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Finite groups in which maximal subgroups of Sylow $p-$subgroups are nearly S-permutable

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Abstract

In a group, A is said to satisfy the N_p-condition when all subgroups of a Sylow p-subgroup P of A are nearly S-permutable in the normalizer of P in A. If A satisfies the N_p-condition for all prime p, then A is called N-group. The objective of this article is to study N-groups. In particular, we explain why every solvable $NSPT$-group is N-group.

We also study groups A in which maximal subgroups of Sylow p-subgroup are nearly S-permutable in the group A. We prove that this class coincides with the class of groups in which maximal subgroups of Sylow p-subgroups are normal in the group A.

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1 Introduction

Throughout this paper, we assume that all groups considered are finite. In [8], Robinson introduced the concept of C_p-property and used this to obtain a description for the class of finite T-groups; i.e., the class of finite groups in which normality is a transitive relation. A group G is said to satisfy the C_p-property if each subgroup of a Sylow p-subgroup P in G is normal $N_G(P)$. The relation between the C_p-property and T-groups was obtained in (Peng[7] and Robinson [8]): A group G is a solvable T-group if and only if G satisfies the C_p-property for all primes p.

Two subgroups X and Y in a group G are said to permute if $XY = YX$. When the subgroup X of G permutes with all subgroups of G, then X is called permutable. The subgroup X that permutes with all Sylow p-subgroups of G is called S-permutable. The class of those groups G which posses a transitive permutability (Sylow quasinormality) is called the PT-groups (PST-groups). The classes of PT-groups and PST-groups were also described through the behavior of their Sylow normalizers using the X_p and Y_p -properties which were established in ([3] and [4]).

Following [2], if H is a subgroup of G, then we say that H is nearly S-permutable in G if for every prime p such that $(p, |H|) = 1$ and for all subgroups K of G containing H, the normalizer $N_K(H)$ contains some p-Sylow subgroup of K. We shall write H nsp G to denote that H is nearly S-permutable in G. The class of groups in which nearly S-permutablity is a transitive relation were introduced in [1] and denoted by $NSPT$-groups. Motivated by the C_p, X_p and Y_p- properties, it seems reasonable to think about the description for the class of solvable $NSPT$-groups in terms of the behavior of their Sylow subgroups. For this reason, we introduced the N_p-property as:

A group G is said to satisfy the N_p-property if every proper subgroup of the Sylow p-subgroup of G is nearly S-permutable in G.

In this article, we shed light on the class of solvable groups satisfying the N_p-property and their relation to the class of $NSPT$-groups.

2 Preliminaries and Basic Facts About Groups

In this section, we list definitions and basic properties of groups that will be needed in the sequel. These results are standard ([5], [10], [6], and [9]).

Definition 2.1. *In a group G, a subgroup H in G is said to be a character-*

istic in G, denoted H char G, if $\varphi(H) = H$ for all $\varphi \in Aut(G)$.

Theorem 2.2. *Let G be any group. Then the following hold:*
(1) If H char G, then $H \trianglelefteq G$.
(2) If H is the unique subgroup of G of a given order, then H char G.
(3) If K char H and $H \trianglelefteq G$, then $K \trianglelefteq G$.

Definition 2.3. *Let G a finite group and Y be a set. An action G on X is a map $* : G \times Y \to Y$ such that*
1. $y.1 = y$ for all $y \in Y$.
2. $(g_1 g_2)(y) = g_1(g_2 y)$ for all $y \in Y$ and all g_1, $g_2 \in G$.
Under these conditions, Y is a G-set.

The following well-known property of group action will be of interest:

Theorem 2.4. *Let G be a p-group of order p^n and let X be a finite G-set. Then $|X| \equiv |X_G|$, where $X_G = \{x \in X : gx = x\}$ for all $g \in G$.*

Definition 2.5. *A group G is nilpotent if it has an upper central series that terminates with G.*

Theorem 2.6. *In a finite group the following are equivalent:*
(1) G is a nilpotent group.
(2) If H is a proper subgroup of G, then H is a proper normal subgroup of $N_G(H)$.
(3) Every Sylow subgroup of G is normal.
(4) G is the direct product of its Sylow subgroups.
(5) G has a normal subgroup of order d for each d dividing the order of G.

We next introduce the solvable groups:

Definition 2.7. *A group G is called solvable if it has a composition series whose factor groups are all abelian.*

Solvable groups have many interesting properties, among which we list:

Theorem 2.8. *Let G be a group. Then the following hold:*
(1) A subgroup and factor group of solvable group are solvable.
(2) G is solvable if and only if both N and G/N are solvable for any normal subgroup N in G.
(3) The class of solvable groups is closed under direct products and extensions.
(4) Every minimal normal subgroup of a solvable group is elementary abelian.

There are many classes between the classes of nilpotent and solvable groups. The class of supersolvable groups is one of the interesting classes between nilpotent and solvable classes.

Definition 2.9. *A group G is supersolvable if there exists a normal series $1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_{s-1} \lhd H_s = G$ such that each quotient group H_{i+1}/H_i for each $i \in \{0, s-1\}$.*

Theorem 2.10. *Let G be a finite group. Then*

(1) The commutator subgroup of a supersolvable group is nilpotent.

(2) Subgroups and quotient groups of supersolvable groups are supersolvable.

(3) Every group of square-free order and every group with cyclic Sylow subgroups are supersolvable.

(4) A finite group is supersolvable if and only if every maximal subgroup has prime index.

(5) Let G have a normal cyclic subgroup H with G/H supersolvable. Then G is supersolvable.

Lemma 2.11. *If A and B are normal subgroups of G, then $\langle A, B\rangle \simeq A \times B \trianglelefteq G$.*

3 Nearly S-permutablity for certain subgroups of the Sylow subgroups

In this section, we discuss some properties that follow from the nearly S-permutablity of certain subgroups.

Lemma 3.1. *Every normal subgroup is nearly S-permutable.*

Proof. This follows from the fact that the normalizer of a normal subgroup is the whole group and so it contains all Sylow p-subgroups of the group. □

Remark 3.2. *A nearly S-permutablity subgroup need not be normal in the group as the following example shows:*

Example 3.3. Let G be the symmetric group S_4. Then $|G| = 2^3.\,3$. If we let H be any of the subgroups of order 6, then H will be nearly S-permutablity in G but H is not normal in G.

Remark 3.4. *If $\pi(G)$ is the set of all prime divisors of the order of the group G and H is a subgroup of G with $\pi(H) = \pi(G)$, then H is nearly S-permutable in G.*

Lemma 3.5. *A group G is nilpotent if and only if every Sylow p-subgroup of G is nearly S-permutable in G.*

Proof. If G is a nilpotent group and $P \in Syl_p(G)$ for every $4p \in \pi(G)$, then, by Theorem 2.6, P must be normal in G. By Lemma 3.1, P is nearly S-quasinormal in G.

Conversely, for every $q \in \pi(G)$, let $Q \in Syl_q(G)$ such that Q is nearly S-permutable in the group G. Since Q is nearly S-permutable in G, $N_G(Q)$ contains every Sylow r-subgroup R for any prime $r \neq q$. But $Q \in N_G(Q)$ and $N_G(Q)$ must be the whole group G. So we have every Sylow subgroup of G is normal in G. Therefore, G must be nilpotent. □

Corollary 3.6. *Let G be a group. Then the following conditions are equivalent:*

(i) Every Sylow p-subgroup of G is nearly S-quasi normal in G.

(ii) Every Sylow p-subgroup of G is normal in G.

Theorem 3.7. *Let G be a group, P be a Sylow p-subgroup of G, and M be a maximal subgroup of P. Then M is a nearly S-permutable subgroup in G if and only if M is normal in G.*

Proof. First, assume that M is a nearly S-permutable subgroup in G. Then the result follows directly from Lemma 3.1.

Assume that M is nearly S-permutable in G. Then $Q \leq N_G(M)$, for every $Q \in Syl_q(G)$ and every prime $q \neq p$. Since P is nilpotent and M is maximal subgroup of P, by Theorem 2.6, we have M is normal in P. Hence $P \leq N_G(M)$ and the normality of M follows. □

Corollary 3.8. *Let G be group in which all maximal subgroups of every Sylow p-subgroup of G is nearly S-permutable in G. Then every Sylow p-subgroup of G is either cyclic or normal in G.*

Proof. Let G be a group in which the maximal subgroup of each Sylow p-subgroup is nearly S-permutable in G. Let $P \in Syl_p(G)$. Then for P we consider two cases:

Case 1: P has a unique maximal subgroup M. Let $x \in P \backslash M$. Then, for the subgroup $\langle x \rangle$ of P, we have $\langle x \rangle \nsubseteq M$. By Zorn's Lemma, $\langle x \rangle$ is contained in some maximal subgroup of P. The uniqueness of M in P forces $\langle x\rangle = P$. Hence, P is cyclic.

Case 2: P has at least two maximal subgroups, say M_1 and M_2. Since M_1 and M_2 are nearly S-quasinormal in G (by Theorem 3.7), M_1 and M_2 are normal in G. Therefore, $P = \langle M_1, M_2\rangle$ must be normal in G by Lemma2.11.

The fact that a group with a unique maximal subgroup must be simple. □

Definition 3.9. *A group G is said to satisfy the N_p-condition if for a Sylow p-subgroup P of G every subgroup H of P is nearly S-permutable in $N_G(P)$. A group G is an N-group if G satisfies the N_p-condition for every prime p.*

Theorem 3.10. *Every solvable $NSPT$-group is an N-group.*

Proof. Let G be a solvable $NSPT$-group. By [1, Theorem B.] G is supersolvable. If G is not N-group, then G does not satisfy the N_p-condition for some prime p. Let us consider two cases for this prime p.

Case 1: p is the largest prime divisor of $|G|$. If $P \in Syl_p(G)$, then $P \trianglelefteq G$. So H is nearly S-permutable in P and P is nearly S-permutable in G. Since G is $NSPT$-group, H is nearly S-permutable in $G = N_G(P)$. Therefore, G satisfies the N_p-condition. This is the desired contradiction.

Case 2: p is not the largest prime divisor of $|G|$. Let q be the largest prime divisor of $|G|$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$ and by induction the group G/Q satisfies the theorem. Therefore, HQ/Q is nearly S-permutable in $N_{G/Q}(PQ/Q) = N_G(P)Q/Q$. From [2, Lemma 2.2.], H is nearly S-permutable in G which is a a contradiction. □

Remark 3.11. *Theorem 3.10 does not hold for non solvable groups as the following example shows:*

Example 3.12. Let $G = A_5 = \langle(1,2,3,4,5),(1,2,3)\rangle$ be the alternating group on 5 letters. Then e and G are the only nearly S-permutable subgroups in G. Hence G is an $NSPT$-group. Now $|G| = 60 = 2^2\cdot 3\cdot 5$. If $P \in Syl_2(G)$, then $P \simeq C_2\times C_2$ and $N_G(P) \simeq A_4$. Let H be any subgroup of order 2 in P. Then H is not nearly S-permutable in $N_G(P)$. Therefore, G does not satisfy the N_2-condition.

Remark 3.13. *The converse of Theorem 3.10 is not true in general as the following example shows:*

Example 3.14. Let $G = S_4$. If $P \in Syl_2(G)$, then $P = N_G(P)$. Hence every subgroup of the p-group P is nearly S-permutable in P. Therefore, G satisfies the N_2-condition. If $Q \in Syl_3(G)$, then $\{e\}$ and Q itself are the only subgroups of Q. So both $\{e\}$ and Q are nearly S-permutable in $N_G(Q)$. Therefore, G satisfies the N_3-condition. Hence, G is an N-group. Note that $\langle(1,2,3)\rangle$ is nearly S-quasinormal in $\langle(1,2,3),(1,2)\rangle \simeq S_3$ and $\langle(1,2,3),(1,2)\rangle \simeq S_3$ is nearly S-permutable in S_4. But $\langle(1,2,3)\rangle$ is not nearly S-permutable in S_4. Therefore, S_4 is not an $NSPT$-group.

4 Future Work

We conclude our paper by posing the following problem for future research:
Problem. What is the relation between the classes $NSPT$-groups, N-groups and the class of groups in which the maximal subgroups of Sylow subgroups are nearly S-permutable in the group?

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