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Synchronization of Multiplicative Noise Dissipative Systems under Discretization

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Abstract

We discuss the discretization influence on multiplicative noise dissipative system. Moreover, using a drift-implicit Euler formula with discretization, we find the synchronization of multiplicative noise dissipative model.

1 Introduction

In physical sciences, the synchronization of coupled systems is common. Strogatz [1] showed a descriptive account of its diversity of occurrence in his recent book, which contains an extensive list of references. Afraimovich and Rodrigues [2] investigated the synchronization of coupled dissipative model in the case of autonomous model. Asymptotically stable equilibria and general attractors have been investigated mathematically by Carvalho, et al. [3] and Rodrigues [4]. Caraballo, et al. [5] proved synchronization of systems under additive noise and gave another matter of random attractors. They considered stochastic stationary solutions instead of deterministic solutions. Moreover, they studied synchronization

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with multiplicative noise. Furthermore, in both synchronization of coupled systems, the noises are the same. Al-Azzawi, et al. [6] analyzed that convergence rate of synchronization of additive nose dissipative coupled systems. Also, they innovated a new method to handle synchronization of general stochastic differential equations. In addition, they [7] gave examples and simulations for the synchronization of coupled dynamical systems.

In this paper, we investigate the influences of discretization with multiplicative noise on a dissipatively coupled model. We use a drift-implicit Euler formula to discretize the system with constant step size and illustrate the synchronization of multiplicative noise dissipative systems under discretization is independent of the used step size.

2 Formulation of the Synchronization Problem

Suppose that

$$\frac{dx}{dt} = f_1(x) \quad and \quad \frac{dy}{dt} = f_2(y) \tag{1}$$

are two autonomous ordinary differential equations in R^d, where f_1 and f_2 are differentiable functions that satisfy one-sided dissipative Lipschitz conditions

$$\langle x_1 - x_2, f_1(x_1) - f_1(x_2)\rangle \leqslant -K|x_1 - x_2|^2, \langle y_1 - y_2, f_2(y_1) - f_2(y_2)\rangle \leqslant -K|y_1 - y_2|^2 \tag{2}$$

on R^d for some $K > 0$. Thus we have a unique equilibria $\acute{x}$ and $\acute{y}$, respectively, which are globally asymptotically stable [8]. We can see that the continuous differentiability of functions f_1 and f_2 and the Lipschitz conditions (2) ensure the existence and uniqueness of solutions to (1). Consider the dissipative coupled system as follows

$$\frac{dx}{dt} = f_1(x) + v(y - x), \qquad \frac{dy}{dt} = f_2(y) + v(x - y), \tag{3}$$

where $v > 0$. Clearly, this coupled system in R^{2d} is fulfilled with the one-sided dissipative Lipschitz condition with a positive constant K. By Afraimovich and Rodrigues [2] and Carvalho, et al. [3], its unique equilibrium is $(\acute{x}^v, \acute{y}^v)$, which is globally asymptotically stable. Also, $(\acute{x}^v, \acute{y}^v)$ converges to $(\acute{z}, \acute{z})$ as $v \to \infty$, such that $\acute{z}$ is the unique globally asymptotically stable equilibrium of the averaged system

$$\frac{dz}{dt} = \frac{1}{2}(f_1(z) + f_2(z)). \tag{4}$$

The phenomenon is called synchronization. Clearly, analogous results are fulfilled for more general autonomous attractors [2, 3] and for nonautonomous model [9] with suitably defined nonautonomous attractors as well as discrete time systems [10]. Caraballo, et al. [5] illustrated that the synchronization effect persists under additive noise, provided equilibria are replaced by stationary random solutions. Specifically, they presented two Ito stochastic differential equations in R^d as follows

$$dX_t = f_1(X_t)dt + \alpha dW_t^1, \qquad dY_t = f_2(Y_t)dt + \beta dW_t^2, \tag{5}$$

for all nonzero constants $\alpha, \beta \in R$, $(W_t^1)_{t\in R}$ and $(W_t^2)_{t\in R}$ are independent two-sided scalar Wiener processes as well as the continuously differentiable f_1 and f_2 satisfy the one-sided dissipative Lipschitz conditions (2). The synchronized system corresponding to the stochastic differential equation (5) follows

$$dX_t = (f_1(X_t) + \nu(Y_t - X_t))dt + \alpha dW_t^1, dY_t = (f_2(Y_t) + \nu(X_t - Y_t))dt + \beta dW_t^2 \tag{6}$$

Thus equation (6) has a unique stationary solution $(\acute{X}_t^\nu, \acute{Y}_t^\nu)_{t\in R}$, which is pathwise globally asymptotically stable. Also,

$$(\acute{X}_t^\nu, \acute{Y}_t^\nu) \to (\acute{Z}_t^\infty, \acute{Z}_t^\infty) \quad as \quad \nu \to \infty,$$

pathwise on finite time intervals $[M_1, M_2]$ of R, where $\acute{Z}_t^\nu$, for all $t \in R$, is the unique pathwise globally asymptotically stable stationary solution of the "averaged" stochastic differential equation in R^d :

$$dZ_t = \frac{1}{2}[f_1(Z_t) + f_2(Z_t)]dt + \frac{1}{2}\alpha dW_t^1 + \frac{1}{2}\beta dW_t^2.$$

3 The synchronization of Systems

We consider two stochastic differential equations in R^d as follows

$$dX_t = f_1(X_t)dt + \sum_{i=1}^{n} \alpha_i X_t dW_t^{(i)}, dY_t = f_2(Y_t)dt + \sum_{i=1}^{n} \beta_i Y_t dW_t^{(i)}, \tag{7}$$

where $W_t^{(1)}, \cdots, W_t^{(n)}$ are two-sided scalar independent Wiener processes, $\alpha_i, \beta_i \in R, i = 1, \cdots, n$ and f_1 and f_2 satisfy the one-sided dissipative Lipschitz conditions (2). By [10] we consider the pathwise random ordinary differential equation as follows

$$\begin{cases} \frac{dx}{dt} = F^{(1)}(x, O_t^{(1)}(W)) = e^{-O_t^{(1)}(W)} f_1(e^{O_t^{(1)}(W)}x) + O_t^{(1)}(W)x, \\ \frac{dy}{dt} = F^{(2)}(y, O_t^{(2)}(W)) = e^{-O_t^{(2)}(W)} f_2(e^{O_t^{(2)}(W)}y) + O_t^{(2)}(W)y. \end{cases} \tag{8}$$

The transformation $x(t,W) = e^{-O_t^{(1)}(W)}X_t(W)$ and $y(t,W) = e^{-O_t^{(2)}(W)}Y_t(W)$ such that

$$O_t^{(1)} = \sum_{i=1}^{n} \alpha_i e^{-t} \int_{-\infty}^{t} e^{\tau} dW_\tau^{(i)}, O_t^{(2)} = \sum_{i=1}^{n} \beta_i e^{-t} \int_{-\infty}^{t} e^{\tau} dW_\tau^{(i)}$$

are two stationary Ornstein-Uhlenbeck processes. We consider the synchronization of coupled random ordinary differential equation

$$\begin{cases} \frac{dx}{dt} = F^{(1)}(x, O_t^{(1)}(W)) = e^{-O_t^{(1)}(W)} f_1(e^{O_t^{(1)}(W)}x) + \nu(y-x), \\ \frac{dy}{dt} = F^{(2)}(y, O_t^{(2)}(W)) = e^{-O_t^{(2)}(W)} f_2(e^{O_t^{(2)}(W)}y) + \nu(x-y). \end{cases} \tag{9}$$

Therefore, equation (9) has a stochastic stationary solution $(\bar{x}^\nu(W), \bar{y}^\nu(W))$. Moreover, $(\breve{x}(W), \breve{y}(W)) \to (\bar{z}(W), \bar{z}(W))$ as $\nu \to \infty$, where $\bar{z}(W)$ is the pathwise globally asymptotically stable stationary solution of the averaged random ordinary differential equation

$$\frac{dz}{dt} = \frac{1}{2}[F^{(1)}(z, O_t^{(1)}) + F^{(2)}(z, O_t^{(2)})]. \tag{10}$$

Furthermore, we have

$$\frac{dz}{dt} = \frac{1}{2}[e^{-O_t^{(1)}(W)} f_1(e^{O_t^{(1)}(W)}z) + e^{-O_t^{(2)}(W)} f_2(e^{O_t^{(2)}(W)}z) + (O_t^{(1)}(W) + O_t^{(2)}(W))z]$$

with equivalent stochastic differential equation

$$dz_t = \frac{1}{2}[e^{-\pi_t} f_1(e^{\pi_t} z_t) + e^{\pi_t} f_2(e^{-\pi_t} z_t)]dt + \frac{1}{2}\sum_{i=1}^{m}(\alpha_i + \beta_i) z_t dW_t^{(i)},$$

where $\pi_t = \frac{1}{2}(O_t^{(1)} - O_t^{(2)})$.

From the original system of (7), the coupled random ordinary differential equation has the scheme

$$\begin{cases} dX_t = (f_1(X_t) + \nu(e^{2\pi_t}Y_t - X_t))dt + \sum_{i=1}^{m} \alpha_i X_t dW_t^{(i)}, \\ dY_t = (f_2(Y_t) + \nu(e^{-2\pi_t}X_t - Y_t))dt + \sum_{i=1}^{m} \beta_i Y_t dW_t^{(i)}. \end{cases} \tag{11}$$

Then equation (11) has a unique stochastic stationary solution $(\bar{x}^\nu\theta_t, \bar{y}^\nu\theta_t)$, which is pathwise globally asymptotically stable. Moreover,

$$(\bar{x}^\nu(\theta_t W), \bar{y}^\nu(\theta_t W)) \to (\bar{z}(\theta_t W)e^{-\pi_t^{(1)}(W)}, \bar{z}(\theta_t W)e^{-\pi_t^{(2)}(W)}) \quad as \quad \nu \to \infty,$$

pathwise on finite time intervals $[T_1, T_2]$ of R. Now, if $\alpha_i = \beta_i$ for all $i = 1, \cdots, m$; (i.e., the driving noises are the same), then we consider the synchronization of the coupled stochastic differential equation as follows

$$dX_t = (f_1(X_t) + v(Y_t - X_t))dt + \sum_{i=1}^{m} \alpha_i X_t dW_t^{(i)},$$

$$dY_t = (f_2(Y_t) + v(X_t - Y_t))dt + \sum_{i=1}^{m} \alpha_i Y_t dW_t^{(i)},$$

in R^{2d}, where $W^{(i)} = (W_t^{(i)})_{t\in R}$ are independent two-sided d-dimensional Wiener Processes and α_i are non zero scalar constants. From equation (11) we consider the Euler-Maruyama scheme as follows

$$\begin{cases} X_{m+1} = X_m + h(f_1(X_{m+1}) + v(Y_{m+1} - X_{m+1})) + \sum_{j=1}^{n} \alpha_j \Delta W_m^{(j)}, \\ Y_{m+1} = Y_m + h(f_2(Y_{m+1}) + v(X_{m+1} - Y_{m+1})) + \sum_{j=1}^{n} \alpha_j W_m^{(j)}, \end{cases} \tag{12}$$

4 Discretization

Here, we rewrite the coupled system of equation (1) as a vector system in R^{2d}

$$\frac{dX}{dt} = U(X) + vAX, \tag{13}$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad U(X) = \begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix},$$

$$A = \begin{bmatrix} -Id & Id \\ Id & -Id \end{bmatrix}$$

with the $d \times d$ identity matrix Id. From the Euler schema with step size $h > 0$, we obtain

$$X_{m+1} = X_m + hU(X_{m+1}) + hvAX_{m+1}, \tag{14}$$

where $m = 0, 1, 2, \cdots$. The equation has a unique equilibrium $\acute{X^v} = (\acute{x^v}, \acute{y^v})$ for any step size $h > 0$. The ball $A_{2d}[0, R]$ in R^{2d} with radius R is defined by

$$R^2 = 1 + \frac{1}{k^2}|U(0)|_{2d}^2.$$

In particular, we get $\acute{X}^v \in A_{2d}[0, R]$. Since R does not depend on v, the equilibria are in the common compact set $A_{2d}[0, R]$. Therefore, $\acute{X}^v \to \acute{X}$ as $v \to \infty$. Let $X_m^v = (x_m^v, y_m^v)$ be a solution sequence of equation (14) in the compact ball $A_{2d}[0, R]$. We have

$$x_{m+1}^v - y_{m+1}^v = x_m^v - y_m^v - 2vh(x_{m+1}^v - y_{m+1}^v) + h(f_1(x_{m+1}^v) - f_2(y_{m+1}^v)).$$

Taking the inner product, we get

$$\begin{aligned} |x_{m+1}^v - y_{m+1}^v|_d^2 &\leqslant |x_m^v - y_m^v|_d |x_{m+1}^v - y_{m+1}^v|_d - 2vh|x_{m+1}^v - y_{m+1}^v|_d^2 + |x_{m+1}^v - y_{m+1}^v|(|f_1(x_{m+1}^v)|_d + |f_2(y_{m+1}^v)|_d) \\ &\leqslant |x_m^v - y_m^v|_d |x_{m+1}^v - y_{m+1}^v|_d - 2vh|x_{m+1}^v - y_{m+1}^v|_d^2 + N_R |x_{m+1}^v - y_{m+1}^v|_d, \end{aligned}$$

where $N_R = \max_{(x,y)\in A_{2d}[0,R]}(|f_1(x)| + |f_2(y)|) < \infty$. Hence

$$|x_{m+1}^v - y_{m+1}^v|_d \leqslant \frac{1}{1+2vh}|x_m^v - y_m^v|_d + \frac{N_R}{1+2vh}.$$

Therefore, we obtain

$$|x_m^v - y_m^v|_d \leqslant \frac{1}{(1+2vh)^m}|x_0 - y_0|_d + \frac{N_R}{2vh}$$

assuming that the initial conditions do not depend on v. As a result, we have that $(x_m^v - y_m^v) \to 0$ as $v \to \infty$ for all $m \geqslant 1$. Let us define $z_m^v = \frac{(x_m^v + y_m^v)}{2}$ and note that

$$z_m^v - x_m^v = y_m^v - z_m^v = \frac{1}{2}(y_m^v - x_m^v) \to 0$$

as $v \to \infty$ for all $m \geqslant 1$. Since $z_m^v \in A_d[0, R]$, there is a convergent subsequence $z_m^v \to z_n^\infty$ as $v \to \infty$ for all $m \geqslant 1$. Adding the x and y components of equation (14) and dividing by 2, we get

$$z_{m+1}^v = z_m^v + \frac{h}{2}(f_1(x_{m+1}^v + f_2(y_{m+1}^v))$$

and hence

$$z_{m+1}^v = z_m^v + \frac{h}{2}(f_1(2z_{m+1}^v - y_{m+1}^v) + f_2(2z_{m+1}^v - x_{m+1}^v)).$$

Taking the limit, we have

$$z_{m+1}^\infty = z_m^\infty + \frac{h}{2}(f_1(z_{m+1}^\infty) + f_2(z_{m+1}^\infty)).$$

5 Main results

Lemma 5.1. $\hat{X}_m^{(h,\nu)} - \hat{Y}_m^{(h,\nu)} \to 0$ *pathwise on any bounded time interval* $[M_1, M_2]$ *as* $\nu \to \infty$.

Proof. From equation (12), the stationary solution $\Upsilon_m^{(h,\nu)}$ gives

$$\hat{X}_{m+1}^{(h,\nu)} - \hat{Y}_{m+1}^{(h,\nu)} = \hat{X}_m^{(h,\nu)} - \hat{Y}_m^{(h,\nu)} - 2\nu h(\hat{X}_{m+1}^{(h,\nu)} - \hat{Y}_{m+1}^{(h,\nu)}) + hf_1(\hat{X}_{m+1}^{(h,\nu)}) - hf_2(\hat{Y}_{m+1}^{(h,\nu)}) + \sum_{j=1}^{n}(\alpha_j - \beta_j)\Delta W_m^{(j)}$$

or

$$\hat{D}_{m+1}^{(h,\nu)} = \rho\hat{D}_m^{(h,\nu)} + \rho h F_{m+1}^{(1)(h,\nu)} - \rho h F_{m+1}^{(2)(h,\nu)} + \rho\sum_{j=1}^{n}(\alpha_j - \beta_j)\Delta W_m^{(j)}$$

with

$$\hat{D}_{m+1}^{(h,\nu)} = \hat{X}_m^{(h,\nu)} - \hat{Y}_m^{(h,\nu)},$$

$$F_{m+1}^{(1)(h,\nu)} = f_1(\hat{X}_{m+1}^{(h,\nu)}),$$

$$F_{m+1}^{(2)(h,\nu)} = f_2(\hat{Y}_{m+1}^{(h,\nu)})$$

and

$$\rho = \frac{1}{(1+2\nu h)}.$$

Therefore, we obtain

$$\hat{D}_m^{(h,\nu)} = \rho^{m-m_0}\hat{D}_{m_0}^{(h,\nu)} + \rho h\sum_{i=m_0}^{m-1}\rho^{m-1-i}F_{i+1}^{(1)(h,\nu)} - \rho h\sum_{i=m_0}^{m-1}\rho^{m-1-i}F_{i+1}^{(2)(h,\nu)} + \rho\sum_{i=m_0}^{m-1}\rho^{m-1-i}\sum_{j=1}^{n}(\alpha_j - \beta_j)\Delta W_m^{(j)}.$$

By the triangle inequality, for $m = M_1, \cdots, M_2$ with $M_1 > m_0$, we have

$$\begin{aligned} |\hat{D}_m^{(h,\nu)}|_d &\leqslant \rho^{m-m_0}|\hat{D}_{m_0}^{(h,\nu)}|_d + \rho h\sum_{i=m_0}^{m-1}\rho^{m-1-i}(|F_{i+1}^{(1)(h,\nu)}|_d + |F_{i+1}^{(2)(h,\nu)}|_d) + \rho\sum_{i=m_0}^{m-1}\rho^{m-1-i}\sum_{j=1}^{n}(\alpha_j - \beta_j)\Delta W_m^{(j)} \\ &\leqslant \frac{1}{(1+2\nu h)^{m-m_0}}|\hat{D}_m^{(h,\nu)}|_d + \frac{h}{(1+2\nu h)}\sum_{i=m_0}^{m-1}\frac{1}{(1+2\nu h)^{m-1-i}}(|F_{i+1}^{(1)(h,\nu)}|_d + |F_{i+1}^{(2)(h,\nu)}|_d) \end{aligned}$$

$$+\frac{\sum_{j=1}^{n}|\alpha_j+\beta_j|}{(1+2vh)}\sup_{m=M_1,\cdots,M2}|\sum_{i=m_0}^{m-1}\frac{1}{(1+2vh)^{m-1-i}}\Delta W_m^{(j)}|_d.$$

From above, the finite sums are finite valued. Hence

$$|\hat{D}_m^{(h,v)}|_d \to 0 \quad as \quad v \to \infty.$$

□

Theorem 5.2.

$$(\hat{X}_m^{(h,v)}, \hat{Y}_m^{(h,v)}) \to (\hat{Z}_m^{(h,\infty)}, \hat{Z}_m^{(h,\infty)})$$

pathwise uniformly on bounded time intervals $[M_1, M_2]$ *as* $v \to \infty$, *where* m *is an integer and* $(\hat{Z}_m^{(h,\infty)})$ *is the discrete time stationary stochastic solution of the averaged random ordinary differential equation with drift-implicit Euler-Marutama formula*

$$dZ_t = \frac{1}{2}[e^{-\pi_t} f_1(e^{\pi_t} Z_t) + e^{\pi_t} f_2(e^{-\pi_t} Z_t)]dt + \frac{1}{2}\sum_{j=1}^{n}(\alpha_j+\beta_j)dW_t^{(j)} \tag{15}$$

with $\pi_t = \frac{1}{2}(O_t^{(1)} - O^{(2)})$; *i.e.,*

$$Z_{m+1} = Z_m + \frac{1}{2}h[e^{-\pi_{m+1}} f_1(e^{\pi_{m+1}} Z_{m+1}) + e^{\pi_{m+1}} f_2(e^{-\pi_{m+1}} Z_{m+1})] + \frac{1}{2}\sum_{j=1}^{n}(\alpha_j+\beta_j)\Delta W_m^{(j)} \tag{16}$$

Proof. Define

$$Z_m^{(h,v)} = \frac{1}{2}(\hat{X}_m^{(h,v)}, \hat{Y}_m^{(h,v)}),$$

where m is integer. We observe that $Z_m^{(h,v)}$ satisfies the equation

$$Z_{m+1}^{(h,v)} = Z_m^{(h,v)} + \frac{1}{2}h[e^{-\pi_{m+1}} f_1(e^{\pi_{m+1}} \hat{X}_{m+1}^{(h,v)}) + e^{\pi_{m+1}} f_2(e^{-\pi_{m+1}} \hat{Y}_{m+1}^{(h,v)})] + \frac{1}{2}\sum_{j=1}^{n}(\alpha_j+\beta_j)\Delta W_m^{(j)} \tag{17}$$

Since the $(\hat{X}_m^{(h,v)}, \hat{Y}_m^{(h,v)})^T$ belong to the $B_{2d}[0, \bar{R}_m]$ in R^{2d} for each m and all $v > 0$, where $\bar{R}_m$ is independent of v, we have $|Z_m^{(h,v)}| \leqslant \bar{R}_m$ for each m and $v > 0$. The ball $B_d[0, \bar{R}_m]$ in R^d is compact. Moreover, for any sequence $v_i \to \infty$ there is a subsequence $Z_m^{(h,v_{ik})} \in B_d[0, \bar{R}_m]$ which converges pathwise uniformly on $[M_1, M_2]$. Now, fix a subsequence v_{ik} and denote its limit by $\hat{Z}_m^{(h,\infty)}$. By Lemma 5.1 we have

$$Z_m^{(h,v_{ik})} - \hat{Y}_m^{(h,v_{ik})} = \frac{1}{2}(\hat{X}^{(h,v_{ik})} - \hat{Y}^{(h,v_{ik})}) \to 0,$$

$$Z_m^{(h,v_{ik})} - \hat{X}_m^{(h,v_{ik})} = \frac{1}{2}(\hat{Y}^{(h,v_{ik})} - \hat{X}^{(h,v_{ik})}) \to 0$$

as $v_{ik} \to \infty$. Then

$$\hat{X}_m^{(h,v_{ik})} = 2Z_m^{(h,v_{ik})} - \hat{Y}_m^{(h,v_{ik})} \to \hat{Z}_m^{(h,\infty)},$$

$$\hat{Y}_m^{(h,v_{ik})} = 2Z_m^{(h,v_{ik})} - \hat{X}_m^{(h,v_{ik})} \to \hat{Z}_m^{(h,\infty)},$$

as $v_{ik} \to \infty$ pathwise uniformly on $[M_1, M_2]$. Taking the limit $v_{ik} \to \infty$ on both sides of equation (17), we get

$$\hat{Z}_{m+1}^{(h,\infty)} = \hat{Z}_m^{(h,\infty)} + \frac{1}{2}h[e^{-\pi_{m+1}} f_1(e^{\pi_{m+1}}\hat{Z}_{m+1}^{(h,\infty)}) + e^{\pi_{m+1}} f_2(e^{-\pi_{m+1}}\hat{Z}_{m+1}^{(h,\infty)})] + \frac{1}{2}\sum_{j=1}^{n}(\alpha_j + \beta_j)\Delta W_m^{(j)}$$

which means that $\hat{Z}_m^{(h,\infty)}$ is a complete solution of the drift-implicit Euler-Maruyama (16) applied to the averaged stochastic differential equation (15). Then $\hat{Z}_m^{(h,\infty)}$ is a discrete time stationary stochastic process. Also, the drift of the averaged stochastic differential equation (15) satisfies a dissipative one-sided Lipschitz condition. So each of equations (15) and (16) has a unique stochastic stationary solution. Then $\hat{Z}_m^{(h,\infty)}$ is the numerical scheme of stochastic stationary solution. Finally, we can see that pathwise all possible subsequences have the same limit and so by Lemma 2.2 in [5] every full sequence $\hat{Z}_m^{(h,v_i)}$ converges to $\hat{Z}_m^{(h,\infty)}$ as $v_i \to \infty$.

□

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