International Journal of Mathematics and
Computer Science, **15**(2020), no. 3, 809–820

On β-Absorbing Submodules

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(Received February 19, 2020, Revised March 4, 2020,
Accepted April 3, 2020)

Abstract

We introduce the concept of β-absorbing submodules of unital left R-module over a commutative ring with nonzero identity. Many properties and characterizations of β-absorbing submodules and weakly β-absorbing submodules are given.

1 Introduction

In this paper all rings are commutative rings with nonzero identity and all modules are unital. A proper submodule P of an R-module M is called prime if for any $r \in R$ and $m \in M$ such that $rm \in P$, then either $r \in (P : M)$ or $m \in P$. Note that the ideal $\{r \in R \mid rM \subseteq P\}$ will be denoted by $(P : M)$ where P is a submodule of an R-module M.

In [2], the definition of β-prime submodule was introduced. A proper submodule P of M is called β-prime if for any element $r \in R$ and $m \in M$ such that $rm \in P$, we have $r + r \in (P : M)$ or $m + m \in P$. So every prime submodule is β-prime. Furthermore, the characterization of β-prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$ is also given.

On the $\mathbb{Z}$-module $\mathbb{Z}$, $n\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ if and only if $n = 0$ or $n = 8$ or n is a prime number or $n = 2p$ where p is a prime number.

Key words and phrases: β-prime submodule, β-absorbing submodule, weakly β-absorbing submodule.
AMS (MOS) Subject Classifications: 16D99.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

The characterization of β-prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$ obtains that β-prime submodules need not to be prime submodules.

A. Y. Darani and F. Soheilnia in [1] introduced the concept of 2-absorbing submodule that a proper submodule P of an R-module M is called a 2-absorbing submodule if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P$, we have $rs \in (P : M)$ or $rm \in P$ or $sm \in P$. The following is the characterization of 2-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}$.

In [4], on the $\mathbb{Z}$-module $\mathbb{Z}$, $n\mathbb{Z}$ is a 2-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0$ or n is a prime number or $n = pq$ where p and q are prime numbers.

Example 1.1. We see that $8\mathbb{Z}$ is β-prime but is not 2-absorbing. However, $9\mathbb{Z}$ is 2-absorbing but is not β-prime.

In this article, we introduce a slightly different notion of prime submodule and call it β-absorbing submodule. In section 2, we investigate characterizations of β-absorbing submodules in arbitrary module. In section 3, we apply the notion of β-absorbing submodules to β-absorbing ideals. Some properties of β-absorbing ideals are given. In section 4, we introduce weakly β-absorbing submodules as a generalization of β-absorbing submodules and prove that if P is a weakly β-absorbing submodule of an R-module M and $(P : M)^2\beta(P) \neq 0$, then P is a β-absorbing submodule of M.

First of all, we start with the definition of β-absorbing submodules.

Definition 1.2. *A proper submodule P of a left R-module M is called a β-absorbing submodule of M if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.*

Example 1.3. Every β-prime submodule of a module is a β-absorbing submodule. However, $9\mathbb{Z}$ is β-absorbing but is not β-prime.

Example 1.4. Every 2-absorbing submodule of a module is a β-absorbing submodule. However, $8\mathbb{Z}$ is β-absorbing but is not 2-absorbing submodule.

Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and let $n \in \mathbb{Z}$. Note that $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$ if and only if for all $r, s, m \in \mathbb{Z}$, if $n \mid rsm$, then $n \mid 2rs$ or $n \mid 2rm$ or $n \mid 2sm$.

Example 1.5. Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and p is a prime number. Then $8p\mathbb{Z}$ and $32\mathbb{Z}$ are β-absorbing submodules of $\mathbb{Z}$.

Next is the characterization of β-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}$.

Example 1.6. Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and let $n \in \mathbb{Z}$. Then $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0$ or $n = 32$ or n is a prime number or $n = pq$ where p and q are prime numbers or $n = 2^3p$ where p is prime number or $n = 2pq$ where p and q are prime numbers.

Proof. Let $n\mathbb{Z}$ be a β-absorbing submodule of $\mathbb{Z}$. Assume that $n \neq 0$ and $n \neq 32$ and n is not a prime number and $n \neq pq$ for all prime number p and q and $n \neq 2^3p$ for all prime number p. Then $n = ab$ where a and b are integers with $1 < a, b < n$. Moreover, a is not a prime number or b is not a prime number. We assume that a is not a prime number. Then $a = cd$ where c and d are integers with $1 < c, d < a$. Hence $n = cdb$. Since $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$, $n \mid 2cd$ or $n \mid 2cb$ or $n \mid 2db$. Since $n = cdb$, $b = 2$ or $d = 2$ or $c = 2$. This shows that $n = 2cd$ or $n = 2cb$ or $n = 2db$. Without loss of generality, we assume that $n = 2cd$. Suppose for a contradiction that c is not a prime number or d is not a prime number. First, we consider c is not a prime number. Then $c = c_1c_2$ where c_1 and c_2 are integers with $1 < c_1, c_2 < c$. Then $n = (2c_1)c_2d$. Since $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$, $n \mid 2(2c_1)c_2$ or $n \mid 2(2c_1)d$ or $n \mid 2c_2d$. Since $n = 2c_1c_2d$, $d = 2$ or $c_2 = 2$.

Case 1. Assume that $d = 2$. Then $n = 4c_1c_2$. Thus $n \mid 8c_1$ or $n \mid 8c_2$. Hence $c_2 = 2$ or $c_1 = 2$. If $c_2 = 2$, then $n = 2^3c_1$. By our assumption, c_1 is not a prime number. Hence $c_1 = c_3c_4$ where c_3 and c_4 are integers with $1 < c_3, c_4 < c_1$. Hence $n = 8c_3c_4$. Since $n\mathbb{Z}$ is a β-absorbing submodule of $\mathbb{Z}$, $n \mid 16c_3$ or $n \mid 16c_4$. Then $c_4 = 2$ or $c_3 = 2$. This means $n = 16c_3$ or $n = 16c_4$. If $n = 16c_3 = 4 \cdot 4 \cdot c_3$, then $n \mid 32$. So $c_3 = 2$. Hence $n = 32$ which is a contradiction. Similarly, if $n = 16c_4$, then $n = 32$ which leads to a contradiction. Analogously, if $c_1 = 2$, then we have the similar of contradiction.

Case 2. Assume that $c_2 = 2$. By the similar proof of Case 1, we have the same arguments.

Also, if we consider d is not a prime number, then the above process imply the same contradiction. Hence $n = 2pq$ where p and q are prime numbers. □

Note that $a\mathbb{Z} \cap b\mathbb{Z} = lcm(a, b)\mathbb{Z}$.

Example 1.7. The intersection of two 2-absorbing submodules need not to be 2-absorbing submodule. For example, $4\mathbb{Z}$ and $6\mathbb{Z}$ are 2-absorbing submodules of $\mathbb{Z}$ but $4\mathbb{Z} \cap 6\mathbb{Z} = 12\mathbb{Z}$ is not a 2-absorbing submodules of $\mathbb{Z}$.

Example 1.8. The intersection of two β-prime submodules need not to be β-absorbing submodule. For example, $8\mathbb{Z}$ and $3\mathbb{Z}$ are β-prime submodules of $\mathbb{Z}$ but $8\mathbb{Z} \cap 3\mathbb{Z} = 24\mathbb{Z}$ is not a β-absorbing submodules of $\mathbb{Z}$.

2 Characterizations of β-absorbing submodules

Let $(G,+)$ be a group and $H \subseteq G$. We denote the symbol $\beta(H)$ by $\{h+h \mid h \in H\}$ and $\alpha(H)$ by $\{h \mid h+h \in H\}$. It is clear that $\beta(H) \subseteq H \subseteq \alpha(H)$. If I is an ideal of R, then $\alpha(I)$ and $\beta(I)$ are ideals of R. Furthermore, if N is a submodule of M, then $\alpha(N)$ and $\beta(N)$ are submodules of M.

Proposition 2.1. *A proper submodule P of an R-module M is β-absorbing if and only if for all submodules N of M and for all $a, b \in R$,*

$$\text{if } abN \subseteq P, \text{ then } ab+ab \in (P:M) \text{ or } a\beta(N) \subseteq P \text{ or } b\beta(N) \subseteq P.$$

Proof. Firstly, assume that P is a β-absorbing submodule of an R-module M. Let $a, b \in R$ and N be a submodule of M such that $abN \subseteq P$. Suppose that $a\beta(N) \nsubseteq P$ and $b\beta(N) \nsubseteq P$. There are elements $n_1, n_2 \in N$ such that $a(n_1+n_1) \notin P$ and $b(n_2+n_2) \notin P$. Since $ab(n_1+n_2) \in abN$ and $abN \subseteq P$, $ab(n_1+n_2) \in P$. Since P is β-absorbing submodule, $ab+ab \in (P:M)$ or $a(n_1+n_2+n_1+n_2) \in P$ or $b(n_1+n_2+n_1+n_2) \in P$. Assume that $a(n_1+n_2+n_1+n_2) \in P$. Since $a(n_1+n_1) \notin P$, $a(n_2+n_2) \notin P$. Since $abn_2 \in P$ and P is β-absorbing submodule, $ab+ab \in (P:M)$. Assume that $b(n_1+n_2+n_1+n_2) \in P$. We know that $b(n_2+n_2) \notin P$. Then $b(n_1+n_1) \notin P$. Since $abn_1 \in P$ and P is β-absorbing submodule, $ab+ab \in (P:M)$. Conversely, let $r, s \in R$ and $m \in M$ be such that $rsm \in P$. Then $rs(Rm) = R(rs)m \subseteq RP \subseteq P$. Hence $rs+rs \in (P:M)$ or $r\beta(Rm) \subseteq P$ or $s\beta(Rm) \subseteq P$. Since $m \in Rm$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$. Therefore P is a β-absorbing submodule of M. □

Lemma 2.2. *Let I be an ideal of R and P be a β-absorbing submodule of M. If $a \in R$ and $m \in M$ and $aIm \subseteq P$, then $a(m+m) \in P$ or $I(m+m) \subseteq P$ or $aI \subseteq \alpha((P:M))$.*

Proof. Let $a \in R$ and $m \in M$ and $aIm \subseteq P$. Suppose that $a(m+m) \notin P$ and $aI \nsubseteq \alpha((P:M))$. Then $ab+ab \notin (P:M)$ for some $b \in I$. Hence $abm \in P$. Since P is a β-absorbing submodule of M, $b(m+m) \in P$. To show that $I(m+m) \subseteq P$, let $c \in I$. Then $a(b+c)m \in P$. This implies that $a(b+c)+a(b+c) \in (P:M)$ or $(b+c)(m+m) \in P$. Assume that $a(b+c)+a(b+c) \in (P:M)$. Since $ab+ab \notin (P:M)$, $ac+ac \notin (P:M)$. Since $acm \in P$ and P is a β-absorbing submodule of M, $c(m+m) \in P$. Next, assume that $(b+c)(m+m) \in P$. Since $b(m+m) \in P$, $c(m+m) \in P$. This obtains that $I(m+m) \subseteq P$. □

Lemma 2.3. *Let I and J be ideals of R and P be a β-absorbing submodule of M. If $m \in M$ and $IJm \subseteq P$, then $I(m+m) \subseteq P$ or $J(m+m) \subseteq P$ or $IJ \subseteq \alpha((P:M))$.*

Proof. Let $m \in M$ and $IJm \subseteq P$. Suppose that $I(m+m) \nsubseteq P$ and $J(m+m) \nsubseteq P$. By Lemma 2.2, we have the following facts,

$$aJ \subseteq \alpha((P:M)) \text{ for all } a \in I \backslash (P:m+m) \tag{2.1}$$

$$bI \subseteq \alpha((P:M)) \text{ for all } b \in J \backslash (P:m+m) \tag{2.2}$$

Let $a \in I$ and $b \in J$ be such that $a(m+m) \notin P$ and $b(m+m) \notin P$. To shows that $IJ \subseteq \alpha((P:M))$, let $x \in I$ and $y \in J$. By equations (2.1) and (2.2), we have

$$ab, ay, bx \in \alpha((P:M)) \tag{2.3}$$

Since $(a+x)(b+y)m \in P$ and P is a β-absorbing submodule of M, $(a+x)(b+y)+(a+x)(b+y) \in (P:M)$ or $(a+x)(m+m) \in P$ or $(b+y)(m+m) \in P$. Assume that $(a+x)(m+m) \in P$. Then $x(m+m) \notin P$. Thus $xJ \subseteq \alpha((P:M))$. So $xy \in \alpha((P:M))$. If $(b+y)(m+m) \in P$, then $y(m+m) \notin P$. This implies that $yI \subseteq \alpha((P:M))$. Hence $xy \in \alpha((P:M))$. Finally, assume that $(a+x)(b+y)+(a+x)(b+y) \in (P:M)$. We know from equation (2.3) that $ab, ay, bx \in \alpha((P:M))$. Therefore $xy \in \alpha((P:M))$. This proves that $IJ \subseteq \alpha((P:M))$. □

Lemma 2.4. *Let I and J be ideals of R and P be a β-absorbing submodule of M. If $IJN \subseteq P$, then $I(m+m) \subseteq P$ for all $m \in N$ or $J(m+m) \subseteq P$ for all $m \in N$ or $IJ \subseteq \alpha((P:M))$.*

Proof. Assume that $IJN \subseteq P$ and $IJ \nsubseteq \alpha((P:M))$. Suppose for a contradiction that there are $m, m' \in N$ such that $I(m+m) \nsubseteq P$ and $J(m'+m') \nsubseteq P$. By Lemma 2.3, $J(m+m) \subseteq P$ and $I(m'+m') \subseteq P$. Then $IJ(m+m') \subseteq P$. By Lemma 2.3, $I(m+m'+m+m') \subseteq P$ or $J(m+m'+m+m') \subseteq P$. Since $J(m+m) \subseteq P$ and $I(m'+m') \subseteq P$, we have that $I(m+m) \subseteq P$ or $J(m'+m') \subseteq P$ which are contradictions. □

Proposition 2.5. *A proper submodule P of an R-module M is β-absorbing if and only if for all ideals I and J of R and for all submodule N of M,*

$$\text{if } IJN \subseteq P, \text{ then } I\beta(N) \subseteq P \text{ or } J\beta(N) \subseteq P \text{ or } IJ \subseteq \alpha((P:M)).$$

Proof. (→) Assume that P is a β-absorbing submodule of M. Let I and J be ideals of R and N be a submodule of M such that $IJN \subseteq P$ and $IJ \nsubseteq \alpha((P:M))$. By Lemma 2.4, we have $I(m+m) \subseteq P$ for all $m \in N$ or $J(m+m) \subseteq P$ for all $m \in N$. Therefore $I\beta(N) \subseteq P$ or $J\beta(N) \subseteq P$.

(←) This part follows from Proposition 2.1. □

For an element $r \in R$ and a submodule N of M, we will denote a submodule $\{m \in M \mid rm \in N\}$ with the short form N_r.

Proposition 2.6. *Let $r, s \in R$. Assume that P is a submodule of M. Then the following statement are equivalent :*

1. *P is a β-absorbing submodule of M.*
2. *If $rs + rs \notin (P : M)$, then $P_{rs} \subseteq \alpha(P_r) \cup \alpha(P_s)$.*
3. *If $rs + rs \notin (P : M)$, then $P_{rs} \subseteq \alpha(P_r)$ or $P_{rs} \subseteq \alpha(P_s)$.*

Proof. (1) → (2) Assume that P is a β-absorbing submodule of M and $rs + rs \notin (P : M)$. Let $m \in P_{rs}$. Then $rsm \in P$. By our assumption, $r(m+m) \in P$ or $s(m+m) \in P$. Thus $m+m \in P_r$ or $m+m \in P_s$. Therefore $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. This means $m \in \alpha(P_r) \cup \alpha(P_s)$.

(2) → (3) This part follows from the fact that $P_{rs}, \alpha(P_r)$ and $\alpha(P_s)$ are submodules of M.

(3) → (1) Assume that the statement (3) holds. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P$ and $rs + rs \notin (P : M)$. By (3), we have $P_{rs} \subseteq \alpha(P_r)$ or $P_{rs} \subseteq \alpha(P_s)$. Since $rsm \in P$, $m \in P_{rs}$. Hence $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. This implies $r(m+m) \in P$ or $s(m+m) \in P$. Therefore P is a β-absorbing submodule of M. □

Note that for a submodule N of M and an ideal I of R, we denote $(N :_M I)$ by $\{m \in M \mid Im \subseteq N\}$.

Proposition 2.7. *Let P be a submodule of M. Then the following statement are equivalent :*

1. *P is a β-absorbing submodule of M.*
2. *If I is an ideal of R such that $IM \nsubseteq P$, then $(P :_M I)$ is a β-absorbing submodule of M.*
3. *If $r \in R$ such that $rM \nsubseteq P$, then $(P :_M Rr)$ is a β-absorbing submodule of M.*

Proof. (1) → (2) Assume that P is a β-absorbing submodule of M. Let I be an ideal of R such that $IM \nsubseteq P$. Then $(P :_M I) \neq M$. To show that $(P :_M I)$ is a β-absorbing submodule of M, let $r, s \in R$ and $m \in M$ be such that $rsm \in (P :_M I)$. Then $I(rsm) \subseteq P$. This implies from Proposition 2.1 that $rs + rs \in (P : M)$ or $r\beta(Im) \subseteq P$ or $s\beta(Im) \subseteq P$. Since $(P : M) \subseteq ((P :_M I) : M)$ and $\beta(Im) = I(m + m)$, we have $rs + rs \in ((P :_M I) : M)$ or $r(m + m) \in (P :_M I)$ or $s(m + m) \in (P :_M I)$. Therefore $(P :_M I)$ is a β-absorbing submodule of M.

(2) → (3) and (3) → (1) are directly. □

3 Properties of β-prime ideals

Definition 3.1. *A β-absorbing ideal of a ring R is a β-absorbing submodule of an R-module R.*

Let I be a proper ideal of a ring R. Then I is a β-absorbing ideal of R if and only if for each $r, s, t \in R$ such that $rst \in I$, we have $r(s + s) \in I$ or $r(t + t) \in I$ or $s(t + t) \in I$.

Proposition 3.2. *If P is a β-absorbing submodule of an R-module M, then $(P : M)$ is a β-absorbing ideal of R.*

Proof. Assume that P is a β-absorbing submodule of an R-module M. Let $r, s, t \in R$ be such that $rst \in (P : M)$. Suppose that $r(t + t) \notin (P : M)$ and $s(t + t) \notin (P : M)$. Thus $(rst)M \subseteq P$ and there are elements $x_1, x_2 \in M$ such that $r(t+t)x_1 \notin P$ and $s(t+t)x_2 \notin P$. Since $rstx_1 \in P$ and $rstx_2 \in P$, $rst(x_1+x_2) \in P$. Since P is a β-absorbing submodule of M, $rs+rs \in (P : M)$ or $r[t(x_1 + x_2) + t(x_1 + x_2)] \in P$ or $s[t(x_1 + x_2) + t(x_1 + x_2)] \in P$.

Suppose that $r[t(x_1 + x_2) + t(x_1 + x_2)] \in P$. Then $r(t + t)(x_1 + x_2) \in P$. Since $r(t+t)x_1 \notin P$, $r(t+t)x_2 \notin P$. Since $rstx_2 \in P$ and P is a β-absorbing submodule of an R-module M, $rs + rs \in (P : M)$.

Suppose that $s[t(x_1 + x_2) + t(x_1 + x_2)] \in P$. Then $s(t + t)(x_1 + x_2) \in P$. Since $s(t+t)x_2 \notin P$, $s(t+t)x_1 \notin P$. Since $rstx_1 \in P$ and P is a β-absorbing submodule of an R-module M, $rs + rs \in (P : M)$.

Therefore $(P : M)$ is a β-absorbing ideal of R. □

Let R be a ring. The Cartesian product $R \times R$ is a ring under componentwise addition and the multiplication $(a, b) * (c, d) = (ac, ad + bc)$. In the following result, we use the notation $R(+)R$ for this ring.

Proposition 3.3. *If P is a β-absorbing ideal of a ring R, then $P \times R$ is a β-absorbing ideal of $R(+)R$.*

Proof. It is straightforward. □

The following example shows that $I \times J$ may be not β-absorbing ideals even if I and J are β-absorbing ideals.

Example 3.4. We know from Example 1.6 that $8\mathbb{Z}$ and $13\mathbb{Z}$ are β-absorbing ideal of a ring $\mathbb{Z}$. Next, we will show that $8\mathbb{Z} \times 13\mathbb{Z}$ is not a β-absorbing ideal of a ring $\mathbb{Z}(+)\mathbb{Z}$. Let $r = (4,3), s = (1,2)$ and $t = (2,1)$. We see that

$$\begin{aligned} rst &= (4,3) * (1,2) * (2,1) = (8,26) \in 8\mathbb{Z} \times 13\mathbb{Z} \\ r(s+s) &= (4,3) * (2,4) = (8,22) \notin 8\mathbb{Z} \times 13\mathbb{Z} \\ r(t+t) &= (4,3) * (4,2) = (16,20) \notin 8\mathbb{Z} \times 13\mathbb{Z} \\ s(t+t) &= (1,2) * (4,2) = (4,10) \notin 8\mathbb{Z} \times 13\mathbb{Z} \end{aligned}$$

Therefore $8\mathbb{Z} \times 13\mathbb{Z}$ is not a β-absorbing ideal of a ring $\mathbb{Z}(+)\mathbb{Z}$.

4 Weakly β-absorbing submodules

Definition 4.1. *A proper submodule P of M is a weakly β-absorbing submodule of M if for each $r, s \in R$ and every $m \in M$ such that $rsm \in P \backslash \{0\}$, we have $rs + rs \in (P : M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.*

Every β-absorbing is a weakly β-absorbing submodule but the converse does not necessarily hold. For example, consider $\mathbb{Z}_{105}$ as a $\mathbb{Z}$-module, $\{\bar{0}\}$ is weakly β-absorbing but is not β-absorbing because $3 \cdot 5 \cdot 7 = \bar{0}$ but $(3 \cdot 5 + 3 \cdot 5)\mathbb{Z}_{105} \nsubseteq \{\bar{0}\}$ and $3(7+7) = 42 \neq \bar{0}$ and $5(7+7) = 70 \neq \bar{0}$.

Let $r \in R$ and N be a submodule of M. Recall that we define $N_r = \{m \in M \mid rm \in N\}$. In case $N = \{0\}$, we denote $\{0\}_r$ by 0_r.

Proposition 4.2. *Let P be a submodule of an R-module M. Then P is a weakly β-absorbing submodule of M if and only if for all $r, s \in R$, if $rs + rs \notin (P : M)$, then $P_{rs} \subseteq \alpha(P_r) \cup \alpha(P_s) \cup 0_{rs}$.*

Proof. ($\rightarrow$) Assume that P is a weakly β-absorbing submodule of M. Let $r, s \in R$ be such that $rs + rs \notin (P : M)$ and $m \in P_{rs}$. Then $rsm \in P$. If $rsm = 0$, then $m \in 0_{rs}$. Assume that $rsm \neq 0$. Since P is a weakly β-absorbing submodule of M and $rs + rs \notin (P : M)$, $r(m+m) \in P$ or

$s(m+m) \in P$. Hence $m \in \alpha(P_r)$ or $m \in \alpha(P_s)$. This implies that $m \in \alpha(P_r) \cup \alpha(P_s) \cup 0_{rs}$.

($\leftarrow$) Assume that for all elements $r, s \in R$, if $rs + rs \notin (P : M)$, then $P_{rs} \subseteq \alpha(P_r) \cup \alpha(P_s) \cup 0_{rs}$. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P \backslash \{0\}$ and $rs + rs \notin (P : M)$. Then $m \in P_{rs}$. By assumptions, $m \in \alpha(P_r) \cup \alpha(P_s)$. Hence $r(m+m) \in P$ or $s(m+m) \in P$. This proves that P is a weakly β-absorbing submodule of M. □

Let $m \in M$ and N be a submodule of M. Then we define the symbol $(N : m) = \{r \in R \mid rm \in N\}$. For a zero submodule $\{0\}$ of M, we denote $(\{0\} : m)$ by $(0 : m)$.

Proposition 4.3. *Let P be a submodule of an R-module M. Then P is a weakly β-absorbing submodule of M if and only if for all $s \in R$ and $m \in M$ with $sm \notin \alpha(P)$, $(P : sm) \subseteq \alpha((P : sM)) \cup \alpha((P : m)) \cup (0 : sm)$.*

Proof. ($\rightarrow$) Assume that P is a weakly β-absorbing submodule of M. Let $r, s \in R$ and $m \in M$ be such that $sm \notin \alpha(P)$ and $r \in (P : sm)$. Then $rsm \in P$. If $rsm = 0$, then $r \in (0 : sm)$. Now, we assume that $rsm \neq 0$. Since P is a weakly β-absorbing submodule and $sm \notin \alpha(P)$, $rs + rs \in (P : M)$ or $r(m+m) \in P$. Then $r \in \alpha((P : sM))$ or $r \in \alpha((P : m))$. Hence $(P : sm) \subseteq \alpha((P : sM)) \cup \alpha((P : m)) \cup (0 : sm)$.

($\leftarrow$) Assume that for all $s \in R$ and $m \in M$ with $sm \notin \alpha(P)$, we have $(P : sm) \subseteq \alpha((P : sM)) \cup \alpha((P : m)) \cup (0 : sm)$. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P \backslash \{0\}$ and $sm \notin \alpha(P)$. Then $r \in (P : sm)$ and $r \notin (0 : sm)$. By assumption, $r \in \alpha((P : sM)) \cup \alpha((P : m))$. Hence $rs + rs \in (P : M)$ or $r(m+m) \in P$. Therefore P is a weakly β-absorbing submodule of M. □

Let R_1 and R_2 be commutative rings with identity and M_i be a unital R_i-module where $i = 1, 2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module under the operation $(r_1, r_2)(m_1, m_1) = (r_1 m_1, r_2 m_2)$ for all $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$. We use these notation for the next two results.

Proposition 4.4. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let N_1 be an R_1-submodule of M_1. Then N_1 is a β-absorbing submodule of M_1 if and only if $N_1 \times M_2$ is a β-absorbing submodule of $M_1 \times M_2$.*

Proof. ($\rightarrow$) Assume that N_1 is a β-absorbing submodule of M_1. Let $r_1, s_1 \in R_1$ and $r_2, s_2 \in R_2$ and $(m_1, m_2) \in M_1 \times M_2$ be such that $(r_1, r_2)(s_1, s_2)(m_1, m_2) \in N_1 \times M_2$. Then $r_1 s_1 m_1 \in N_1$. Since N_1 is a β-absorbing submodule of M_1, we have $r_1 s_1 + r_1 s_1 \in (N_1 : M_1)$ or $r_1(m_1 + m_1) \in N_1$ or $s_1(m_1 + m_1) \in N_1$.

This implies that $(r_1, r_2)(s_1, s_2) + (r_1, r_2)(s_1, s_2) \in (N_1 \times M_1 : M_1 \times M_2)$ or $(r_1, r_2)((m_1, m_2) + (m_1, m_2)) \in N_1 \times M_2$ or $(s_1, s_2)((m_1, m_2) + (m_1, m_2)) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is a β-absorbing submodule of $M_1 \times M_2$.

($\leftarrow$) Assume that $N_1 \times M_2$ is a β-absorbing submodule of $M_1 \times M_2$. Let $r, s \in R$ and $m \in M_1$ be such that $rsm \in N_1$. Then $(r, 0)(s, 0)(m, 0) = (rsm, 0) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a β-absorbing submodule of $M_1 \times M_2$, we have that $(r, 0)(s, 0)+(r, 0)(s, 0) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(r, 0)((m, 0)+(m, 0)) \in N_1 \times M_2$ or $(s, 0)((m, 0)+(m, 0)) \in N_1 \times M_2$. Hence $rs+rs \in (N_1 : M)$ or $r(m + m) \in N_1$ or $s(m + m) \in N_1$. Therefore N_1 is a β-absorbing submodule of M_1. □

Proposition 4.5. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let N_1 be an R_1-submodule of M_1. Consider the following statements.*

1. *N_1 is a β-absorbing submodule of M_1.*
2. *$N_1 \times M_2$ is a β-absorbing submodule of $M_1 \times M_2$.*
3. *$N_1 \times M_2$ is a weakly β-absorbing submodule of $M_1 \times M_2$.*

Then $(1) \to (2) \to (3)$. Moreover, if $M_2 \neq \{0\}$, then $(1), (2)$ and (3) are equivalent.

Proof. It implies from Proposition 4.4 that $(1) \leftrightarrow (2)$. For the part $(2) \to (3)$, this directly follows from definition of β-absorbing submodules and weakly β-absorbing submodules. To prove $(3) \to (1)$, we assume that (3) holds and $y \in M_2\backslash\{0\}$. Let $r, s \in R_1$ and $m \in M_1$ be such that $rsm \in N_1$. Then $(r, 1)(s, 1)(m, y) = (rsm, y) \in \Big(N_1 \times M_2\Big)\backslash\{(0, 0)\}$. Hence $(r, 1)(s, 1) + (r, 1)(s, 1) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(r, 1)((m, y) + (m, y)) \in N_1 \times M_2$ or $(s, 1)((m, y)+(m, y)) \in N_1 \times M_2$. Then $rs+rs \in (N_1 : M_1)$ or $r(m+m) \in N_1$ or $s(m + m) \in N_1$. Therefore N_1 is a β-absorbing submodule of M_1. □

The following example show the condition $M_2 \neq \{0\}$ is necessary for proving the part $(3) \to (1)$ of Proposition 4.5.

Example 4.6. To show that $\{\bar{0}\}$ in $\mathbb{Z}_{45}$ as a $\mathbb{Z}$-module is not a β-absorbing submodule, we consider $3 \cdot 3 \cdot \bar{5} = \bar{45} = \bar{0}$ but $(3 \cdot 3 + 3 \cdot 3)\mathbb{Z}_{45} = 18\mathbb{Z}_{45} \neq \{\bar{0}\}$ and $3 \cdot (\bar{5} + \bar{5}) = \bar{30} \neq \bar{0}$. Hence $\{\bar{0}\}$ is not a β-absorbing submodule of $\mathbb{Z}_{45}$ as a $\mathbb{Z}$-module. Furthermore, $\{\bar{0}\} \times \{0\}$ is a weakly β-absorbing submodule of $\mathbb{Z}_{45} \times \{0\}$ as an $(\mathbb{Z} \times \mathbb{Z})$-module.

Lemma 4.7. *Let P be a weakly β-absorbing submodule of M and $rsm = 0$ where $r, s \in R$ and $m \in M$. If $k \in (P : M)$ and $krm \neq 0$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.*

Proof. Assume that $k \in (P : M)$ and $krm \neq 0$. Then $r(k + s)m = rkm + rsm = rkm \neq 0$. Since P is a weakly β-absorbing submodule of M, $r(k + s) + r(k + s) \in (P : M)$ or $r(m + m) \in P$ or $(k + s)(m + m) \in P$. Since $k \in (P : M)$, $rk \in (P : M)$ and $km \in P$. Hence $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. □

Proposition 4.8. *If P is a weakly β-absorbing submodule of an R-module M and $(P : M)^2\beta(P) \neq 0$, then P is a β-absorbing submodule of M.*

Proof. Assume that P is a weakly β-absorbing submodule of an R-module M and $(P : M)^2\beta(P) \neq 0$. Let $r, s \in R$ and $m \in M$ be such that $rsm \in P$. If $rsm \neq 0$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Assume that $rsm = 0$. We consider two cases as follow.

Case 1. $rs\beta(P) \neq 0$.

Let $n_0 \in P$ be such that $rs(n_0 + n_0) \neq 0$. Then $rs(m + n_0 + n_0) = rs(n_0 + n_0) \neq 0$. Since P is a weakly β-absorbing submodule of M and $n_0 \in P$, $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

Case 2. $rs\beta(P) = 0$.

Subcase 2.1. $s(P : M)m \neq 0$.

Let $t \in (P : M)$ be such that $stm \neq 0$. Then $s(r + t)m = rsm + stm = stm \neq 0$. Since P is a weakly β-absorbing submodule of M, $s(r+t)+s(r+t) \in (P : M)$ or $s(m + m) \in P$ or $(r + t)(m + m) \in P$. Since $t \in (P : M)$, $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

Subcase 2.2. $s(P : M)m = 0$.

Since $(P : M)^2\beta(P) \neq 0$, $kf(n + n) \neq 0$ for some $k, f \in (P : M)$ and $n \in P$.

By Lemma 4.7 obtains that if $krm \neq 0$ or $rfm \neq 0$, then $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Hence P is a β-absorbing submodule of M. Now, we assume that $krm = rfm = 0$. If $kfm \neq 0$, then $(k+r)(f+s)m = kfm+ksm+rfm+rsm = kfm \neq 0$. Since P is a weakly β-absorbing submodule of M, $(k + r)(f + s) + (k + r)(f + s) \in (P : M)$ or $(k + r)(m + m) \in P$ or $(f + s)(m + m) \in P$. Since $k, f \in (P : M)$, $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$. Hence P is a β-absorbing submodule of M. Next, we assume that $kfm = 0$ and consider two subsubcases as follow.

Subsubcase 2.2.1 $kr(n + n) \neq 0$ or $sf(n + n) \neq 0$.

Firstly, assume that $kr(n+n) \neq 0$. Then $r(s+k)(n+n+m) = rs(n+n)+rsm+rk(n+n)+rkm = rk(n+n) \neq 0$. Since P is a weakly β-absorbing submodule of M, $r(s+k)+r(s+k) \in (P:M)$ or $r(n+n+m+n+n+m) \in P$ or $(s+k)(n+n+m+n+n+m) \in P$. Since $k \in (P:M)$ and $n \in P$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$. Similarly, If $sf(n+n) \neq 0$, then $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Subsubcase 2.2.2 $kr(n+n) = sf(n+n) = 0$. Then $(s+k)(r+f)(m+n+n) = srm+sr(n+n)+sfm+sf(n+n)+krm+kr(n+n)+kfm+kf(n+n) = kf(n+n) \neq 0$. Since P is a weakly β-absorbing submodule of M, $(s+k)(r+f)+(s+k)(r+f) \in (P:M)$ or $(s+k)(m+n+n+m+n+n) \in P$ or $(r+f)(m+n+n+m+n+n) \in P$. Since $k, f \in (P:M)$ and $n \in P$, $rs+rs \in (P:M)$ or $r(m+m) \in P$ or $s(m+m) \in P$.

Therefore P is a β-absorbing submodule of M. □

Acknowledgements. This work was supported by Faculty of Science, King Mongkut's Institute of Technology Ladkrabang under Grant no. 2563-02-05-03.

References

[1] A. Darani, F. Soheilnia, 2-absorbing and weakly 2-absorbing submodules, Thai Journal of Mathematics, **9,** (2011), 577–584.

[2] T. Khumprapussorn, On β-prime submodules, The Journal of the Indonesian Mathematical Society, **25,** (2019), 128–138.

[3] Sh. Payrovi, S. Babaei, On the 2-absorbing submodules, Iranian Journal of Mathematical Sciences and Informatics, **10,** (2015), 131–137.

[4] Sh. Payrovi, S. Babaei, On 2-absorbing submodules, Algebra Colloquium (Spec 1), **19,** (2012), 913–920.