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Generalized Subsemigroups and Fuzzy Subsemigroups in Semigroups

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Abstract

In this paper, we investigate some properties of n-subsemigroups and fuzzy n-subsemigroups of semigroups. Moreover, we define almost n-subsemigroups and fuzzy almost n-subsemigroups of semigroups, investigate properties of them and give relationships between almost n-subsemigroups and fuzzy almost n-subsemigroups.

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1 Introduction and Preliminaries

The generalization of classical algebraic structures to n-ary structures was first initiated by Kasner [3] in 1904. First, we recall the definition of an n-ary groupoids which is a nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$. If the operation f is associative, then S is called an n-ary semigroup. In 1965, the fundamental concept of a fuzzy set was introduced by Zadeh [9]. Since then, it opened up applications in various fields. Fuzzification of n-ary groupoids and n-ary semigroups was studied (for example, [1], [4], [5], [6] and [8]). Now, we recall some basic facts in fuzzy set theory. A fuzzy subset of a set S is a function from S into the closed interval $[0, 1]$. For any two fuzzy subsets f and g of S,

1. $f \cap g$ is a fuzzy subset of S defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$ for all $x \in S$,

2. $f \cup g$ is a fuzzy subset of S defined by $(f \cup g)(x) = \max\{f(x), g(x)\}$ for all $x \in S$ and

3. $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

The characteristic mapping of a subset A of S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

For a fuzzy subset f of S and $t \in (0, 1]$, a t-level set of f is defined by

$$f_t = \{x \in S \mid f(x) \geq t\}.$$

For a fuzzy subset f of S, the support of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

Let f and g be fuzzy subsets of a semigroup S. The product of f and g is a fuzzy subset $f \circ g$ defined as follows:

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab} \min\{f(a), g(b)\} & \text{if } x = ab \text{ for some } a, b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer $k \geq 2$, $f^k := \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ terms}}$.

In section 2, we focus on n-subsemigroups and fuzzy n-subsemigroups. The notion of almost ideals of semigroups was introduced in 1980 by Grosek and Satko [2]. Recently, almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups were studied in [7]. These are interesting and they became an inspiration of studying about almost n-subsemigroups and fuzzy almost n-subsemigroups in section 3.

2 n-Subsemigroups and Fuzzification

Definition 2.1. Let n be a positive integer such that $n \geq 2$. A nonempty subset A of a semigroup S is called an *n-subsemigroup* of S if A is an n-ary groupoid; that is, $A^n \subseteq A$.

Example 2.1. Consider the semigroup $(\mathbb{Z}, \cdot)$ and let $A = \{-1\}$. Then A is an n-subsemigroup of S if n is any odd number but not in the other cases.

Remark 2.1. Let S be a semigroup.

(1) If A is a 2-subsemigroup of S, then A is an n-subsemigroup of S for all $n \geq 2$.

(2) If A is a 3-subsemigroup of S, then A is an n-subsemigroup of S for all odd integers $n \geq 3$.

(3) S is itself an n-subsemigroup of S for all $n \geq 2$.

(4) Let A and B be n-subsemigroups of S. If $A \cap B \neq \emptyset$, then $A \cap B$ is an n-subsemigroup of S.

Let $\mathcal{I}(S)$ be the set of all n-subsemigroups of a semigroup S. By Remark 2.1 (3), $\mathcal{I}(S) \neq \emptyset$. From Remark 2.1 (4), if the result of the intersection is empty, then it is sure that $A \cap B$ is not an n-subsemigroup of S. Hence, from now on, we consider two cases.

Case 1 : $A \cap B \neq \emptyset$ for any two n-subsemigroups A and B of S. Then $\mathcal{I}(S)$ forms a semigroup under the binary operation of intersection.

Case 2 : $A \cap B = \emptyset$ for some two n-subsemigroups A and B of S. Then $\mathcal{I}(S) \cup \{\emptyset\}$ forms a semigroup under the binary operation of intersection.

Nevertheless, these statements are not true in general under the binary operation of union as we shown in the following example.

Example 2.2. Consider the semigroup $(\mathbb{Z}, \cdot)$, let $A = \{-1\}$ and $B = \{2^k \mid k \in \mathbb{N}\}$. Clearly, A and B are 3-subsemigroups but $A \cup B$ is not a

3-subsemigroup. Then the union of two n-subsemigroups needs not be an n-subsemigroup.

Example 2.3. Let S be a commutative semigroup. For any two n-subsemigroups A and B of S, we have that AB is also an n-subsemigroup of S. Then the set of all n-subsemigroups of a commutative semigroup forms a semigroup.

Definition 2.2. A fuzzy subset f of a semigroup S is called a *fuzzy n-subsemigroup* of S if $f(x_1x_2 \cdot \ldots \cdot x_n) \geq \min\{f(x_1), f(x_2), \ldots, f(x_n)\}$ for all $x_1, x_2, \ldots, x_n \in S$.

The three following theorems follow by properties of the fuzzification of n-ary subsemigroups in n-ary semigroups.

Theorem 2.3. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy n-subsemigroup of S if and only if $f^n \subseteq f$.*

Theorem 2.4. *Let A be a nonempty subset of a semigroup S. Then A is an n-subsemigroup of S if and only if C_A is a fuzzy n-subsemigroup of S.*

Theorem 2.5. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy n-subsemigroup of S if and only if for all $t \in (0, 1]$, if $f_t \neq \emptyset$, then f_t is an n-subsemigroup of S.*

3 Almost n-Subsemigroups and Fuzzification

Definition 3.1. Let n be a positive integer such that $n \geq 2$. A nonempty subset A of a semigroup S is called an *almost n-subsemigroup* of S if $A^n \cap A \neq \emptyset$.

Remark 3.1. Let S be a semigroup and let A be a nonempty subset of S.

(1) If A is an n-subsemigroup of S, then A is an almost n-subsemigroup of S. The converse is not true in general.

(2) If A contains an idempotent element, then A is an almost n-subsemigroup of S for all $n \geq 2$.

Example 3.1. Consider the semigroup $\mathbb{Z}$ under multiplication.

(1) Let $A = \{-1\}$. Then A is an almost n-subsemigroup of S if n is any odd number but not in the other cases.

(2) Let $A = \{2, 4\}$ and $B = \{4, 16\}$. Then A and B are almost 2-subsemigroups but $A \cap B = \{4\}$ is not an almost 2-subsemigroup.

By Example 3.1 (2), we can conclude that the intersection of two almost n-subsemigroups need not be an almost n-subsemigroup.

Theorem 3.2. *Let A be an almost n-subsemigroup of a semigroup S. If B is a subset of S containing A, then B is an almost n-subsemigroup of S.*

Proof. Since $A^n \cap A \neq \emptyset$ and $A^n \cap A \subseteq B^n \cap B$, $B^n \cap B \neq \emptyset$. Therefore, B is an almost n-subsemigroup of S. □

Corollary 3.3. *If A and B are almost n-subsemigroups of a semigroup S, then $A \cup B$ is an almost n-subsemigroup of S.*

Proof. The proof is clear by Theorem 3.2. □

From Corollary 3.3, the set of all almost n-subsemigroups of a semigroup S forms a semigroup under the binary operation of union. However, this statement is not true in general under the binary operation of intersection as we have shown in Example 3.1 (2).

The above statements contrasts with the results in the previous section because, under the two cases in section 2, the set of all n-subsemigroups of a semigroup S forms a semigroup under the binary operation of intersection but need not be true under the binary operation of union.

Definition 3.4. A fuzzy subset f of a semigroup S is called a *fuzzy almost n-subsemigroup* of S if $f^n \cap f \neq 0$.

Theorem 3.5. *Let f be a fuzzy almost n-subsemigroup of a semigroup S. If g is a fuzzy subset of S such that $f \subseteq g$, then g is a fuzzy almost n-subsemigroup of S.*

Proof. Since $f^n \cap f \neq 0$ and $f^n \cap f \subseteq g^n \cap g$, $g^n \cap g \neq 0$. Hence g is a fuzzy almost n-subsemigroup of S. □

Corollary 3.6. *If f and g are fuzzy almost n-subsemigroups of a semigroup S, then $f \cup g$ is a fuzzy almost n-subsemigroup.*

Proof. This follows from Theorem 3.5. □

Finally, we provide the results showing the connection between almost n-subsemigroups and fuzzy almost n-subsemigroups.

Theorem 3.7. *Let A be a nonempty subset of a semigroup S. Then A is an almost n-subsemigroup of S if and only if C_A is a fuzzy almost n-subsemigroup of S.*

Proof. Assume that A is an almost n-subsemigroup of S. Then $A^n \cap A \neq \emptyset$. Then there exists $x \in A^n \cap A$. Hence $(C_A)^n(x) = C_{A^n}(x) = 1$ and $C_A(x) = 1$. So $(C_A)^n \cap C_A \neq 0$. Therefore, C_A is a fuzzy almost n-subsemigroup of S. Conversely, suppose that C_A is a fuzzy almost n-subsemigroup of S. Then $(C_A)^n \cap C_A \neq 0$ and so there exists $x \in S$ such that $((C_A)^n \cap C_A)(x) \neq 0$. This implies $x \in A^n \cap A$. Consequently, A is an almost n-subsemigroup of S. □

Example 3.2. From Example 3.1 (2), we know that $A = \{2, 4\}$ and $B = \{4, 16\}$ are almost 2-subsemigroups of $(\mathbb{Z}, \cdot)$. So C_A and C_B are fuzzy almost 2-subsemigroups by Theorem 3.7. However, $C_A \cap C_B = C_{A\cap B}$ is not a fuzzy almost 2-subsemigroup because $A \cap B$ is not an almost 2-subsemigroup by Example 3.1 (2) and Theorem 3.7.

Now, we can conclude that the set of all fuzzy almost n-subsemigroups of a semigroup S forms a semigroup under the binary operation of union by Corollary 3.6 but this statement is not true in general under the binary operation of intersection as we have shown in Example 3.2. Moreover, we obtain this conclusion in a similar way as the case of almost n-subsemigroups.

Theorem 3.8. *Let f be a nonzero fuzzy subset of a semigroup S. Then f is a fuzzy almost n-subsemigroup of S if and only if $supp(f)$ is an almost n-subsemigroup of S.*

Proof. Let f be a fuzzy almost n-subsemigroup of S. Then $f^n \cap f \neq 0$ and so there exists $x \in S$ such that $(f^n \cap f)(x) \neq 0$. Then $f^n(x) \neq 0$ and $f(x) \neq 0$. Since $f^n(x) \neq 0$, there exist $a_1, a_2, \ldots, a_n \in supp(f)$ such that $x = a_1a_2 \cdots a_n$. So $x \in (supp(f))^n \cap supp(f)$. This implies that $supp(f)$ is an almost n-subsemigroup of S. Conversely, assume that $supp(f)$ is an almost n-subsemigroup of S. Thus $(supp(f))^n \cap supp(f) \neq \emptyset$ and so there exists $x \in (supp(f))^n \cap supp(f)$. Since $x \in (supp(f))^n$, there exist $a_1, a_2, \ldots, a_n \in supp(f)$ such that $x = a_1a_2 \cdots a_n$. Then $f^n(x) \neq 0$. Hence $(f^n \cap f)(x) \neq 0$. As a result, f is a fuzzy almost n-subsemigroup of S. □

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References

[1] W. A. Dudek, Fuzzification of n-ary groupoids, Quasigroups Relat. Syst., **7**, (2000), 45–66.

[2] O. Grosek, L. Satko, A new notion in the theory of semigroups, Semigroup Forum, **20**, (1980), 233–240.

[3] R. Kasner. An extension of the group concepts, Bull. American Math. Soc., **10**, (1904), 290–291.

[4] D. R. Prince Williams, Intuitionistic fuzzy n-ary subgroups, Thai J. Math., **8**, (2010), 391—404.

[5] J. P. F. Solano, S. Suebsung, R. Chinram, On almost i-ideals and fuzzy almost i-ideals in n-ary semigroups, JP J. Algebra Number Theory Appl., **40**, (2018), 833–842.

[6] J. P. F. Solano, S. Suebsung, R. Chinram, On ideals of fuzzy points n-ary semigroups, Int. J. Math. Comput. Sci., **13**, (2018), 179–186.

[7] S. Suebsung, K. Wattanatripop, R. Chinram, On almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups, J. Taibah Univ. Sci., **13**, (2019), 897–902.

[8] Q. Wang, X. Zhou, J. Zhan, A novel study of soft sets over n-ary semigroups, Italian J. Pure Appl. Math., **37**, (2017), 583–594.

[9] L. A. Zadeh, Fuzzy sets, Inform. Control, **8**, (1965), 338–353.