International Journal of Mathematics and
Computer Science, **15**(2020), no. 4, 1073–1090

Some Results Concerning Multiplicative (Generalized)-Derivations and Multiplicative Left Centralizers

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(Received June 9, 2020, Revised August 6, 2020,
Accepted September 2, 2020)

Abstract

In ring theory, many significant studies have raised connections between the derivations and the structures of rings. Derivations of rings developed gradually more than half a century ago. In particular, the generalizations of the derivation concept play an important role in the calculation of the eigenvalues of matrices, which is important in mathematics and other sciences, business, engineering, and quantum physics. The main goal of this article is to introduce identities in prime and semiprime rings concerning left multiplicative centralizers and multiplicative (generalized)-derivations that have descriptions of these mappings. Some properties of the proposed identities are proven, and the relationships between these identities in terms of the notions of the multiplicative (generalized)-derivation (MG-D) and the multiplicative left centralizer (MLC) for an associative ring S are studied. If the condition $\zeta(\kappa_1\kappa_2)\pm[\tau(\kappa_1),\kappa_2]\pm\kappa_1\kappa_2 = 0$ is held for all κ_1 and κ_2 in an ideal $\mathcal{J}(\neq 0)$ of S, then either ζ is an MLC on S or S is commutative. Furthermore, examples are given to show that semiprimeness and primeness are irreplaceable conditions.

Key words and phrases: multiplicative (generalized)-derivation, multiplicative left centralizer, semiprime ring, prime ring, generalized derivation, left centralizer.
AMS (MOS) Subject Classifications: 47B47, 17A36, 16W25, 16N60
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

1 Introduction

Ring theory is a rarity field connecting different scientific branches of an applied nature, as it enters into construction techniques that are utilized to perform various calculations to solve many contemporary problems. Throughout this article, $C(S)$ always denotes the center of an associative ring S. Set $\kappa_1, \kappa_2 \in S$, the symbol $[\kappa_1, \kappa_2]$ refers to the commutator $\kappa_1\kappa_2 - \kappa_2\kappa_1$. A ring S is prime if, for any $\kappa_1, \kappa_2 \in S, \kappa_1 S \kappa_2 = 0$, infers either $\kappa_1 = 0$ or $\kappa_2 = 0$, and S is semiprime in the case that $\kappa_1 S \kappa_1 = 0$ infers $\kappa_1 = 0$. Recall that an additive map $d : S \to S$ (that preserves the addition operation) is said to be a derivation if it satisfies $d(\kappa_1\kappa_2) = d(\kappa_1)\kappa_2 + \kappa_1 d(\kappa_2)$ for all $\kappa_1, \kappa_2 \in S$. There are various types of derivations such as in [1] and [2]. The following principle identities of the commutator will be used widely throughout this article to perform various calculations and simplifications to achieve the main results:$[\kappa_1, \kappa_2\kappa_3] = [\kappa_1, \kappa_2]\kappa_3 + \kappa_2[\kappa_1, \kappa_3]$ and $[\kappa_1\kappa_2, \kappa_3] = \kappa_1[\kappa_2, \kappa_3] + [\kappa_1, \kappa_3]\kappa_2$ for all $\kappa_1, \kappa_2, \kappa_3 \in S$. The concept of derivation has been expanded in various ways by many authors over recent decades. Brešar [3] was apparently the first to present the term of generalized derivation (GD, for short) as one of these methods of expansion. An additive mapping $\zeta : S \to S$ which satisfies $\zeta(\kappa_1\kappa_2) = \zeta(\kappa_1)\kappa_2 + \kappa_1 d(\kappa_2)$ for all $\kappa_1, \kappa_2 \in S$ is called is GD, where d is a derivation of S into itself. Clearly, the notion of GD involves the notion of derivation. Following [4], a left centralizer(multiplier) (LC, for short) is an additive mapping $\tau : S \to S$ which satisfies $\tau(\kappa_1\kappa_2) = \tau(\kappa_1)\kappa_2$ for all $\kappa_1, \kappa_2 \in S$. Obviously, the concepts of GD and LC are identical when $d = 0$. According to Daif [5], a multiplicative derivation of S (MD, for short) is a map $d : S \to S$ (not necessarily additive) such that $d(\kappa_1\kappa_2) = d(\kappa_1)\kappa_2 + \kappa_1 d(\kappa_2)$ for all $\kappa_1, \kappa_2 \in S$. The idea of MD was stimulated by the study of Martindale [6]. Clearly, the concept of MD contains the concept of derivation. Moreover, an extensive study of these maps was discussed in [7]. Motivated by Daif [5], the concept of MD was expanded to multiplicative generalized derivation (MGD, for short) as follows: a not-necessarily additive map $\zeta : S \to S$ that satisfies $\zeta(\kappa_1\kappa_2) = \zeta(\kappa_1)\kappa_2 + \kappa_1 d(\kappa_2)$ for all $\kappa_1, \kappa_2 \in S$ where $d : S \to S$ is a derivation is called an MGD [8]. Obviously, the concept of MGD includes the concept of GD, LC and MD. In [9], the authors made a simple modification to the idea of MGD as a generalization for this term; thus, the thought of multiplicative (generalized)-derivation (MG-D, for short) was defined as a mapping $\zeta : S \to S$ (not assumed to be additive) and is said to be MG-D if $\zeta(\kappa_1\kappa_2) = \zeta(\kappa_1)\kappa_2 + \kappa_1\eta(\kappa_2)$ for all $\kappa_1, \kappa_2 \in S$, where d is any map (not assumed to be a derivation nor additive) and for convenience, it is denoted by

(ζ, η). Clearly, MG-D is more precise than the concept of MGD. Thus, one can see that every MGD is an MG-D. Tammam et al. [10] created a slight amendment to the definition of LC by canceling the additive condition as a generalization to the concept of LC by introducing the concept of the multiplicative left centralizer (MLC, for short). Accordingly, a map $\tau : S \rightarrow S$ (not assumed to be additive) is called an MLC if $\tau(\kappa_1\kappa_2) = \tau(\kappa_1)\kappa_2$ for all $\kappa_1, \kappa_2 \in S$. During the last few years, many mathematicians have discussed derivations or generalized derivation in semiprime and prime (one may see the work in [11, 12, 13, 14], where further references can be found). In [11], the authors studied the commutativity of the prime ring, and they demonstrated that a prime ring S that possesses a derivation d is necessarily commutative if $d(\kappa_1\kappa_2) - \kappa_1\kappa_2 \in C(S)$ or $d(\kappa_1\kappa_2) + \kappa_1\kappa_2 \in C(S)$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$, where $\mathcal{J}$ is a nonzero ideal of S. This result encouraged Ashraf et al. [15] to prove the cases with GD instead of derivation. Further, some of these results are extended on semiprime rings for MG-D [9]. Further, Tiwari et al. [16] investigated several identities for MG-D mappings (ζ, η) and (Ω, ψ) in a semiprime ring such as $\Omega(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\zeta(\kappa_2) \pm \kappa_2\kappa_1 = 0$, $\Omega(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\zeta(\kappa_2) \pm \kappa_1\kappa_2 = 0$, and $\zeta([\kappa_1, \kappa_2]) \pm [\eta(\kappa_1), \kappa_2] = 0$, ... etc. Sandhu et al. [17] showed that a semiprime ring S possessing MG-D mappings (ζ, η) and (Ω, ψ) must be commutative according to some formulas contained in $C(S)$. Camci and Aydin [18] proved some identities involve MLC and MG-D mappings to study the commutativity of crucial types of rings, both prime and semiprime. Since the notion of MG-D is an extended version of GD and the notion of MLC is an extension of LC, the study of these mappings could be more general and interesting. For the cases as shown in [18], a natural curiosity arises regarding the results induced from using MG-D and MLC instead of MG-D only according to several different identities which appeared in [16, 19, 20, 21] and have not been addressed previously. For this, this paper aims to introduce several identities that contain both mappings of MG-D and MLC, such as $\zeta(\kappa_1\kappa_2) \pm [\tau(\kappa_1), \kappa_2] \pm \kappa_1\kappa_2 = 0$, $\zeta([\kappa_1, \kappa_2]) \pm [\tau(\kappa_1), \kappa_2] = 0$, $\zeta(\kappa_1\kappa_2) + [\tau(\kappa_1), \kappa_2] \pm \kappa_2\kappa_1 = 0$ etc. in addition to the identities that belong to the center of S, such as $\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\tau(\kappa_2) \in C(S)$, $(\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_2)\tau(\kappa_1))\kappa \in C(S)$ and $(\zeta(\kappa_1\kappa_2) \pm \eta(\kappa_2)\tau(\kappa_1))\kappa \in C(S)$ etc., and to prove some properties regarding these different identities. The organization of the paper is as follows. Section 2 is devoted to investigating extensively the relationships between the created identities: the mappings of MG-D and MLC defined on prime and semiprime rings. However, in Section 3 the properties of the identities of the MG-D mapping (ζ, η) and the MLC mapping τ with zero value are explored. Theorems and examples are proved in both sections. Several acronyms are used for ease of expression.

2 Identities of MG-D and MLC Mappings That Lie in the Center of S

This section is devoted to proving the main properties of many proposed identities related to the concepts of MG-D and MLC mappings of prime and semiprime rings.

Regarding Corollary 3 in [16], Tiwari et al. studied a case for an MG-D mapping (ζ, η) satisfying either $\zeta(\kappa_1\kappa_2) - \zeta(\kappa_1)\zeta(\kappa_2) = 0$ or $\zeta(\kappa_1\kappa_2) - \zeta(\kappa_2)\zeta(\kappa_1) = 0$, for all κ_1, κ_2 in a suitable subset $\mathcal{J}$ of S. The next theorem extends Tiwari's first condition to $\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\tau(\kappa_2) \in C(S)$ for all κ_1, κ_2 in $\mathcal{J}$ with an MG-D mapping (ζ, η) and an MLC mapping τ on S. Further, Dhara et al. in [19, Theorem 3.1] discussed the same condition that has been adopted in the next result, such that $\zeta = \tau$ is considered to be MGD mapping. Although the above cases differed, the same result has been obtained for an ideal case. It is worth noting that, for MG-D and MLC mappings on $S, (\zeta, \eta)$ and τ, respectively, the authors in [20, Theorem 3.1] studied the condition $\zeta(\kappa_1\kappa_2) \pm \tau(\kappa_1)\tau(\kappa_2) = 0$, for all κ_1, κ_2 in an ideal of S.

Theorem 2.1. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a semiprime ring S such that the condition $\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\tau(\kappa_2) \in C(S)$ is valid for all elements κ_1, κ_2 of a nonzero left ideal $\mathcal{J}$ of S. Then, $\mathcal{J}[\eta(\kappa_3), \kappa_3] = 0$ for all $\kappa_3 \in \mathcal{J}$. Moreover, if $\mathcal{J}$ is an ideal of S, then $[\eta(\kappa_3), \kappa_3] = 0$ for all $\kappa_3 \in \mathcal{J}$.*

Proof.
Suppose that

$$\zeta(\kappa_1\kappa_2) + \zeta(\kappa_1)\tau(\kappa_2) \in C(S) \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{2.1}$$

In Equation (2.1), substituting $\kappa_2\kappa_3$ rather than κ_2 with $\kappa_3 \in \mathcal{J}$, we get

$$\begin{aligned}&\zeta(\kappa_1\kappa_2\kappa_3) + \zeta(\kappa_1)\tau(\kappa_2\kappa_3)\\ &= (\zeta(\kappa_1\kappa_2) + \zeta(\kappa_1)\tau(\kappa_2))\kappa_3 + \kappa_1\kappa_2\eta(\kappa_3) \in C(S) \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}.\end{aligned} \tag{2.2}$$

Commuting Equation (2.2) with κ_3 and then applying Equation (2.1) leads to

$$[\kappa_1\kappa_2\eta(\kappa_3), \kappa_3] = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{2.3}$$

Substituting $\eta(\kappa_3)\kappa_1$ and $[\eta(\kappa_3),\kappa_3]\kappa_2$ rather than κ_1 and κ_2, respectively, in Equation (2.3), the equation

$$[\eta(\kappa_3),\kappa_3]\kappa_1[\eta(\kappa_3),\kappa_3]\kappa_2\eta(\kappa_3)+\eta(\kappa_3)[\kappa_1[\eta(\kappa_3),\kappa_3]\kappa_2\eta(\kappa_3),\kappa_3]=0$$

for all $\kappa_1,\kappa_2,\kappa_3\in\mathcal{J}$ is obtained. Because of Equation (2.3), the last equation can be reduced to

$$[\eta(\kappa_3),\kappa_3]\kappa_1[\eta(\kappa_3),\kappa_3]\kappa_2\eta(\kappa_3)=0 \text{ for all } \kappa_1,\kappa_2,\kappa_3\in\mathcal{J}. \quad (2.4)$$

Putting $\kappa_2=\kappa_2\kappa_3$ in Equation (2.4) and right multiplication by κ_3 to Equation (2.4) and then subtracting one from other gives

$$[\eta(\kappa_3),\kappa_3]\kappa_1[\eta(\kappa_3),\kappa_3]\kappa_2[\eta(\kappa_3),\kappa_3]=0 \text{ for all } \kappa_1,\kappa_2,\kappa_3\in\mathcal{J},$$

the last equation yields that $(\mathcal{J}[\eta(\kappa_3),\kappa_3])^3=0$, hence supposing to the contrary that $\mathcal{J}[\eta(\kappa_3),\kappa_3]\neq 0$, gives a nonzero nilpotent left ideal, which is a contradiction.

Likewise, one can demonstrate a similar inference for $\zeta(\kappa_1\kappa_2)-\zeta(\kappa_1)\tau(\kappa_2)\in C(S)$ for all $\kappa_1,\kappa_2\in\mathcal{J}$.

Further, in case $\mathcal{J}$ is an ideal of S, then because of [22, Lemma 2.1], $[\eta(\kappa_3),\kappa_3]=0$ for all $\kappa_3\in\mathcal{J}$. □

Throughout the following, $[\eta(\kappa_1),\kappa_1]_2$ will denote the commutator $[[\eta(\kappa_1),\kappa_1],\kappa_1]$.

Lemma 2.2. *Let the MG-D mapping (ζ,η) be defined on a semiprime ring S such that $\kappa_1\kappa_2\eta(\kappa_2)\kappa_3\in C(S)$ for all elements $\kappa_1,\kappa_2,\kappa_3\in\mathcal{J}$, where $\mathcal{J}$ is a nonzero ideal of S. Then, $[\eta(\kappa_1),\kappa_1]_2\eta(\kappa_1)=2[\eta(\kappa_1),\kappa_1]\kappa_1\eta(\kappa_1)$ for all $\kappa_1\in\mathcal{J}$.*

Proof.
By hypothesis,

$$[\kappa_1\kappa_2\eta(\kappa_2)\kappa_3,\omega]=0 \text{ for all } \kappa_1,\kappa_2,\kappa_3\in\mathcal{J},\omega\in S. \quad (2.5)$$

Substituting $\eta(\kappa_2)$ and $\kappa_1\kappa_2\eta(\kappa_2)\kappa_4$ rather than ω and κ_1, respectively, in Equation (2.5) yields for all $\kappa_1,\kappa_2,\kappa_3,\kappa_4\in\mathcal{J}$,

$$[\kappa_1\kappa_2\eta(\kappa_2),\eta(\kappa_2)]\kappa_4\kappa_2\eta(\kappa_2)\kappa_3=0. \quad (2.6)$$

Replacing κ_4 and κ_3 by $\kappa_4\kappa_1$ and $\eta(\kappa_2)\kappa_3$, respectively in Equation (2.6) implies that

$$[\kappa_1\kappa_2\eta(\kappa_2), \eta(\kappa_2)]\kappa_4\kappa_1\kappa_2\eta(\kappa_2)\eta(\kappa_2)\kappa_3 = 0. \tag{2.7}$$

Again, replacing $\kappa_4\eta(\kappa_2)\kappa_1$ instead of κ_4 in Equation(2.6) and then comparing it with Equations(2.7), we infer that for all $\kappa_1, \kappa_3, \kappa_4, \kappa \in \mathcal{J}$,

$$[\kappa_1\kappa_2\eta(\kappa_2), \eta(\kappa_2)]\kappa_4[\kappa_1\kappa_2\eta(\kappa_2), \eta(\kappa_2)]\kappa_3 = 0. \tag{2.8}$$

Putting $\kappa_4 = \kappa_3\kappa_4$ and as an application of [22, Lemma 2.1] twice in Equation (2.8) yields

$$\begin{aligned} 0 &= [\kappa_1\kappa_2\eta(\kappa_2), \eta(\kappa_2)] \\ &= \kappa_1[\kappa_2, \eta(\kappa_2)]\eta(\kappa_2) + [\kappa_1, \eta(\kappa_2)]\kappa_2\eta(\kappa_2) \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \end{aligned} \tag{2.9}$$

Specifically, when $\kappa_1 = \kappa_2$ in Equation (2.9), we conclude that

$$\kappa_1[\kappa_1, \eta(\kappa_1)]\eta(\kappa_1) = -[\kappa_1, \eta(\kappa_1)]\kappa_1\eta(\kappa_1) \text{ for all } \kappa_1 \in \mathcal{J}. \tag{2.10}$$

By Equation (2.10),

$$\begin{aligned} [\eta(\kappa_1), \kappa_1]_2\eta(\kappa_1) &= [[\eta(\kappa_1), \kappa_1], \kappa_1]\eta(\kappa_1) = ([\eta(\kappa_1), \kappa_1]\kappa_1 - \kappa_1[\eta(\kappa_1), \kappa_1])\eta(\kappa_1) \\ &= ([\eta(\kappa_1), \kappa_1]\kappa_1 + \kappa_1[\kappa_1, \eta(\kappa_1)])\eta(\kappa_1) = ([\eta(\kappa_1), \kappa_1]\kappa_1 - [\kappa_1, \eta(\kappa_1)]\kappa_1)\eta(\kappa_1) \\ &= ([\eta(\kappa_1), \kappa_1]\kappa_1 + [\eta(\kappa_1), \kappa_1]\kappa_1)\eta(\kappa_1) = 2[\eta(\kappa_1), \kappa_1]\kappa_1\eta(\kappa_1) \text{ for all } \kappa_1 \in \mathcal{J}. \end{aligned}$$

□

According to [19], the authors hypothesized that, for an MGD mapping ζ associated with a derivation η in Theorem 3.3 and Theorem 3.4, the conditions $\zeta(\kappa_1\kappa_2) - \zeta(\kappa_2)\zeta(\kappa_1) \in C(S)$ and $\zeta(\kappa_1\kappa_2) + \zeta(\kappa_2)\zeta(\kappa_1) \in C(S)$, respectively, for all κ_1, κ_2 in some appropriate subset $\mathcal{J}$ of S obtain $\mathcal{J}\kappa_1[\kappa_1, \eta(\kappa_1)]_2 = 0$ for all $\kappa_1 \in \mathcal{J}$. After modifying the conditions adopted in [19], it is interesting to study these modified conditions to involve both MG-D and MLC mappings instead of MGD mapping alone, as shown in the following theorem.

Theorem 2.3. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a semiprime ring S such that the statement $(\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_2)\tau(\kappa_1))k \in C(S)$ is held for all elements $\kappa_1, \kappa_2, \kappa$ of a nonzero ideal $\mathcal{J}$ of S. Then $[\kappa_1, \eta(\kappa_1)]_2\eta(\kappa_1) = 0$ for all $\kappa_1 \in \mathcal{J}$. Moreover, if S is prime, then either $\kappa_1\eta(\kappa_1) = 0$ or $[\kappa_1, \eta(\kappa_1)] \in C(S)$ for all $\kappa_1 \in \mathcal{J}$.*

Proof.
Assume that

$$(\zeta(\kappa_1\kappa_2) + \zeta(\kappa_2)\tau(\kappa_1))\kappa \in C(S) \text{ for all } \kappa_1, \kappa_2, \kappa \in \mathcal{J}. \tag{2.11}$$

In Equation (2.11), substituting $\kappa_3\kappa_1$ rather than κ_1, for $\kappa_3 \in \mathcal{J}$ yields

$$\begin{aligned}(\zeta(\kappa_3\kappa_1\kappa_2) + \zeta(\kappa_2)\tau(\kappa_3\kappa_1))\kappa &= (\zeta(\kappa_3\kappa_1\kappa_2) + \zeta(\kappa_2)\tau(\kappa_3)\kappa_1 + \zeta(\kappa_3\kappa_2)\kappa_1 - \zeta(\kappa_3\kappa_2)\kappa_1)\kappa \\ &= (\zeta(\kappa_3\kappa_1\kappa_2) - \zeta(\kappa_3\kappa_2)\kappa_1)\kappa \\ = (\zeta(\kappa_3\kappa_1)\kappa_2 + \kappa_3\kappa_1\eta(\kappa_2) &- \zeta(\kappa_3\kappa_2)\kappa_1)\kappa \in C(S) \text{ for all } \kappa_1, \kappa_2, \kappa_3, \kappa \in \mathcal{J}.\end{aligned} \tag{2.12}$$

Specifically, $\kappa_1 = \kappa_2$ in Equation (2.12) implies $\kappa_3\kappa_1\eta(\kappa_1)\kappa \in C(S)$ for all $\kappa_1, \kappa_3, \kappa \in \mathcal{J}$ which means that

$$[\kappa_3\kappa_1\eta(\kappa_1)\kappa, \omega] = 0 \text{ for all } \kappa_1, \kappa_3, \kappa \in \mathcal{J}, \omega \in S. \tag{2.13}$$

Replacing κ_3 by $\eta(\kappa_1)\kappa_1\kappa_3$ in Equation (2.13) allows us to deduce that

$$[\eta(\kappa_1)\kappa_1, \omega]\kappa_3\kappa_1\eta(\kappa_1)\kappa = 0 \text{ for all } \kappa_1, \kappa_3, \kappa \in \mathcal{J}, \omega \in S. \tag{2.14}$$

Again, replacing κ_3 by $\kappa_1\eta(\kappa_1)\kappa_3$ in Equation (2.13) and then comparing it with Equation (2.14), we infer that

$$[[\eta(\kappa_1), \kappa_1], \omega]\kappa_3\kappa_1\eta(\kappa_1)\kappa = 0 \text{ for all } \kappa_1, \kappa_3, \kappa \in \mathcal{J}, \omega \in S. \tag{2.15}$$

Multiplying both sides of Equation (2.15) by $\kappa_1\eta(\kappa_1)$ from the left and substituting $[[\eta(\kappa_1), \kappa_1], \omega]\kappa$ instead of κ implies that
$\kappa_1\eta(\kappa_1)[[\eta(\kappa_1), \kappa_1], \omega]\kappa_3\kappa_1\eta(\kappa_1)[[\eta(\kappa_1), \kappa_1], \omega]\kappa = 0$ for all $\kappa_1, \kappa_3, \kappa \in \mathcal{J}, \omega \in S$. In the last Equation, replacing κ_3 by $\kappa\kappa_3$ and since $\mathcal{J}$ is considered as semiprime, we find that

$$\kappa_1\eta(\kappa_1)[[\eta(\kappa_1), \kappa_1], \omega] = 0 \text{ for all } \kappa_1, \kappa_3 \in \mathcal{J}, \omega \in S. \tag{2.16}$$

Substituting $\omega\upsilon$ rather than ω, where $\upsilon \in S$ in Equation (2.16), we obtain

$$\kappa_1\eta(\kappa_1)\omega[[\eta(\kappa_1), \kappa_1], \upsilon] = 0 \text{ for all } \kappa_1 \in \mathcal{J}, \omega, \upsilon \in S. \tag{2.17}$$

Multiplication to Equation (2.17) by $2[\kappa_1, \eta(\kappa_1)]$ from left and by $\eta(\kappa_1)$ from the right, and then as an application of Lemma 2.2, we find

$$[[\kappa_1, \eta(\kappa_1)], \kappa_1]\eta(\kappa_1)\omega[[\eta(\kappa_1), \kappa_1], \upsilon]\eta(\kappa_1) = 0 \text{ for all } \kappa_1 \in \mathcal{J}, \omega, \upsilon \in S. \tag{2.18}$$

In particular, when $\upsilon = \kappa_1$ in Equation (2.18), given the semiprime hypothesis of S, this gives $[\kappa_1, \eta(\kappa_1)]_2\eta(\kappa_1) = 0$ for all $\kappa_1 \in \mathcal{J}$.
In the same way, we can conclude the same result to be $(\zeta(\kappa_1\kappa_2) - \zeta(\kappa_2)\tau(\kappa_1))\kappa \in C(S)$ for all $\kappa_1, \kappa_2, \kappa \in \mathcal{J}$.
Now, suppose that S is a prime ring; then, because of Equation (2.17) we get either $\kappa_1\eta(\kappa_1) = 0$ for all $\kappa_1 \in \mathcal{J}$ or $[\kappa_1, \eta(\kappa_1)] \in C(S)$ for all $\kappa_1 \in \mathcal{J}$. □

Corollary 2.4. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a semiprime ring S such that the statement $(\zeta(\kappa_1\kappa_2) \pm \eta(\kappa_2)\tau(\kappa_1))\kappa \in C(S)$ is held for all elements $\kappa_1, \kappa_2, \kappa$ of a nonzero ideal $\mathcal{J}$ of S. Then $[\kappa_1, \eta(\kappa_1)]_2\eta(\kappa_1) = 0$ for all $\kappa_1 \in \mathcal{J}$. Besides, if S is prime, then either $\kappa_1\eta(\kappa_1) = 0$ or $[\kappa_1, \eta(\kappa_1)] \in C(S)$ for all $\kappa_1 \in \mathcal{J}$.*

Proof.
It is an immediate result of Theorem 2.3. □

3 Identities of the MG-D Mapping (ζ, η) and the Mapping τ of the MLC Type with Zero Value

In the following, (ζ, η) represents MG-D mapping ζ associated with a mapping η, and τ is an MLC mapping, rather than both of them being MG-D mappings as in [16].

Theorem 3.1. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a semiprime ring S such that the statement $\zeta(\kappa_1\kappa_2) \pm [\tau(\kappa_1), \kappa_2] \pm \kappa_1\kappa_2 = 0$ is valid for all elements κ_1, κ_2 of a nonzero left ideal $\mathcal{J}$ of S. Then $\mathcal{J}[\eta(\mathcal{J}), \mathcal{J}] = 0$. Also, $\eta(\mathcal{J})$ is central when $\mathcal{J}$ is an ideal of S.*

Proof.
Firstly, suppose that

$$\zeta(\kappa_1\kappa_2) + [\tau(\kappa_1), \kappa_2] + \kappa_1\kappa_2 = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.19}$$

In Equation (3.19), substituting $\kappa_2\kappa_3, \kappa_3 \in \mathcal{J}$ instead of κ_2 yields

$$\begin{aligned} 0 &= \zeta(\kappa_1\kappa_2\kappa_3) + [\tau(\kappa_1), \kappa_2\kappa_3] + \kappa_1\kappa_2\kappa_3 \\ &= \zeta(\kappa_1\kappa_2)\kappa_3 + \kappa_1\kappa_2\eta(\kappa_3) + \kappa_2[\tau(\kappa_1), \kappa_3] + [\tau(\kappa_1), \kappa_2]\kappa_3 + \kappa_1\kappa_2\kappa_3 \end{aligned}$$

Applying Equation (3.19) implies

$$\kappa_1\kappa_2\eta(\kappa_3) + \kappa_2[\tau(\kappa_1), \kappa_3] = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.20}$$

Replacing κ_2 by $\eta(\kappa_3)\kappa_2$ in Equation (3.20) allows us to infer that

$$\kappa_1\eta(\kappa_3)\kappa_2\eta(\kappa_3) + \eta(\kappa_3)\kappa_2[\tau(\kappa_1), \kappa_3] = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.21}$$

Now, using left multiplication by $\eta(\kappa_3)$ to Equation (3.20) and subtracting from Equation (3.21), we deduce that

$$[\eta(\kappa_3), \kappa_1]\kappa_2\eta(\kappa_3) = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.22}$$

Replacing κ_2 by $\kappa_2\kappa_1$ in Equation (3.22), we find that

$$[\eta(\kappa_3), \kappa_1]\kappa_2\kappa_1\eta(\kappa_3) = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.23}$$

Multiplying both sides of Equation (3.22) from the right by κ_1 and then comparing with Equation (3.23), we deduce that $[\eta(\kappa_3), \kappa_1]\kappa_2[\eta(\kappa_3), \kappa_1] = 0$ for all $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}$, which infers that $0 \neq \mathcal{J}[\eta(\kappa_3), \kappa_1]$ is nilpotent left ideal, but this contradicts the fact that a semiprime ring is always without a nonzero nilpotent left ideal. Thus, $\mathcal{J}[\eta(\kappa_3), \kappa_1] = 0$ for all $\kappa_1, \kappa_3 \in \mathcal{J}$ which means $\mathcal{J}[\eta(\mathcal{J}), \mathcal{J}] = 0$. Likewise, one can demonstrate a similar inference for $\eta(\kappa_1\kappa_2) - [\tau(\kappa_1), \kappa_2] - \kappa_1\kappa_2 = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$. For the ideal case, in view of [23, Lemma 2], one can show that $\eta(\mathcal{J}) \subseteq C(\mathcal{J}) \subseteq C(S)$ which means that $\eta(\mathcal{J})$ is central. □

Notice that for a nonzero ideal $\mathcal{J}$ in the case of a prime ring S, the following corollary can be concluded from Theorem 3.1.

Corollary 3.2. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a prime ring S such that the statement $\zeta(\kappa_1\kappa_2) \pm [\tau(\kappa_1), \kappa_2] \pm \kappa_1\kappa_2 = 0$ is held for all elements κ_1, κ_2 of a nonzero ideal $\mathcal{J}$ of S. Then either ζ is an MLC on S or S is commutative.*

Proof.
From Theorem 3.1,

$$[\eta(\kappa_1), \kappa_3] = 0 \text{ for all } \kappa_1, \kappa_3 \in \mathcal{J}. \tag{3.24}$$

Substituting $\kappa_1\kappa_2$ rather than κ_1 in Equation (3.24), where $\kappa_2 \in \mathcal{J}$, as an application of [18, Lemma 2], this implies that

$$0 = [\eta(\kappa_1), \kappa_3]\kappa_2 + \eta(\kappa_1)[\kappa_2, \kappa_3] + [\kappa_1, \kappa_3]\eta(\kappa_2) + \kappa_1[\eta(\kappa_2), \kappa_3]$$

As an application of Equation (3.24), reduce it to

$$\eta(\kappa_1)[\kappa_2,\kappa_3]+[\kappa_1,\kappa_3]\eta(\kappa_2)=0 \text{ for all } \kappa_1,\kappa_2,\kappa_3\in\mathcal{J}. \tag{3.25}$$

Specifically, for $\kappa_1=\kappa_3$ in Equation (3.25), we deduce that

$$\eta(\kappa_1)[\kappa_2,\kappa_3]=0 \text{ for all } \kappa_1,\kappa_2\in\mathcal{J}. \tag{3.26}$$

In Equation (3.26), replacing κ_2 by $\omega\kappa_2$ and then by Theorem 3.1 and by Equation (3.26), $\eta(\kappa_1)[\omega,\kappa_1]\kappa_2=0$ for all $\kappa_1,\kappa_2\in\mathcal{J},\omega\in S$. Using the hypothesis of S and $\mathcal{J}$, we conclude that either $\eta(\mathcal{J})=0$ or $\mathcal{J}$ is a central ideal. Now, for all $\kappa_1\in\mathcal{J},\omega,k\in S$; $\zeta(\omega)k\kappa_1+\omega\eta(k\kappa_1)=\zeta(\omega k\kappa_1)=\zeta(\omega k)\kappa_1+\omega k\eta(\kappa_1)$. As an application of the first case and primeness of S, ζ is an MLC, while the second case using [23, Lemma 2] implies that S is commutative. □

In [21] and [9], the formulas $\zeta([\kappa_1,\kappa_2])\pm[\zeta(\kappa_1),\kappa_2]\in C(S)$ and $\zeta([\kappa_1,\kappa_1])\pm[\eta(\kappa_1),\kappa_2]=0$ are studied, respectively. However, Ali et al. [20] took the properties of the formula $\zeta([\kappa_1,\kappa_2])\pm\tau[\kappa_1,\kappa_2]=0$. The next result investigates the behavior of a proposed case that studies the conditions $\zeta([\kappa_1,\kappa_2])\pm[\tau(\kappa_1),\kappa_2]=0$ and $\zeta(\kappa_1\kappa_2)+[\tau(\kappa_1),\kappa_2]\pm\kappa_2\kappa_1=0$.

Theorem 3.3. *Let the MG-D mapping (ζ,η) and the MLC mapping τ be defined on a semiprime ring S. For all elements κ_1,κ_2 of a nonzero left ideal $\mathcal{J}$ of S, if one of the following statements is fulfilled:*

(a) $\zeta([\kappa_1,\kappa_2])\pm[\tau(\kappa_1),\kappa_2]=0$.

(b) $\zeta(\kappa_1\kappa_2)+[\tau(\kappa_1),\kappa_2]\pm\kappa_2\kappa_1=0$.

Then $\mathcal{J}[\kappa_1,\eta(\kappa_1)]=0$ for all $\kappa_1\in\mathcal{J}$. Further, $[\eta(\kappa_1),\kappa_1]=0$ for all $\kappa_1\in\mathcal{J}$ is satisfied when $\mathcal{J}$ is an ideal of S.

Proof.

(a) Firstly, consider that

$$\zeta([\kappa_1,\kappa_2])+[\tau(\kappa_1),\kappa_2]=0 \text{ for all } \kappa_1,\kappa_2\in\mathcal{J}. \tag{3.27}$$

Putting $\kappa_2\kappa_1$ instead of κ_2 in Equation (3.27) yields

$$\begin{aligned} 0 &= \zeta([\kappa_1,\kappa_2]\kappa_1)+[\tau(\kappa_1),\kappa_2\kappa_1] \\ &= \zeta([\kappa_1,\kappa_2])\kappa_1+[\kappa_1,\kappa_2]\eta(\kappa_1)+[\tau(\kappa_1),\kappa_2]\kappa_1+\kappa_2[\tau(\kappa_1),\kappa_1] \end{aligned}$$

$$= [\kappa_1, \kappa_2]\eta(\kappa_1) + \kappa_2[\tau(\kappa_1), \kappa_1] \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \quad (3.28)$$

In Equation (3.28), putting $\omega\kappa_2$ rather than κ_2 for $\omega \in S$ and using left multiplication by ω to Equation (3.28) and then subtracting one from the other gives

$$[\kappa_1, \omega]\kappa_2\eta(\kappa_1) = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}, \omega \in S. \quad (3.29)$$

Especially, when $\omega = \eta(\kappa_1)$ in Equation (3.29), we obtain $[\kappa_1, \eta(\kappa_1)]\kappa_2\eta(\kappa_1) = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$. This is the same as Equation (3.22) in Theorem 2.3. Then, by the same argument as we have used in Theorem 2.3, the conclusion follows.

Likewise, one can demonstrate a similar inference for $\zeta([\kappa_1, \kappa_2]) - [\tau(\kappa_1), \kappa_2] = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$.

(b) Suppose that

$$\zeta(\kappa_1\kappa_2) + [\tau(\kappa_1), \kappa_2] + \kappa_2\kappa_1 = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \quad (3.30)$$

Using the same technique of Equation (3.27) in (a), we deduce that

$$\kappa_1\kappa_2\eta(\kappa_1) + \kappa_2[\tau(\kappa_1), \kappa_1] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \quad (3.31)$$

Substituting $\omega\kappa_2$, for $\omega \in S$ in place of κ_2 in Equation (3.31) yields

$$\omega\kappa_1\kappa_2\eta(\kappa_1) + \omega\kappa_2[\tau(\kappa_1), \kappa_1] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}, \omega \in S. \quad (3.32)$$

Using left multiplication by ω in Equation (3.32) and then comparing it with Equation (3.32) implies that

$$[\kappa_1, \omega]\kappa_2\eta(\kappa_1) = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}, \omega \in S. \quad (3.33)$$

In the analogous manner of Equation (3.29) in (a), the proof of the desired result is completed.

In a similar way, the same conclusion can be obtained for the equation $\zeta(\kappa_1\kappa_2) + [\tau(\kappa_1), \kappa_2] - \kappa_2\kappa_1 = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$.

Further, if $\mathcal{J}$ is considered as an ideal of S, then by applying [22, Lemma 2.1], we obtain $[\kappa_1, \eta(\kappa_1)] = 0$ for all $\kappa_1 \in \mathcal{J}$. □

In the case of the prime ring, the next result proves the possibility that the ring is commutative when MG-D and MLC are mappings on S.

Proposition 3.4. *Let the MG-D mapping (ζ, η) and the MLC mapping τ be defined on a prime ring S. For all elements κ_1, κ_2 of a nonzero left ideal $\mathcal{J}$ of S, if one of the next statements fulfills:*

(a) $\zeta([\kappa_1, \kappa_2]) \pm [\tau(\kappa_1), \kappa_2] = 0.$

(b) $\zeta(\kappa_1\kappa_2) + [\tau(\kappa_1), \kappa_2] \pm \kappa_2\kappa_1 = 0.$

(c) $\zeta(\kappa_1\kappa_2) \pm \tau(\kappa_1)\zeta(\kappa_2) \pm \kappa_2\kappa_1 = 0.$

(d) $\zeta(\kappa_1\kappa_2) \pm \tau(\kappa_1)\zeta(\kappa_2) \pm [\kappa_1, \kappa_2] = 0.$

Then, either ζ is an MLC or S is commutative.

Proof.

(a) As an application of Equation (3.29) in Theorem 3.3 and [23, Lemma 2], the required result has been proven.

(b) Because of Equation (3.33) in Theorem 3.3 and [23, Lemma 2], the conclusion holds.

(c) Firstly, assume that

$$\zeta(\kappa_1\kappa_2) + \tau(\kappa_1)\zeta(\kappa_2) + \kappa_2\kappa_1 = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.34}$$

Substituting $\kappa_2\kappa_3$ for κ_2, where $\kappa_3 \in \mathcal{J}$ in Equation (3.34), an application of Equation (3.34) implies

$$\begin{aligned} 0 &= \zeta(\kappa_1\kappa_2)\kappa_3 + \kappa_1\kappa_2\eta(\kappa_3) + \tau(\kappa_1)\zeta(\kappa_2)\kappa_3 + \tau(\kappa_1)\kappa_2\eta(\kappa_3) + \kappa_2\kappa_3\kappa_1 \\ &= -\kappa_2\kappa_1\kappa_3 + \kappa_1\kappa_2\eta(\kappa_3) + \tau(\kappa_1)\kappa_2\eta(\kappa_3) + \kappa_2\kappa_3\kappa_1 \end{aligned}$$

$$= \kappa_2[\kappa_3, \kappa_1] + \kappa_1\kappa_2\eta(\kappa_3) + \tau(\kappa_1)\kappa_2\eta(\kappa_3) \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.35}$$

Especially, $\kappa_1 = \kappa_3$ in Equation (3.35) yields

$$(\kappa_1 + \tau(\kappa_1))\kappa_2\eta(\kappa_1) = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.36}$$

Replacing $\omega\kappa_2$ instead of κ_2 for $\omega \in S$ in Equation (3.36), the hypothesis of S implies that either $\tau(\kappa_1) + \kappa_1 = 0$ or $\kappa_2\eta(\kappa_1) = 0$ for all $\kappa_1 \in \mathcal{J}$. For the first case, $\tau(\kappa_1) + \kappa_1 = 0$ for all $\kappa_1 \in \mathcal{J}$, then as an application of Equation (3.35) and [23, Lemma 2], the commutativity of S has to be obtained while the second case and primeness of S yields ζ is an MLC. In a similar way, the desired result can be proved for $\zeta(\kappa_1\kappa_2) - \tau(\kappa_1)\zeta(\kappa_2) - \kappa_2\kappa_1 = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$.

(d) Suppose that $\zeta(\kappa_1\kappa_2) \pm \tau(\kappa_1)\zeta(\kappa_2) \pm [\kappa_1, \kappa_2] = 0$, as the same steps in the proof of (c), the result is proved. □

Theorem 3.5. *Let the MG-D mapping* (ζ, η) *and the MLC mapping* τ *be defined on a semiprime ring* S*. For all elements* κ_1, κ_2 *of a nonzero left ideal* $\mathcal{J}$ *of* S*, if one of the next statements fulfills:*

(a) $[\zeta(\kappa_1), \kappa_2] + [\kappa_1, \tau(\kappa_2)] = 0.$

(b) $\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\tau(\kappa_2) \pm \kappa_2\kappa_1 = 0.$

(c) $\tau(\kappa_1\kappa_2) + \zeta([\kappa_1, \kappa_2]) \pm [\kappa_1, \kappa_2] = 0.$

Then $\mathcal{J}[\kappa_1, \eta(\kappa_1)] = 0$ *for all* $\kappa_1 \in \mathcal{J}$*.*

Proof.

(a) By assumption,

$$[\zeta(\kappa_1), \kappa_2] + [\kappa_1, \tau(\kappa_2)] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.37}$$

Especially, when $\kappa_1 = \kappa_2$, Equation (3.37) can be reduced to

$$[\zeta(\kappa_1), \kappa_1] + [\kappa_1, \tau(\kappa_1)] = 0 \text{ for all } \kappa_1 \in \mathcal{J}. \tag{3.38}$$

Substituting $\kappa_2\kappa_3$, $\kappa_3 \in \mathcal{J}$ instead of κ_1 in Equation (3.38) and using (3.37) implies

$$\kappa_2[\zeta(\kappa_1), \kappa_3] + \tau(\kappa_2)[\kappa_1, \kappa_3] = 0 \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \tag{3.39}$$

Especially, $\kappa_1 = \kappa_3$ Equation (3.39) can be simplified to

$$\kappa_2[\zeta(\kappa_1), \kappa_1] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.40}$$

From Equations (3.38) and (3.40), the statement

$$\kappa_2[\kappa_1, \tau(\kappa_1)] = 0 \text{ is valid for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.41}$$

Substituting $\kappa_1\kappa_3$ in place of κ_1 in Equation (3.37) yields

$$\zeta(\kappa_1)[\kappa_3, \kappa_2] + [\kappa_1, \kappa_2]\eta(\kappa_3) + \kappa_1[\eta(\kappa_3), \kappa_2] + \kappa_1[\kappa_3, \tau(\kappa_2)] = 0 \tag{3.42}$$

for all $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}$. Putting $\kappa_2 = \kappa_3$ in Equation (3.42) yields

$$[\kappa_1, \kappa_2]\eta(\kappa_2) + \kappa_1[\eta(\kappa_2), \kappa_2] + \kappa_1[\kappa_2, \tau(\kappa_2)] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.43}$$

An application of Equation (3.41) in Equation (3.43) gives

$$[\kappa_1\eta(\kappa_2), \kappa_2] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.44}$$

In Equation (3.44), putting $\eta(\kappa_2)\kappa_1$ in place κ_1 leads us to

$$[\eta(\kappa_2), \kappa_2]\kappa_1\eta(\kappa_2) = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.45}$$

By simple computations, Equation (3.45) implies $\kappa_3[\eta(\kappa_2), \kappa_2]\kappa_1[\eta(\kappa_2), \kappa_2] = 0$ for all $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}$. That is, $0 \neq \mathcal{J}[\eta(\kappa_2), \kappa_2] \subseteq S$ is nilpotent left ideal for all $\kappa_2 \in \mathcal{J}$ but this contradicts with the fact of S is semiprime. So, we must have $\mathcal{J}[\eta(\kappa_2), \kappa_2] = 0$ for all $\kappa_2 \in \mathcal{J}$.

(b) By hypothesis

$$\zeta(\kappa_1\kappa_2) + \zeta(\kappa_1)\tau(\kappa_2) + \kappa_2\kappa_1 = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.46}$$

In Equation (3.46), substituting $\kappa_2\kappa_3$, for $\kappa_3 \in \mathcal{J}$ in place κ_2 implies

$$\begin{aligned} 0 &= \zeta(\kappa_1\kappa_2)\kappa_3 + \kappa_1\kappa_2\eta(\kappa_3) + \zeta(\kappa_1)\tau(\kappa_2)\kappa_3 + \kappa_2\kappa_3\kappa_1 \\ &= \kappa_2[\kappa_3, \kappa_1] + \kappa_1\kappa_2\eta(\kappa_3) \text{ for all } \kappa_1, \kappa_2, \kappa_3 \in \mathcal{J}. \end{aligned} \tag{3.47}$$

Putting $\kappa_1 = \kappa_3$ in Equation (3.47) yields

$$\kappa_1\kappa_2\eta(\kappa_1) = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.48}$$

Substituting $\eta(\kappa_1)\kappa_2$ instead of κ_2 in Equation (3.48) implies

$$\kappa_1\eta(\kappa_1)\kappa_2\eta(\kappa_1) = 0 \text{ is valid for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.49}$$

Using left multiplication by $\eta(\kappa_1)$ to Equation (3.48) and then comparing it with Equation (3.49) implies that the statement $[\kappa_1, \eta(\kappa_1)]\kappa_2\eta(\kappa_1) = 0$ is true for all $\kappa_1, \kappa_2 \in \mathcal{J}$. This situation is the same as Equation (3.45); thus, the conclusion follows. In the same way, we can obtain the same result for $\zeta(\kappa_1\kappa_2) - \zeta(\kappa_1)\tau(\kappa_2) - \kappa_2\kappa_1 = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$.

(c) Suppose that

$$\tau(\kappa_1\kappa_2) + \zeta([\kappa_1, \kappa_2]) + [\kappa_1, \kappa_2] = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.50}$$

In Equation (3.50), substituting $\kappa_1\kappa_2$ instead of κ_1 leads us to

$$[\kappa_1, \kappa_2]\eta(\kappa_2) = 0 \text{ for all } \kappa_1, \kappa_2 \in \mathcal{J}. \tag{3.51}$$

Substituting $\eta(\kappa_2)\kappa_1$ in place κ_1 in Equation (3.51) yields $[\eta(\kappa_2), \kappa_2]\kappa_1\eta(\kappa_2) = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$ which is the same as Equation (3.45); thus, we get the required result.
Likewise, the same result can be obtained for $\tau(\kappa_1\kappa_2) + \zeta([\kappa_1, \kappa_2]) - [\kappa_1, \kappa_2] = 0$ for all $\kappa_1, \kappa_2 \in \mathcal{J}$. □

The proof of the next corollary is an immediate application of Theorem 3.5 and [22, Lemma 2.1].

Corollary 3.6. *Let the MG-D mapping* (ζ, η) *and the MLC mapping* τ *be defined on a semiprime ring* S*. For all elements* κ_1, κ_1 *of a nonzero ideal* $\mathcal{J}$ *of* S*, if one of the following statements fulfills:*

(a) $[\zeta(\kappa_1), \kappa_2] + [\kappa_1, \tau(\kappa_2)] = 0.$

(b) $\zeta(\kappa_1\kappa_2) \pm \zeta(\kappa_1)\tau(\kappa_2) \pm \kappa_2\kappa_1 = 0.$

(c) $\tau(\kappa_1\kappa_2) + \zeta([\kappa_1, \kappa_2]) \pm [\kappa_1, \kappa_2] = 0.$

Then, $[\eta(\kappa_1), \kappa_1] = 0$ *for all* $\kappa_1 \in \mathcal{J}$*.*

The following example explains the necessary assumption of primeness and semiprimeness. Throughout the following, **Z** will represent the set of integers.

Example 3.7. Consider

$$S = \left\{ \begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \kappa_1, \kappa_2, \kappa_3 \in \mathbf{Z} \right\}.$$

Let ζ, η, and τ be mappings of S defined by

$$\zeta\left(\begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & \kappa_2 & \kappa_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\eta\left(\begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -\kappa_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau\left(\begin{pmatrix}\kappa_1 & \kappa_2 & \kappa_3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}-\kappa_1 & -\kappa_2 & -\kappa_3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It is a simple matter to check that ζ is an MG-D on S and τ is an MLC on S. Since $\left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right) S \left(\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)$, then S is not semiprime.

For all elements $\mathcal{J}_1, \mathcal{J}_2$ of the left ideal

$$\mathcal{J}=\left\{\begin{pmatrix}\kappa_1 & 0 & \kappa_2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} : \kappa_1, \kappa_2 \in \mathbf{Z}\right\} \subseteq S,$$

the statement $\zeta([\mathcal{J}_1, \mathcal{J}_2]) + [\tau(\mathcal{J}_1), \mathcal{J}_2] = 0$ is valid, but there exists $\mathcal{J}_1 = \left(\begin{smallmatrix}\kappa_1&0&\kappa_1\\0&0&0\\0&0&0\end{smallmatrix}\right) \in S$ where $0 \neq \kappa_1 \in \mathbf{Z}$ such that $\mathcal{J}[\zeta(\kappa_1), \kappa_1] \neq 0$. Therefore, the semiprimeness assumption in (a) of Theorem 3.3 is essential.

Example 3.8. Consider S, ζ, η, and τ are as in the above Example. It is a simple matter to check that ζ is an MG-D on S and τ is an MLC on S. In view of the previous Example, S is not a prime ring. For all elements $\mathcal{J}_1, \mathcal{J}_2$ of the ideal $\mathcal{J} = \left\{\left(\begin{smallmatrix}0&0&\kappa_3\\0&0&0\\0&0&0\end{smallmatrix}\right) : \kappa_3 \in \mathbf{Z}\right\} \subseteq S$, the following statements are fulfilled:

(a) $\zeta(\mathcal{J}_1\mathcal{J}_2) \pm [\tau(\mathcal{J}_1), \mathcal{J}_2] \pm \mathcal{J}_1\mathcal{J}_2 = 0$

(b) $\zeta([\mathcal{J}_1, \mathcal{J}_2]) \pm [\tau(\mathcal{J}_1), \mathcal{J}_2] = 0$

(c) $\zeta(\mathcal{J}_1\mathcal{J}_2) + [\tau(\mathcal{J}_1), \mathcal{J}_2] \pm \mathcal{J}_2\mathcal{J}_1 = 0$

(d) $\zeta(\mathcal{J}_1\mathcal{J}_2) \pm \tau(\mathcal{J}_1)\zeta(\mathcal{J}_2) \pm \mathcal{J}_2\mathcal{J}_1 = 0$

(e) $\zeta(\mathcal{J}_1\mathcal{J}_2) \pm \tau(\mathcal{J}_1)\zeta(\mathcal{J}_2) \pm [\mathcal{J}_1, \mathcal{J}_2] = 0$,

but neither is S commutative nor is ζ an MLC on S. Therefore, the primeness assumption in Corollary 3.2 and Proposition 3.4 is essential.

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