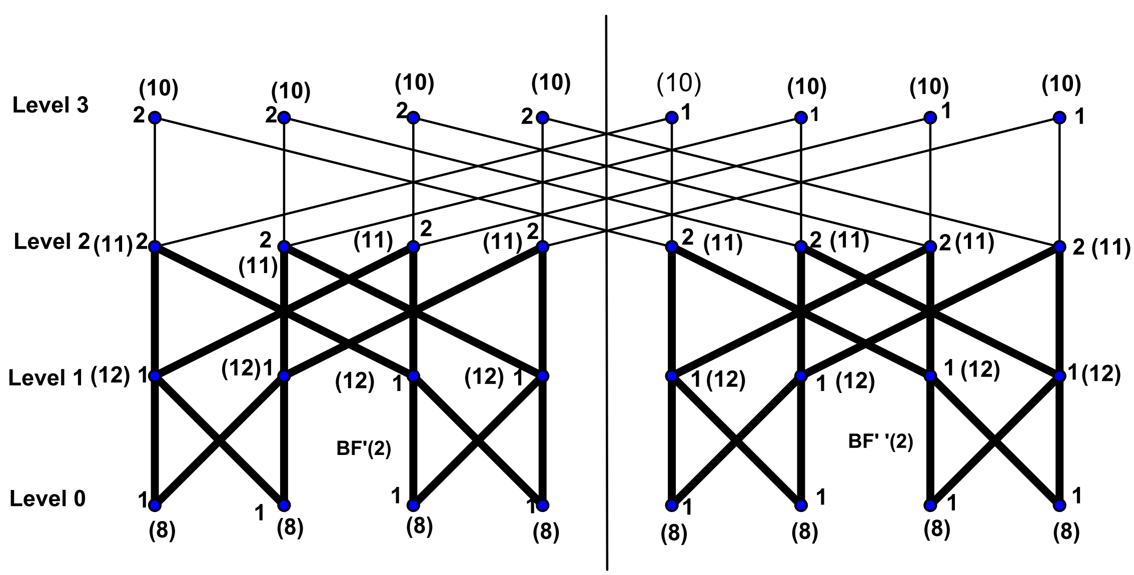

Figure 4: e-lucky labeling of $BF(3)$

4 Book Graph

Definition 4.1. *The n-book graph is defined as the graph Cartesian product $B_n = S_{n+1} \times P_2$, where S_n is a star graph and P_2 is the path graph on two vertices [5].*

Theorem 4.2. *For $B_n, n \geq 3, \eta_{el}(B_n) = 2$.*

Proof.
Since $BF(1)$ is a subgraph of $B_n, \eta_{el}(B_n) \geq 2$. For an upper bound, we consider the following:
A Book graph B_n consists of vertices of degree 2 and $(n+1)$. Name the vertices of degree 2 as $v_1, v_2, ..., v_{2n}$ and vertices of degree $(n+1)$ as u_0 and v_0 respectively. Define $l : V(B_n) \to \{1, 2\}$ by $l(v_i) = 2$, $1 \leq i \leq (2n-2)$; $l(v_i) = 1$, $(2n-1) \leq i \leq 2n$; $l(v_0) = 1$ and $l(u_0) = 2$.
Because of the above-defined labeling pattern, the book graph B_n admits an e-lucky labeling. Hence $\eta_{el}(B_n) = 2$.

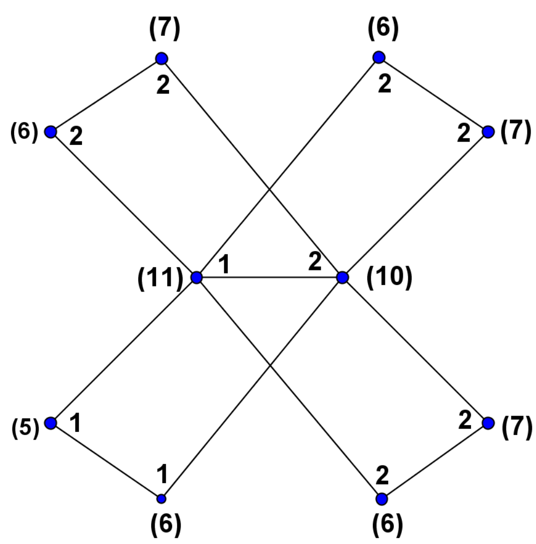

Figure 5: e-lucky labeling of B_4

Theorem 4.3. *Any regular bipartite self-centred graph G admits e-lucky labeling and $\eta_{el}(G) = 2$.*

Proof.
Let $|V(G)| = n$. We define a partition of $V(G)$ by setting $V_1 = \{u_i \in V(G), i = 1, 2, ..., \frac{n}{2}\}$ and $V_2 = \{v_i \in V(G), i = 1, 2, ..., \frac{n}{2}\}$ such that every edge in G joins a vertex in V_1 with vertex V_2. Define a labeling $l : V(G) \to \{1, 2\}$ by $l(u_i) = 1$, for $i = 1, 2, ..., \frac{n}{2}$ and $l(v_i) = 2$, for $i = 1, 2, ..., \frac{n}{2}$. Since each $u_i \in V_1$, $d_G(u_i) = k$ and for k adjacent vertices v_i of u_i with $l(v_i) = 2$, we have $el(u_i) = 2k + 1$ for $i = 1, 2, ..., \frac{n}{2}$.
Similarly, each vertex $v_i \in V_2$, $d_G(v_i) = k$ and for k adjacent vertices u_i of v_i with $l(u_i) = 1$, we have $el(v_i) = k+2$ for $i = 1, 2, ..., \frac{n}{2}$. Since no two vertices in V_1 and V_2 are adjacent, G admits e-lucky labeling and hence $\eta_{el}(G) = 2$.
Open Problem: Any biregular bipartite graph G admits an e-lucky labeling and $\eta_{el}(G) = 2$.

5 Conclusion

In this paper, we have computed e-lucky number for book graphs, complete graphs, butterfly network and a sharp bound for any regular bipartite self-centered graphs. The e-lucky number for interconnection networks is under investigation.

References

[1] M. Miller, I. Rajasingh, D. A. Emilet, D. A. Jemilet, *d*-Lucky Labeling of Graphs, Procedia Computer Science, **57,** (2015), 766–771.

[2] D. A. Emilet, I. Rajasingh, *d*-Lucky Labeling of Cycle of Ladder, *n*-sunlet and Helm graphs, Int. J. Pure App. Math., **10,** no. 109, (2016), 219–227.

[3] S. Klavzar, I. Rajasingh, D. A. Emilet, A Lower Bound and Several Exact Results on the *d*-Lucky Number, App. Math. Comput., **366,** (2020), 124760.

[4] J. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Boston, 2013.

[5] F. Harary, Graph Theory, Narosa Publishing House, 2001.