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On Tetranacci Function and Tetranacci Numbers

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Abstract

The Fibonacci sequence is one of the most fascinating sequences in the field of mathematics. It has been researched on a huge scale in the realm of Number Theory, Applied Mathematics, Computer Science, Biology and Physics. Through this paper, we conduct an indepth study of Tetranacci functions on the real numbers $\mathbb{R}$; i.e., functions $\phi : \mathbb{R} \Rightarrow \mathbb{R}$ such that
$\phi(y+4) = \phi(y+3) + \phi(y+2) + \phi(y+1) + \phi(y), \forall y \in \mathbb{R}$.
We obtain results for the Tetranacci functions using the concept of f-even and f-odd functions. We also deduce that, if ϕ is a Tetranacci function, then $\lim_{y\to\infty} \frac{\phi(y+n+i)}{\phi(y+n)} = \alpha^i$, where α is a positive real root of the equation $y^4 - y^3 - y^2 - y - 1 = 0$ for which $\alpha > 1$.

1 Introduction

Fibonacci numbers [12, 13] were discovered by the Italian mathematician Leonardo Pisano Bigollo and appeared in his book Liber Abaci. They are one of the most well known numbers in the stream of mathematics. Their properties have generated many amazing possibilities. It is interesting to note that the Fibonacci sequence possesses many generalizations that can be

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applied to each and every area of science. A Fibonacci sequence is present in nature. We can visualize it in an arrangement of leaves, in the floral patterns and in the pattern of a pineapple. The Fibonacci numbers can be applied to the growth of every living being like a single cell, a grain of wheat, a bee hive, or a human body.

1.1 Fibonacci Sequence

The Fibonacci sequence is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1. \tag{1}$$

1.1.1 Binet's formula for Fibonacci sequence

Binet's formula for the Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{2}$$

where α and β are the roots of the polynomial $x^2 - x - 1 = 0$.

1.2 Fibonacci Function

A function f defined on the real numbers $\mathbb{R}$ is said to be a Fibonacci Function if it satisfies the following relation

$$f(x+2) = f(x+1) + f(x), \forall x \in \mathbb{R}. \tag{3}$$

First and foremost, Elmore [8] and Spickerman [14] discovered useful results of the Fibonacci function. Parker [3] researched, at length, the derivation of the Fibonacci functions. In [6], an additional dimension of the Fibonacci sequence was promoted. In the above mentioned research study, a Fibonacci function $f : \mathbb{R} \to \mathbb{R}$ was defined by

$$f(x+n) = F_n f(x+1) + F_{n-1} f(x), \forall x \in \mathbb{R} \tag{4}$$

The limit value of the Fibonacci function is approximately 1.618. Many renowned researchers such as Sroysang, Han, Kim, and Neggers, et al. [2, 5] have devoted their study to the analysis of many properties of the Fibonacci function. They highlighted some properties on the Fibonacci functions with period k using the concept of f-even and f-odd functions with period k.

In addition, they developed some properties on the odd Fibonacci functions with period k.

Fibonacci numbers have been scrutinized at a large scale and authors always attempt to propose some directions to mathematical calculations using these special numbers. A lot of papers dealing with a variety of generalizations of the Fibonacci sequence have been published. First, in The Fibonacci Quarterly, Horadam [1] was one of the pioneers to begin this research when he changed the two initial terms of the Fibonacci sequence from 0 and 1 to arbitrary integers H_0 and H_1. He discovered that there are two basic ways in which this sequence can be generalized: either the recurrence relation can be modified or the initial terms can be changed. Of these two changes, a change in the recurrence relation seems to lead to greater complexity in the properties of the resulting sequence. One of these alterations goes through to the Tribonacci numbers. In fact, Tribonacci numbers have been initially defined by Feinberg [9] in 1963.

1.3 Tribonacci Sequence

Similar to the Fibonacci sequence with two predetermined terms, the Tribonacci sequence starts with three pre-determined terms and each term afterwards is the sum of the preceding three terms.

$$M_n = M_{n-1} + M_{n-2} + M_{n-3}, M_0 = M_1 = 0, M_2 = 1. \tag{5}$$

The first few Tribonacci numbers are 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768,...

1.3.1 Binet's formula for Tribonacci sequence

Binet's formula for Tribonacci sequence [10] is given by

$$M_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\beta-\gamma)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\gamma-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\alpha-\beta)}, \tag{6}$$

where α, β and γ are the three roots of the polynomial $x^3 - x^2 - x - 1 = 0$.

1.4 Tribonacci Function

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tribonacci function if it satisfies the following relation

$$f(x) = f(x-1) + f(x-2) + f(x-3), \forall x \in \mathbb{R}. \tag{7}$$

Recently, Maryam and Madjid [10] have developed the notion of Tribonacci functions using the concept of f-even and f-odd functions. They concluded that, if f is a Tribonacci function, then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \beta$, where β is s positive real root of the $x^3 - x^2 - x - 1 = 0$ for which β is greater than 1.

2 Tetranacci Numbers and Tetranacci Function

The second variation of the Fibonacci sequence is a sequence in which each number is the sum of the preceding four numbers, known as Tetranacci sequence.

2.1 Tetranacci Sequence

A Tetranacci sequence P_n is defined by

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4}, P_0 = P_1 = 0, P_2 = P_3 = 1. \tag{8}$$

The first few Tetranacci numbers are 0,0,1,1,2,4,8,15,29,56,108,208...

2.1.1 Binet's Formula for the Tetranacci Sequence

Binet's Formula for the Tetranacci Sequence [7, 11] is given by

$$T_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \tag{9}$$

where α, β, γ and δ are the four roots of the polynomial $x^4-x^3-x^2-x-1=0$
This results in $\alpha = 1.92756$ (Tetranacci constant), β=-0.774084, γ=-0.0764+.08147 i and δ=-0.0764-0.814 i

2.2 Quotient of Tetranacci Numbers

Zaveri, et al. [11] have given that $\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \alpha$ and, in general, $\lim_{n\to\infty} \frac{P(n+i)}{P(n)} = \alpha^i$.

2.3 Tetranacci Function

Definition 2.1. *A function ϕ defined on the real numbers $\mathbb{R}$ is called a Tetranacci function if*

$$\phi(y) = \phi(y-1) + \phi(y-2) + \phi(y-3) + \phi(y-4), \forall y \in \mathbb{R}. \tag{10}$$

2.4 Example

Let $\phi(y) = k^y$ be a Tetranacci function on $\mathbb{R}$, where k is positive. Then

$$k^{y+4} = \phi(y+4) = \phi(y+3) + \phi(y+2) + \phi(y+1) + \phi(y) \tag{11}$$

$$\Rightarrow k^{y+4} = k^{y+3} + k^{y+2} + k^{y+1} + k^y \tag{12}$$

$$\Rightarrow k^4 = k^3 + k^2 + k + 1. \tag{13}$$

Since $k > 0$, $k^4 - k^3 - k^2 - k - 1 = 0$, and $k = \alpha$ such that α is the positive root of the equation $k^4 - k^3 - k^2 - k - 1 = 0$ and $\alpha > 1$, we have $f(y) = k^y$ is a Tetranacci function on $\mathbb{R}$.

2.5 Example

Let $(p_n)_{-\infty}^{\infty}$, $(q_n)_{-\infty}^{\infty}$, $(r_n)_{-\infty}^{\infty}$ and $(s_n)_{-\infty}^{\infty}$ be full Tetranacci sequences. We define a function

$\phi(y) = p_{[y]} + q_{[y]}t + r_{[y]}t^2 + s_{[y]}t^3$, where $t = y - [y] \in (0, 1)$. Then

$\phi(y+4) = p_{[y]+4} + q_{[y]+4}t + r_{[y]+4}t^2 + s_{[y]+4}t^3$

$= \left[p_{[y]+3} + p_{[y]+2} + p_{[y]+1} + p_{[y]}\right] + \left[q_{[y]+3} + q_{[y]+2} + q_{[y]+1} + q_{[y]}\right] t + \left[r_{[y]+3} + r_{[y]+2} + r_{[y]+1} + r_{[y]}\right] t^2 + \left[s_{[y]+3} + s_{[y]+2} + s_{[y]+1} + s_{[y]}\right] t^3$

$= \left[p_{[y]+3} + q_{[y]+3}t + r_{[y]+3}t^2 + s_{[y]+3}t^3\right] + \left[p_{[y]+2} + q_{[y]+2}t + r_{[y]+2}t^2 + s_{[y]+2}t^3\right] + \left[p_{[y]+1} + q_{[y]+1}t + r_{[y]+1}t^2 + s_{[y]+1}t^3\right] + \left[p_{[y]} + q_{[y]}t + r_{[y]}t^2 + s_{[y]})t^3\right]$

$= \phi(y+3) + \phi(y+2) + \phi(y+1) + \phi(y) \forall y \in \mathbb{R}$,

which shows that ϕ is a Tetranacci function.

2.6 Theorem

Proposition 1. *Let $\phi(y)$ be Tetranacci function. Then $\psi(y) = \phi(y + t + t^2 + t^3)$, where $y, t \in \mathbb{R}$ is also a Tetranacci function.*

Proof. We have $\psi(y) = \phi(y+t+t^2+t^3)$. Then $\psi(y+4) = \phi(y+4+t+t^2+t^3)$
$= \phi(y+3+t+t^2+t^3)+\phi(y+2+t+t^2+t^3)+\phi(y+1+t+t^2)+\phi(y+t+t^2+t^3)$
$= \psi(y + 3) + \psi(y + 2) + \psi(y + 1) + \psi(y)$,
which implies that $\psi(y) = \phi(y+t+t^2+t^3)$ is also a Tetranacci function. □

2.7 Example

Since $\phi(y) = \alpha^y$ is a Tetranacci function $\psi(y) = \phi(y + t + t^2 + t^3)$ which implies that $\alpha^{(y+t+t^2+t^3)}$ =$\alpha^{(t+t^2+t^3)}\alpha^y = \alpha^{(t+t^2+t^3)}$. Consequently, $\phi(y)$ is also a Tetranacci function.

2.8 Theorem

Let $\phi(y)$ be a Tetranacci function and let $a_n, b_n, c_n and\ d_n$ be sequence of Tetranacci numbers $P_0 = 0 = P_1, P_2 = 1 = P_3$. Then

$$\phi(y + n) = a_n\phi(y + 3) + b_n\phi(y + 2) + c_n\phi(y + 1) + d_n\phi(y), \tag{14}$$

where

$$a_n = P_{n-1} \tag{15}$$
$$b_n = P_{n-2} + P_{n-3} + P_{n-4} \tag{16}$$
$$c_n = P_{n-2} + P_{n-3} \tag{17}$$
$$d_n = P_{n-2}, \tag{18}$$

$\forall\ y \in \mathbb{R}, n \geq 4$.

Proof. There is a close connection between induction and recursive definitions. Induction is the most natural way to express recursive processes. Here we extend this theme by using the induction method to prove this result.
If $n = 4$, then $\phi(y+4) = \phi(y+3)\phi(y+2)+\phi(y+1)+\phi(y) = a_4\phi(y + 3) + b_4\phi(y + 2) + c_4\phi(y + 1)$
Now we will show that statement is true for $n > 4$.
Assuming that the result is true for $n = k$,
$\phi(y + k) = a_k\phi(y + 3) + b_k\phi(y + 2) + c_k\phi(y + 1) + d_k\phi(y)$

$$= P_k\phi(y + 3) + [P_{k-1} + P_{k-2} + P_{k-3}]\,\phi(y + 2) + [P_{k-1} + P_{k-2}]\,\phi(y + 1) + P_{k-1}\phi(y) \tag{19}$$

For $n = k + 1$, we have
$\phi(y + k + 1) = a_{k+1}\phi(y + 3) + b_{k+1}\phi(y + 2) + c_{k+1}\phi(y + 1) + d_{k+1}\phi(y)$
$\phi(y+k+1) = P_{k+1}\phi(y+3)+[P_k + P_{k-1} + P_{k-2}]\,\phi(y+2)+[P_k + P_{k-1}]\,\phi(y+1) + P_k\phi(y)$
Adding $\phi(y + k - 1)$, $\phi(y + k - 2)$ and $\phi(y + k - 3)$ to both sides in (19), we have
$\phi(y + k) + \phi(y + k - 1) + \phi(y + k - 2) + \phi(y + k - 3) = P_k\phi(y + 3) + [P_{k-1} + P_{k-2} + P_{k-3}]\,\phi(y + 2)$
$+ [P_{k-1} + P_{k-2}]\,\phi(y+1)+P_{k-1}\phi(y)+\phi(y+k-1)+\phi(y+k-2)+\phi(y+k-3)$
By definition

$$\phi(y + k - 1) = a_{k-1}\phi(y + 3) + b_{k-1}\phi(y + 2) + c_{k-1}\phi(y + 1) + d_{k-1}\phi(y) \tag{20}$$

$$\phi(y + k - 2) = a_{k-2}\phi(y + 3) + b_{k-2}\phi(y + 2) + c_{k-2}\phi(y + 1) + d_{k-2}\phi(y) \tag{21}$$

$$\phi(y + k - 3) = a_{k-3}\phi(y + 3) + b_{k-3}\phi(y + 2) + c_{k-3}\phi(y + 1) + d_{k-3}\phi(y) \tag{22}$$

which implies
$\phi(y+k+1) = [a_k + a_{k-1} + a_{k-2} + a_{k-3}]\,\phi(y+3)+[b_k + b_{k-1} + b_{k-2} + b_{k-3}]\,f(x+2) + [c_k + c_{k-1} + c_{k-2} + c_{k-3}]\,\phi(y + 1) + [d_k + d_{k-1} + d_{k-2} + d_{k-3}]\,\phi(y)$
and so $\phi(y + k + 1) = a_{k+1}\phi(y + 3) + b_{k+1}\phi(y + 2) + c_{k+1}\phi(y + 1) + d_{k+1}\phi(y)$
Hence the theorem follows.

□

2.9 Corollary

Let P_n be the sequence of Tetranacci numbers such that $P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4}, n \geq 4$, $P_0 = 0 = P_1, P_2 = 1 = P_3$ and $\alpha > 1$ is the positive root of the equation $y^4 - y^3 - y^2 - y - 1 = 0$. Then

$$\alpha^n = a_n\alpha^3 + b_n\alpha^2 + c_n\alpha + d_n \tag{23}$$

Proof. We know $\phi(y) = \alpha^y$ is Tetranacci function. Hence

$$\begin{aligned}\alpha^{(y+n)} = \phi(y + n) &= a_n\phi(y + 3) + b_n\phi(y + 2) + c_n\phi(y + 1) + d_n\phi(y)\\ &= a_n\alpha^{(y+3)} + b_n\alpha^{(y+2)} + c_n\alpha^{(y+1)} + d_n\alpha^y.\end{aligned}$$

As a result $\alpha^n = a_n\alpha^3 + b_n\alpha^2 + c_n\alpha + d_n$. □

3 f-even and f-odd functions

In 2012, Han, et al., [5], defined f-even and f-odd functions as follows:

Let $a(x)$ be a real valued function of real variables such that if $a(x)h(x) = 0$ and $h(x)$ is continuous. Then $h(x) \equiv 0$.
$f(x)$ is said to be an f-even function if $a(x+1) = a(x)$ and f-odd function if $a(x+1) = -a(x)$, $\forall x \in \mathbb{R}$.

3.1 Theorem

Let $f(y)$=$\mu(y)\lambda(y)$ be a function, where $\mu(y)$ is an f- even function and $\lambda(y)$ is continuous function. Then $f(y)$ is a Tetranacci function iff $\lambda(y)$ is a Tetranacci function.

Proof. If $f(y)$ is a Tetranacci function, then
$\mu(y)\lambda(y+4) = \mu(y+4)\lambda(y+4) = f(y+4)$.
So
$f(y+4) = f(y+3) + f(y+2) + f(y+1) + f(y)$
$= \mu(y)\left[\lambda(y+3) + \lambda(y+2) + \lambda(y+1) + \lambda(y)\right]$.
Hence
$\mu(y)[\lambda(y+4) - \lambda(y+3) - \lambda(y+2) - \lambda(y+1) - \lambda(y)] \equiv 0$.
Therefore
$\lambda(y+4) - \lambda(y+3) - \lambda(y+2) - \lambda(y+1) - \lambda(y) \equiv 0$.
As a result
$\lambda(y+4) = \lambda(y+3) + \lambda(y+2) + \lambda(y+1) + \lambda(y)$.
Consequently
$\lambda(y)$ is Tetranacci function.
Conversely, if $\lambda(y)$ is any Tetranacci function, then $\lambda(y+4) = \lambda(y+3) + \lambda(y+2) + \lambda(y+1) + \lambda(y)$.
So
$\mu(y)\lambda(y+4) = \mu(y)\left[\lambda(y+3) + \lambda(y+2) + \lambda(y+1) + \lambda(y)\right]$
$= \mu(y+3)\lambda(y+3) + \mu(y+2)\lambda(y+2) + \mu(y+1)\lambda(y+1) + \mu(y)\lambda(y)$
$= f(y+3) + f(y+2) + f(y+1) + f(y)$.
Hence f is also a Tetranacci function. □

4 Ratio of Tetranacci function

4.1 Theorem

If $\phi(y)$ is a Tetranacci function, then the limit of quotient $\frac{\phi(y+n+i)}{\phi(y+n)}$ exists.

Proof. Recently, we have constructed the result
$\phi(y+n) = a_n\phi(y+3) + b_n\phi(y+2) + c_n\phi(y+1) + d_n\phi(y)$
So, we can write

$$\lim_{y\to\infty} \frac{\phi(y+n+i)}{\phi(y+n)} = \lim_{n\to\infty} \frac{a_{n+i}\phi(y+3)+b_{n+i}\phi(y+2)+c_{n+i}\phi(y+1)+d_{n+i}\phi(y)}{a_n\phi(y+3)+b_n\phi(y+2)+c_n\phi(y+1)+d_n\phi(y)}$$

$$= \lim_{n\to\infty} \frac{P_{n+i}\phi(y+3)+\{P_{n+i-1}+P_{n+i-2}+P_{n+i-3}\}\phi(y+2)+\{P_{n+i-1}+P_{n+i-2}\}\phi(y+1)+P_{n+i-1}\phi(y)}{P_n\phi(y+3)+\{P_{n-1}+P_{n-2}+P_{n-3}\}\phi(y+2)+\{P_{n-1}+P_{n-2}\}\phi(y+1)+P_{n-1}\phi(y)}$$

$$= \lim_{n\to\infty} \frac{\frac{P_{n+i}}{P_n}\phi(y+3)+\frac{P_{n+i-1}+P_{n+i-2}+P_{n+i-3}}{P_n}\phi(y+2)+\frac{P_{n+i-1}+P_{n+i-2}}{P_n}\phi(y+1)+\frac{P_{n+i-1}}{P_n}f(x)}{P_n\phi(y+3)+\{P_{n-1}+P_{n-2}+P_{n-3}\}\phi(y+2)+\{P_{n-1}+P_{n-2}\}\phi(y+1)+P_{n-1}\phi(y)}$$

we know that

$$\lim_{n\to\infty} \frac{P_{n+i}}{P_n} = \alpha^i.$$

Hence

$$\lim_{y\to\infty} \frac{\phi(y+n+i)}{\phi(y+n)} = \frac{\alpha^i\phi(y+3)+\{\alpha^{i-1}+\alpha^{i-2}+\alpha^{i-3}\}\phi(y+2)+\{\alpha^{i-1}+\alpha^{i-2}\}\phi(y+1)+\alpha^{i-1}\phi(y)}{\phi(y+3)+\{\alpha^{-1}+\alpha^{-2}+\alpha^{-3}\}\phi(y+2)+\{\alpha^{-1}+\alpha^{-2}\}\phi(y+1)+\alpha^{-1}\phi(y)}$$
$= \alpha^i.$

□

5 Conclusion

In this paper, we have extended the concept of a Fibonacci function and a Tribonacci function to that of the Tetranacci function for the Tetranacci numbers. We obtained some fundamental properties of Tetranacci functions using the induction method and Binet's form. We conclude our paper with a very interesting question: is it possible to do the same for Pentanacci or higher orders? We hope that our paper provides leads in that direction.

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