International Journal of Mathematics and
Computer Science, **15**(2020), no. 3, 945–962

On the solvability of a third-order p-Laplacian m-point boundary value problem at resonance on the half-line with two dimensional kernel

O. F. Imaga, S. A. Iyase, S. A. Bishop

Department of Mathematics
Covenant University
Ota, Ogun State, Nigeria

email: imaga.ogbu@covenantuniversity.edu.ng,
samuel.iyase@covenantuniversity.edu.ng,
sheila.bishop@covenantuniversity.edu.ng

(Received March 2, 2020, Accepted April 1, 2020)

Abstract

The solvability of a third-order boundary value problem at resonance on the half-line is considered in this work. By using a semi-projector and the Ge and Ren extension of Mawhin's coincidence degree theory, existence results are established for the problem, where dim ker L=2. An example will be used to illustrate our result.

1 Introduction

Boundary value problems with integral and multi-point boundary conditions in an infinite interval have many real life applications, for instance, in the study of many physical phenomenons such as unsteady flow of fluid through a semi-infinite porous media and radially symmetric solutions of nonlinear elliptic equations. They are also found in plasma physics and the study of drain flows see [1].

Key words and phrases: Completed zeta function, Coincidence degree, half-line, integral boundary value problem, m-point, p-Laplacian, resonance.
AMS (MOS) Subject Classifications: 34B10, 34B15, 34B40.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

If the corresponding homogeneous part of a boundary value problem has a non-trivial solution, then the boundary value problem is said to be at resonance. Resonant problems with both linear and p-Laplacian differential operators have been studied by many authors using Mawhin's coincidence degree theorem [12] and Ge and Ren [3] extension of the coincidence degree theorem see [14, 9, 6, 5, 7, 2, 13, 10].

However, to the best of our knowledge, only few authors in the literature have considered p-Laplacian boundary value problems on the half-line with two dimensional kernel. Motivated by this, we study the solvability for the following p-Laplacian third-order boundary value problem having integral and m-point boundary conditions at resonance on the half-line with two dimensional kernel:

$$(\sigma(t)\varphi_p(u''(t)))' + v(t)w(t, u(t), u'(t), u''(t)) = 0,\ t \in (0, \infty), \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} v(t)u(t)dt,\ u'(0) = \int_0^{\infty} v(t)u'(t)dt,\ \lim_{t\to\infty}(\sigma(t)\varphi_p(u''(t))) = 0, \tag{1.2}$$

where $w : [0, \infty)\times\mathbb{R}^3 \to \mathbb{R}$ is a v-Carathéodory function, $0 \le \xi_i < \infty$, $\xi_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$, $v(t) \in L^1[0, \infty)$, $v(t) > 0$, $\sigma \in C[0, \infty) \cap C^2(0, \infty)$, $\sigma(t) > 0$, $\varphi_q\left(\frac{1}{\sigma}\right) \in L^1[0, \infty)$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$ and $\varphi_q = \varphi_p^{-1}$.

In section 2 of this work, necessary lemmas, theorems and definitions will be given while section 3 will be dedicated to stating and proving the existence result. An example will be given to corroborate the result obtained.

2 Preliminaries

Definition 2.1. *A map $w : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ is v-Carathéodory if the following conditions are satisfied:*

(i) for each $(d, e, f) \in \mathbb{R}^3$, the mapping $t \to w(t, d, e, f)$ is Lebesgue measurable,

(ii) for a.e. $t \in [0, \infty)$, the mapping $(d, e, f) \to w(t, d, e, f)$ is continuous on $\mathbb{R}^3$,

(iii) for each $k > 0$ and $v \in L^1[0, \infty)$, there exists $\psi_k(t) : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty v(t)\psi_k(t)dt < \infty$ such that, for a.e. $t \in [0, \infty)$ and every $(d, e, f) \in [-k, k]$, we have $|w(t, d, e, f)| \le \psi_k(t)$.

Definition 2.2. *[3] Let $(U, \|\cdot\|_U)$ and $(Z, \|\cdot\|_Z)$ be two Banach spaces. The continuous operator $M : U \cap dom\ M \to Z$, is quasi-linear if the following hold*

(i) *$Im\ M = M(U \cap dom\ M)$ is a closed subset of Z;*

(ii) *$\ker M = \{u \in U \cap dom\ M : Mu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.*

Definition 2.3. *[4] Let U be a Banach space and $U_1 \subset U$ a subspace. Let $Q : U \to U_1$ be an operator. Then Q is a semi-projector if*

(i) *$Q^2 = Q$,*

(ii) *$Q(\lambda u) = \lambda Q u$, where $u \in U$, $\lambda \in \mathbb{R}$.*

Let $U_1 = \ker M$ and U_2 be the complement space of U_1 in U, then $U = U_1 \oplus U_2$. Similarly, if Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z, then $Z = Z_1 \oplus Z_2$. Let $P : U \to U_1$ be a projector, $Q : Z \to Z_1$ be a semi-projector and $\Omega \subset U$ an open bounded set with $\theta \in \Omega$ the origin. Also, let N_1 be denoted by N and let $N_\lambda : \overline{\Omega} \to Z$, where $\lambda \in [0, 1]$ is a continuous operator and $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$.

Definition 2.4. *[8] Let U be the space of all continuous and bounded vector-valued functions on $[0, \infty)$ and $X \subset U$. Then X is said to be relatively compact if the following statements hold:*

(i) *X is bounded in U,*

(ii) *all functions from X are equicontinuous on any compact subinterval of $[0, \infty)$,*

(ii) *all functions from X are equiconvergent at ∞; i.e., $\forall\ \epsilon > 0$, $\exists$ a $T = T(\epsilon)$ such that $\|A(t) - A(\infty)\|_{R^n} < \epsilon\ \forall\ t > T$ and $A \in X$.*

Definition 2.5. *[3] Let $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ be a continuous operator. The operator N_λ is said to be M-compact in $\overline{\Omega}$ if there exists a vector subspace $Z_1 \in Z$ such that $\dim Z_1 = \dim U_1$ and a compact and continuous operator $R : \overline{\Omega} \times [0, 1] \to U_2$ such that for $\lambda \in [0, 1]$, the following hold:*

(i) *$(I - Q)N_\lambda(\overline{\Omega}) \subset Im\ M \subset (I - Q)Z$,*

(ii) *$QN_\lambda u = 0 \Leftrightarrow QNu = 0$, $\lambda \in (0, 1)$,*

(iii) $R(\cdot, u)$ *is the zero operator and* $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,

(iv) $M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda$.

Lemma 2.6. *[4] The following are properties of the function* $\varphi_p : \mathbb{R} \to \mathbb{R}$*:*

(i) It is continuous, monotonically increasing and invertible with inverse $\varphi_p^{-1} = \varphi_q$, *where* $q > 1$ *and satisfies* $\frac{1}{p} + \frac{1}{q} = 1$,

(ii) For any $x, y > 0$,

(a) $\varphi_p(x + y) \le \varphi_p(x) + \varphi_p(y)$, *if* $1 < p < 2$,

(b) $\varphi_p(x + y) \le 2^{p-2}(\varphi_p(x) + \varphi_p(y))$, *if* $p \ge 2$.

Theorem 2.7. *[3] Let* $(U, \|\cdot\|_U)$ *and* $(Z, \|\cdot\|_Z)$ *be two Banach spaces and* $\Omega \subset U$ *an open and bounded set. If the following hold*

(B_1) the operator $M : U \cap \text{dom } M \to Z$ *is a quasi-linear,*

(B_2) the operator $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ *is M-compact,*

(B_3) $Mu \ne N_\lambda u$, $\lambda \in [0, 1]$, $u \in \partial\Omega$,

(B_4) $\deg\{JQN, \Omega \cap \ker M, 0\} \ne 0$,

then the equation $Mu = Nu$ *has at least one solution in* $\overline{\Omega}$, *where* $N = N_1$ *and the operator* $J : Z_1 \to U_1$ *is a homeomorphism with* $J(\theta) = \theta$.

Let

$$U = \left\{ u \in C^2[0, \infty) : u, u', \sigma\varphi_p(u'') \in AC[0, \infty), \lim_{t\to\infty} e^{-t}|u^{(i)}(t)| \text{ exist, } i = 0, 1, 2 \right\},$$

with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ defined on U, where $\|u\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|u|$. We claim that the space $(U, \|\cdot\|)$ is a Banach Space.

Let $Y = L^1[0, \infty)$ with the norm $\|y\|_{L^1} = \int_0^\infty |y(t)|dt$ defined on it and $Z = \{y : [0, \infty) \to \mathbb{R} : \int_0^\infty v(t)|y(t)|dt < \infty\}$ with the norm $\|z\|_Z = \int_0^\infty v(t)|z(v)|dv$. Define M as a continuous operator such that $M : \text{dom } M \subset U \to Z$, where

$$\text{dom } M = \left\{ u \in U : \varphi_p(u'') \in L^1[0, \infty), u(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t)u(t)dt, \right.$$
$$\left. u'(0) = \int_0^\infty v(t)u'(t)dt, \lim_{t\to\infty} (\sigma(t)\varphi_p(u''(t))) = 0 \right\},$$

and $Mu = (\sigma(t)\varphi_p(u''(t)))'$. We will define the operator $N_\lambda u : \overline{\Omega} \to Z$ by

$$N_\lambda u = -\lambda f(t, u(t), u'(t), u''(t)) \quad \lambda \in [0,1],\ t \in [0,\infty),$$

where $\Omega \subset U$ is an open and bounded set. Then the boundary value problem (1.1) in abstract form is $Mu = Nu$.

The following are the assumptions made in this work:

(ϕ_1) $\int_0^\infty v(t)dt = 1,\ \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t)dt = 1,\ \sum_{i=1}^m \alpha_i \int_0^{\xi_i} tv(t)dt = 0;$

(ϕ_2) $G = \begin{vmatrix} Q_1 e^{-t} & Q_2 e^{-t} \\ Q_1 te^{-t} & Q_2 te^{-t} \end{vmatrix} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, where

$Q_1 z = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} v(t) \int_0^t \int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdxdt$ and

$Q_2 z = \int_0^\infty v(t) \int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdt.$

By simple calculation, we see that $\ker M = \{a + bt : a, b \in \mathbb{R},\ t \in [0,\infty)\}$ and $\dim\ker M = 2$.

Lemma 2.8. *The operator $M : dom\ M \subset U \to Z$ is quasi-linear.*

Proof. Clearly, $\ker M = \{u \in \text{dom } M : u = a + bt,\ a,\ b \in \mathbb{R}\}$. Next, we obtain $\text{Im } M$. Let $u \in \text{dom } M$ and consider the problem

$$(\sigma(t)\varphi_p(u''(t)))' = -vz,\ t \in [0,\infty), \tag{2.3}$$

Integrating (2.3) from t to ∞, we have

$$\sigma(t)\varphi_p u''(t) = \lim_{t\to\infty} \sigma(t)\varphi_p u''(t) + \int_t^\infty v(r)z(r)dr, \tag{2.4}$$

from (1.2), we obtain

$$u''(t) = \varphi_q\Big(\frac{1}{\sigma(t)}\Big)\varphi_q\Big(\int_t^\infty v(r)z(r)dr\Big). \tag{2.5}$$

Integrating (2.5) from 0 to t yields

$$u'(t) = u'(0) + \int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)ds. \tag{2.6}$$

It follows from (2.6) that

$$\int_0^\infty v(t)u'(t)dt = \int_0^\infty v(t)u'(0)dt + \int_0^\infty v(t)\int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdt$$

and applying boundary conditions (1.2) gives

$$u'(0) = u'(0)\int_0^\infty v(t)dt - \int_0^\infty v(t)\int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdt.$$

Since $\int_0^\infty v(t)dt = 1$, $\int_0^\infty v(t)\int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdt = 0.$
Integrating (2.6) from 0 to t gives

$$u(t) = u(0) + u'(0)t + \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdx. \quad (2.7)$$

Applying boundary conditions (1.2) gives

$$u(0) = \sum_{i=1}^m \alpha_i\int_0^{\xi_i} v(t)\left(u(0) + u'(0)t + \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdx\right)dt.$$

Since $\sum_{i=1}^m \alpha_i\int_0^{\xi_i} v(t)dt = 1$ and $\sum_{i=1}^m \alpha_i\int_0^{\xi_i} tv(t)dt = 0$,

$$\sum_{i=1}^m \alpha_i\int_0^{\xi_i} v(t)\int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdxdt = 0$$

and

$$u(t) = a + bt + \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^\infty v(r)z(r)dr\Big)dsdx,$$

where a and b are arbitrary constants and $u(t)$ is a solution to (2.3) satisfying (1.2). So $\ker M = 2 < \infty$ and $M \subset (U \cap \operatorname{dom} M) \subset Z$ is closed. Therefore, M is quasi-linear. □

Let the projector $P : U \to U_1$ be defined as

$$Pu(t) = u(0) + u'(0)t, \quad u \in U \quad (2.8)$$

and the operators $\Delta_1,\ \Delta_2 : Z \to Z_1$ as

$$\Delta_1 y = \frac{1}{G}(\delta_{11}Q_1y + \delta_{12}Q_2y)e^{-t},\ \Delta_2 y = \frac{1}{G}(\delta_{21}Q_1y + \delta_{22}Q_2y)e^{-t},$$

where δ_{ij} is the co-factor of g_{ij}, $i, j = 1, 2$. Then, the operator $Q : Z \to Z_1$ will be defined as

$$Qy = (\Delta_1 y) + (\Delta_2 y)\cdot t, \quad (2.9)$$

where Z_1 is the complement space of Im M in Z.

Lemma 2.9. *The operator $Q : Z \to Z_1$ is a semi-projector.*

Proof. It can be shown that $\Delta_1((\Delta_1 z)) = (\Delta_1 z)$, $\Delta_1\Big((\Delta_2 z)t\Big) = 0$, $\Delta_2((\Delta_1 z)) = 0$ and $\Delta_2\Big((\Delta_2 z)t\Big) = \Delta_2 z$. Thus, $Q^2 z = Q\Big[(\Delta_1 z) + (\Delta_2 z)\cdot t\Big] = (\Delta_1 z) + 0 \cdot t + 0 \cdot t^2 + (\Delta_2 z) \cdot = Qz$ or $Q^2 = Q$. Also, for $\lambda \in \mathbb{R}$, $\Delta_1 \lambda z = \frac{1}{G}(\delta_{11} Q_1 \lambda z + \delta_{12} Q_2 \lambda z)e^{-t} = \frac{\lambda}{G}(\delta_{11} Q_1 z + \delta_{12} Q_2 z)e^{-t} = \lambda \Delta_1 z$ and $\Delta_2 \lambda z = \frac{1}{G}(\delta_{21} Q_1 \lambda z + \delta_{22} Q_2 \lambda z)e^{-t} = \frac{\lambda}{G}(\delta_{21} Q_1 z + \delta_{22} Q_2 z)e^{-t} = \lambda \Delta_2 z$. Hence,

$$Q(\lambda z) = (\Delta_1 \lambda z) + (\Delta_2 \lambda z) \cdot t = \lambda((\Delta_1 z) + (\Delta_2 z) \cdot t) = \lambda Q z.$$

Therefore, by definition 2.3, $Q : Z \to Z_1$ is a semi-projector. □

Let the operator $R : U \times [0, 1] \to U_2$ be defined by

$$R(u, \lambda)(t) = \int_0^t \int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right) \varphi_q\bigg(\int_s^\infty (\lambda v(f(r, u(r), u'(r), u''(r)) + QN_\lambda u(r))dr\bigg)dsdx,$$

where $\varphi_p^{-1} = \varphi_q$, U_2 is the complement space of $\ker M$ in U.

Lemma 2.10. *If w is a v-Carathéodory function, then $R : U \times [0, 1] \to U_2$ is M-compact.*

Proof. Let $\Omega \subset U$ be nonempty, open and bounded. Then, for $u \in \overline{\Omega}$, there exists a constant $k > 0$ such that $\|u\| < k$. Since w is an v-Carathéodory function, there exists $\psi_k : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty v(t)\psi_k(t)dt < \infty$ such that for a.e. $t \in [0, \infty)$ and $\lambda \in [0, 1]$, we have

$$|N_\lambda u(t)| = |-\lambda w(t, u(t), u'(t), u''(t))| \le |Nu(t)| = |w(t, u(t), u'(t), u''(t))| \le \psi_k(t),$$

$$\|N_\lambda u\|_Z = \int_0^\infty v(r)|N_\lambda u(r)|dr \ \le \int_0^\infty v(r)|\psi_k(r)|dr \le \|\psi_k\|_Z,$$

$$|QN_\lambda u(t)| = |\lambda QNu(t)| \le |QNu(t)|,$$

and

$$\|QN_\lambda u\|_Z = \int_0^\infty v(r)|QN_\lambda u(r)|dr \le \|QNu\|_Z.$$

Now for any $u \in \overline{\Omega}$, $\lambda \in [0,1]$, we have

$$\|R(u,\lambda)\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|R(u,\lambda)(t)| \leq \frac{1}{e}\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|Nu\|_Z + \|QNu\|_Z)$$
$$\leq \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{Z}\varphi_q(\|\psi_k\|_Z + \|QNu\|_Z) < \infty, \tag{2.10}$$

$$\|R'(u,\lambda)\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|R'(u,\lambda)(t)| \leq \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_Z + \|QNu\|_Z) < \infty \tag{2.11}$$

and

$$\|R''(u,\lambda)\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|R''(u,\lambda)(t)| \leq \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{\infty}\varphi_q(\|\psi_k\|_Z + \|QNu\|_Z) < \infty. \tag{2.12}$$

Therefore it follows from (2.10), (2.11) and (2.12) that $R(u,\lambda)\overline{\Omega}$ is uniformly bounded.
Next we show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous in a compact set. Let $u \in \overline{\Omega}$, $\lambda \in [0,1]$. For any $T \in [0,\infty)$, with $t_1, t_2 \in [0,T]$, where $t_1 < t_2$, we have

$$|R(u,\lambda)(t_2) - R(u,\lambda)(t_1)|$$
$$= \Bigg| \int_0^{t_2}\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx$$
$$- \int_0^{t_1}\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx\Bigg|$$
$$\leq \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_Z + \|QNu\|_Z)(t_2 - t_1) \to 0, \quad \text{as } t_1 \to t_2, \tag{2.13}$$

$$|R'(u,\lambda)(t_2) - R'(u,\lambda)(t_1)|$$
$$= \Bigg| \int_0^{t_2} \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)ds$$
$$- \int_0^{t_1} \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)ds\Bigg|$$
$$\leq \varphi_q(\|\psi_k\|_Z + \|QNu\|_Z)\int_{t_1}^{t_2}\left|\varphi_q\left(\frac{1}{\sigma(s)}\right)\right|ds \to 0, \quad \text{as } t_1 \to t_2 \tag{2.14}$$

and

$$\begin{aligned}
&|R''(u,\lambda)(t_2) - R''(u,\lambda)(t_1)| \\
&\quad = \left|\varphi_q\left(\frac{1}{\sigma(t_2)}\right)\varphi_q\bigg(\int_{t_2}^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)ds\right. \\
&\quad \left. - \varphi_q\left(\frac{1}{\sigma(t_1)}\right)\varphi_q\bigg(\int_{t_1}^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)ds\right| \\
&\quad + \left|\varphi_q\left(\frac{1}{\sigma(t_1)}\right)\right|\varphi_q\bigg(\int_{t_1}^{t_2} v(r)|Nu(r) + QNu(r)|dr\bigg) \to 0, \quad \text{as } t_1 \to t_2.
\end{aligned} \tag{2.15}$$

Thus, (2.13), (2.14) and (2.15) show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous on $[0,T]$.

$$\begin{aligned}
&|R(u,\lambda)(t_2) - R(u,\lambda)(t_1)| \\
&\quad = \left|\int_0^{t_2}\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\bigg)dsdx\right. \\
&\quad \left. - \int_0^{t_1}\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\bigg)dsdx\right| \\
&\quad \le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1} + \|QNu\|_{L^1})(t_2 - t_1) \to 0, \quad \text{as } t_1 \to t_2,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&|R'(u,\lambda)(t_2) - R'(u,\lambda)(t_1)| \\
&\quad = \left|\int_0^{t_2}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\bigg)ds\right. \\
&\quad \left. - \int_0^{t_1}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda f(v,u(v),u'(v),u''(v) + QN_\lambda u(v))dv\bigg)ds\right| \\
&\quad \le \varphi_q(\|\psi_k\|_{L^1} + \|QNu\|_{L^1})\int_{t_1}^{t_2}\left|\varphi_q\left(\frac{1}{\sigma(s)}\right)\right|ds \to 0, \quad \text{as } t_1 \to t_2
\end{aligned} \tag{2.17}$$

and

$$
\begin{aligned}
&|R''(u,\lambda)(t_2) - R''(u,\lambda)(t_1)| \\
&\quad = \left|\varphi_q\left(\frac{1}{\sigma(t_2)}\right)\varphi_q\bigg(\int_{t_2}^{\infty}(\lambda f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\bigg)ds \right.\\
&\quad \left. - \varphi_q\left(\frac{1}{\sigma(t_1)}\right)\varphi_q\bigg(\int_{t_1}^{\infty}(\lambda f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\bigg)ds\right| \\
&\quad \le \varphi_q\left|\frac{1}{\sigma(t_2)} - \frac{1}{\sigma(t_1)}\right|\varphi_q(\|\psi_k\|_{L^1} + \|QNu\|_{L^1}) \\
&\quad + \left|\varphi_q\left(\frac{1}{\sigma(t_1)}\right)\right|\varphi_q\bigg(\int_{t_1}^{t_2}|Nu(v) + QNu(v)|dv\bigg) \to 0, \quad \text{as } t_1 \to t_2.
\end{aligned} \tag{2.18}
$$

Thus, (2.16), (2.17) and (2.18) show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous on $[0,T]$.

We will now prove that $R(u,\lambda)\overline{\Omega}$ is equiconvergent at ∞. We need

$$
\begin{aligned}
R(u,\lambda)(\infty) &= \lim_{t\to\infty} R(u,\lambda)(t) \\
&= \int_0^{\infty}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda v(r)(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx,
\end{aligned}
$$

$$
\begin{aligned}
R'(u,\lambda)(\infty) &= \lim_{t\to\infty} R'(u,\lambda)(t) \\
&= \int_0^{\infty}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)ds
\end{aligned}
$$

and

$$
\begin{aligned}
R''(u,\lambda)(\infty) &= \lim_{t\to\infty} R''(u,\lambda)(t) \\
&= \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_{\infty}^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx = 0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|R(u,\lambda)(t) - R(u,\lambda)(\infty)| \\
&\quad = \left|\int_0^{t}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx \right.\\
&\quad \left. - \int_0^{\infty}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\bigg(\int_s^{\infty}(\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\bigg)dsdx\right| \\
&\quad \le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_Z + \|QNu\|_Z)\int_t^{\infty}dx \to 0, \text{ uniformly as } t\to\infty,
\end{aligned} \tag{2.19}
$$

$$\begin{aligned}
&|R'(u,\lambda)(t) - R'(u,\lambda)(\infty)| \\
&\quad = \Bigg| \int_0^t \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\Bigg)ds \\
&\quad - \int_0^\infty \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\Bigg)ds\Bigg| \\
&\quad \le \varphi_q(\|\psi_k\|_Z + \|QN_\lambda u\|_Z)\int_t^\infty \varphi_q\left(\frac{1}{\sigma(s)}\right)ds \ \to 0, \text{ uniformly as } t\to\infty
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&|R''(u,\lambda)(t) - R''(u,\lambda)(\infty)| \\
&\quad = \left|\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\Bigg(\int_t^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\Bigg)ds - 0\right| \\
&\quad = \left|\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\Bigg(\int_t^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) + QN_\lambda u(r))dr\Bigg)ds\right| \\
&\quad \le \left|\varphi_q\frac{1}{\sigma(t)}\right|\varphi_q\left(\int_t^\infty v(r)|Nu(r) + QNu(r)|dr\right) \ \to 0, \text{ unifromly as } t\to\infty.
\end{aligned} \tag{2.21}$$

Therefore $R(u,\lambda)\overline{\Omega}$ is equiconvergent at ∞. It then follows from definition 2.4 that $R(u,\lambda)$ is compact. □

Lemma 2.11. *The operator N_λ is M-compact.*

Proof. To prove this lemma, we will need to show that the conditions of definition 2.5 hold. Since w is v-Carathéodry, N_λ is continuous. Let the homeomorphism $J : \operatorname{Im} Q \to \ker M$ be defined by $J(a+bt) = a+bt$. Then $\dim U_1 = \dim Z_1 = 2$.

Since Q is a semi-projector, $Q(I-Q)N_\lambda(\overline{\Omega}) = 0$. Hence, $(I-Q)N_\lambda(\overline{\Omega}) \subset \ker Q = \operatorname{Im} M$. Conversely, let $z \in \operatorname{Im} M$. Then $z = z - Qz = (I-Q)z \in (I-Q)Z$. Hence, condition (i) of definition (2.5) is satisfied. It is easy to show that condition (ii) of definition (2.5) holds as well.

Let $u \in \Sigma_\lambda$, $Mu = N_\lambda u$. Then $QN_\lambda u = (\Delta_1 N_\lambda u) + (\Delta_2 N_\lambda u)\cdot t = 0$ and $R(u,\lambda)(t)$ becomes

$$R(u,\lambda)(t) = \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\left(\int_s^\infty \lambda v(r)w(r,u(r),u'(r),u''(r))dr\right)dsdx.$$

Then

$$R(u,0)(t) = \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\left(\int_s^\infty 0\cdot v(r)w(r,u(r),u'(r),u''(r))dr\right)dsdx = 0$$

and

$$\begin{aligned}R(u,\lambda)(t) &= \int_0^t \int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\left(\int_s^\infty \lambda v(r)w(r,u(r),u'(r),u''(r))dr\right)dsdx \\ &= \int_0^t \int_0^x u''(s)dsdx = u(t)-u(0)-u'(0)t = u(t)-Pu(t) = [(I-P)u](t).\end{aligned} \tag{2.22}$$

Therefore, condition (iii) of definition (2.5) holds.

Let $u \in \overline{\Omega}$. Since $Mu = \frac{1}{v(t)}(\sigma(t)\varphi_p(u''(t)))'$, we have

$$\begin{aligned}M[Pu+R(u,\lambda)](t) &= \frac{1}{v(t)}(\sigma(t)\varphi_p([Pu+R(u,\lambda)]''(t)))' \\ &= \frac{1}{v(t)}\Bigg(\sigma(t)\varphi_p\Bigg[u(0)+u'(0)t+\int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Bigg(\int_s^\infty (\lambda v(r)w(r,u(r),u'(r),u''(r)) \\ &\quad + (QN_\lambda(r))dr\Bigg)dsdx\Bigg]''\Bigg)' \\ &= \frac{1}{v(t)}\left(\sigma(t)\varphi_p\left[\varphi_q\left(\frac{1}{\sigma(t)}\int_t^\infty [v(r)(-I+Q)N_\lambda](r))dr\right)\right]\right)' \\ &= \frac{1}{v(t)}\left(\int_t^\infty v(r)[(-I+Q)N_\lambda](r))dr\right)' \\ &= -\frac{1}{v(t)}v(r)[(-I+Q)N_\lambda](t) = [(I-Q)N_\lambda](t);\end{aligned}$$

i.e., (iv) of definition (2.5) holds. Hence, N_λ is M-compact in $\overline{\Omega}$. □

3 Existence Result

Theorem 3.1. *Suppose the following conditions hold:*

(H_1) there exists functions $x_1(t), x_2(t), x_3(t), x_4(t) \in L^1[0,\infty)$ such that for all $(u,v,w) \in \mathbb{R}^3$ and a.e. $t \in [0,\infty)$,

$$|f(t,u,u',u'')| \le e^{-t}(x_1(t)|u|^{p-1} + x_2(t)|u'|^{p-1} + x_3(t)|u''|^{p-1}) + x_4(t), \tag{3.23}$$

(H_2) for $u \in$ dom M there exist a constant $A_0 > 0$, $l > 0$ such that if $|u(t)| > A_0$ for $t \in [0,l]$ or $|u'(t)| > A_0$ for $t \in [0,\infty)$, then either

$$Q_1Nu(t) \neq 0 \quad or \quad Q_2Nu(t) \neq 0, \quad t \in [0,\infty). \tag{3.24}$$

(H_3) *there exists a constant $B > 0$ such that for $|a| > B$ or $|b| > B$ either*

$$Q_1 N(a+bt) + Q_2 N(a+bt) < 0, \quad t \in (0,\infty), \tag{3.25}$$

or

$$Q_1 N(a+bt) + Q_2 N(a+bt) > 0, \quad t \in (0,\infty), \tag{3.26}$$

where $a, b \in \mathbb{R}$, $|a| + |b| > B$ and $t \in [0,\infty)$.
Then the boundary value problem (1.1)*-*(1.2) *has at least one solution provided*

$$\Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) < 1 \quad \text{for } p \geq 2, \tag{3.27}$$

$$2^{2q-4}\Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) < 1 \quad \text{for } 1 < p < 2, \tag{3.28}$$

where $\Lambda = \max\left\{(2+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, (1+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}.$

The following lemmas are also needed to prove our main result.

Lemma 3.2. *The set $U_1 = \{u \in dom\ M : Mu = N_\lambda u$ for some $\lambda \in (0,1)\}$ is bounded.*

Proof. Let $u \in U_1$. Then $N_\lambda u \in \operatorname{Im} M = \ker Q$. Hence, $QN_\lambda u = 0$ and $QNu = 0$. It follows from H_2 that there exist $t_0 \in [0,l]$ and $t_1 \in [0,\infty)$ such that $|u(t_0)| \leq A_0$, $|u'(t_1)| \leq A_0$. From $u(0) = u(t_0) + \int_0^{t_0} u'(v)dv$, we have $|u(0)| = \left|u(t_0) - \int_0^{t_0} u'(v)dv\right| \leq A_0 + d\|u'\|_\infty$. Also, from $u'(t) = u'(t_1) - \int_t^{t_1} u''(v)dv$, we get $|u'(t)| = \left|u(t_1) - \int_t^{t_1} u'(v)dv\right| \leq A_0 + \|u''\|_{L^1}$. Then

$$|u'(0)| \leq A_0 + \|u''\|_{L^1} \tag{3.29}$$

and

$$\|u'\|_\infty = \sup_{t\in[0,\infty)} e^{-t}|u'(t)| \leq A_0 + \|u''\|_{L^1}. \tag{3.30}$$

Hence, from (3.29) and (3.30), we have

$$|u(0)| \leq 2A_0 + \|u''\|_{L^1}. \tag{3.31}$$

Since $Mu = N_\lambda u$, from (2.5) we have

$$\begin{aligned}\|u''\|_{L^1} &= \int_0^\infty \left| -\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\left(\int_t^\infty \lambda v(r)w(r,u(r),u'(r),u''(r))dr\right)\right| dt \\ &\leq \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} \varphi_q(\|Nu\|_Z).\end{aligned}$$

Considering (H_1) and statement (ii) of lemma 2.6, if $1 < p < 2$, we have

$$\begin{aligned}\varphi_q(\|Nu\|_Z) &\leq \varphi_q\left(\int_0^\infty v(t)|f(t,u(t),u'(t),u''(t))|dt\right) \\ &\leq 2^{q-2}[\varphi_q(\|x_1\|_Z\|u\|^{p-1} + \|x_2\|_Z\|u\|^{p-1}) + \varphi_q(\|x_3\|_Z\|u\|^{p-1} + \|x_4\|_Z)] \\ &\leq 2^{2q-4}\|u\|(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) + 2^{2q-4}\|x_4\|_Z^{q-1}.\end{aligned} \tag{3.32}$$

Similarly, for $p \geq 2$ we have

$$\begin{aligned}\varphi_q(\|Nu\|_Z) &\leq \varphi_q\left(\int_0^\infty v(t)|f(t,u,u',u'')|dt\right) \\ &\leq \|u\|(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1}) + \|x_4\|_Z^{q-1}.\end{aligned} \tag{3.33}$$

Since $QNu = 0$ for $u \in U_1$ and using (2.10), (2.11) and (2.12), we have

$$\|R(u,\lambda)\| \leq \max\left\{\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}\varphi_q(\|Nu\|_Z).$$

Also,

$$\|Pu\| \leq A_0(2+l) + (1+l)\varphi_q(\|Nu\|_Z)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}. \tag{3.34}$$

In addition, for $u \in \Omega_1$, and in view of (2.22), we have

$$u(t) = Pu(t) + (I-P)u(t) = Pu(t) + R(u,\lambda)u(t).$$

Therefore,

$$\begin{aligned}\|u\| = \|Pu\| + \|R(u,\lambda)\| &\leq A_0(2+l) + (1+l)\varphi_q(\|Nu\|_Z)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} \\ &+ \max\left\{\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}(\varphi_q(\|Nu\|)_Z) = A_0(2+l) \\ &+ \max\left\{(2+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, (1+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}\varphi_q(\|Nu\|)_Z).\end{aligned}$$

Set $\Lambda = \max\left\{(2+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, (1+l)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}$. We have

$$\|u\| \leq A_0(2+l) + \Lambda\varphi_q(\|Nu\|_Z). \tag{3.35}$$

If $1 < p < 2$, then in view of (3.32)

$$\|u\| \leq \frac{A_0(2+l) + 2^{2q-4}\Lambda\|x_4\|_Z^{q-1}}{1 - 2^{2q-4}\Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1})},$$

If $p \geq 2$, then in view of (3.33)

$$\|u\| \leq \frac{A_0(2+d) + \Lambda\|x_4\|_Z^{q-1}}{1 - \Lambda(\|x_1\|_Z^{q-1} + \|x_2\|_Z^{q-1} + \|x_3\|_Z^{q-1})}.$$

Therefore Ω_1 is bounded. □

Lemma 3.3. *Assuming that (H_3) holds, the set* $\Omega_2 = \{u \in \ker M : QNu = 0\}$ *is bounded.*

Proof. Let $u \in \Omega_2$. Then $u = a + bt,\ a, b \in \mathbb{R}$, $QNu = 0$ and $Nu \in \text{Im } M$. Hence, from lemma 2.9, $Q_1N(a + bt) = Q_2N(a + bt) = 0$. From ($H_3$) it follows that $|a| < B$ and $|b| < B$. Hence $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\} = \max\{2B, B, 0\} \leq 2B$. So Ω_2 is bounded. □

Lemma 3.4. *If* $\Omega_3 = \{u \in \ker M : -\lambda u + (1-\lambda)JQNu = 0,\ \lambda \in [0, 1]\}$, $J : Im\ Q \to \ker M$ *is a homeorphisim, then* Ω_2 *is bounded.*

Proof. For $a, b \in R$, let $J : \text{Im } Q \to \ker M$ be defined by

$$J(a + bt) = \frac{1}{G}[\delta_{11}|a| + \delta_{12}|b| + (\delta_{21}|a| + \delta_{22}|b|)t)]e^{-t}. \tag{3.36}$$

If (3.25) holds, for any $u(t) = a + bt \in \Omega_3$, from $-\lambda u + (1-\lambda)JQNu = 0$, we obtain

$$\begin{aligned} \lambda|a| &= (1-\lambda)Q_1N(a + bt), \\ \lambda|b| &= (1-\lambda)QN_2(a + bt). \end{aligned} \tag{3.37}$$

From (3.37), when $\lambda = 1,\ a = b = 0$. When $\lambda = 0$, $Q_1N(a + bt) + Q_2N(a + bt) = 0$ which contradicts (3.25) and (3.26.) Hence from (H_3), $|a| \leq B$ and $|b| \leq B$. For $\lambda \in (0, 1)$, $a < B$, $b < B$, from (3.37), we have $\lambda(|a| + |b|) = (1-\lambda)[Q_1N(a+bt) + Q_2N(a+bt)] < 0$, which contradicts $\lambda(|a| + |b|) \geq 0$. Hence, ($H_3$), $|a| \leq B$ and $|b| \leq B$. Thus $\|u\| \leq 2B$. Therefore Ω_3 is bounded. □

Lemma 2.9 shows that condition (B_1) of theorem 2.7 holds. Moreover, Lemma 2.11 implies (B_2). Furthermore, lemmas 3.2 and 3.3 imply (B_3). We will next prove (B_4).

Theorem 3.5. *Assuming that (H_1)-(H_3) hold, the boundary value problem* (1.1) *-* (1.2) *has at least one solution in dom* $M \cap \partial\Omega$.

Proof. Let $\Omega \supset \Omega_1 U \Omega_2 U \Omega_3$ be a nonempty, open and bounded set, $u \in$ dom $M \cap \partial\Omega$ and $H(u,\lambda) = -\lambda u + (1-\lambda)JQNu$, where J is as defined above. Then $H(u,\lambda) \neq 0$ Therefore, by the homotopy property of the Brouwer degree

$$\begin{aligned}\deg\{JQN|_{\overline{\Omega}\cap\ker M}, \Omega\cap\ker M, 0\} &= \deg\{H(\cdot,0), \Omega\cap\ker M, 0\}\\ &= \deg\{H(\cdot,1), \Omega\cap\ker M, 0\}\\ &= \deg\{-I, \Omega\cap\ker M, 0\} \neq 0.\end{aligned}$$

Hence, (B_4) of theorem 2.7 holds. □

Since all the conditions of theorem 2.7 are satisfied, the abstract equation $Mu = Nu$ has at least one solution in $\overline{\Omega} \cap$ dom M. Hence, (1.1) - (1.2) has at least one solution in U.

Example: Consider the following boundary value problem

$$(e^{-5t+2}\varphi_{\frac{4}{3}}(u''(t)))' + 2e^{-2t}w(t,u(t),u'(t),u''(t)), \quad t \in (0,\infty), \tag{3.38}$$

$$\begin{aligned} u(0) &= 54.9397\int_0^{1/25} e^{-t}u(t)dt - 32.2679\int_0^{1/19} e^{-t}u(t)dt,\\ u'(0) &= \int_0^\infty 2e^{-2t}u'(t)dt,\ \lim_{t\to\infty} e^{-5t+2}\varphi_{\frac{4}{3}}(u''(t)) = 0,\end{aligned} \tag{3.39}$$

where

$$f(t,u,v,w) = \begin{cases} 0, & 0 \leq t \leq 1,\\ e^{-2t-1}\sqrt[3]{u(0)} + e^{-2t-2}\sin\sqrt[3]{u'} + e^{-3t-3}\sin\sqrt[3]{u''} + \frac{1}{2}e^{-2t}, & t > 1.\end{cases}$$

Here $\sigma(t) = e^{-5t+2}$, $p = \frac{4}{3}$, $q = 4$, $\alpha_1 = 54.9397$, $\alpha_2 = -32.2679$, $\xi_1 = \frac{1}{25}$, $\xi_2 = \frac{1}{19}$, $v(t) = 2e^{-2t}$. Clearly, $G \neq 0$, $\sum_{i=1}^{2}\alpha_i\int_0^{\xi_i} v(t)dt = 1$, $\sum_{i=1}^{2}\alpha_i\int_0^{\xi_i} tv(t)dt = 0$ and $\int_0^\infty v(t)dt = 1$. Hence, (ϕ_1) and (ϕ_2) hold.
$\|x_1\|_Z^{4-1} = \frac{1}{e^3}$, $\|x_2\|_Z^{4-1} = \frac{1}{8e^6}$, $\|x_3\|_Z^{4-1} = \frac{8}{125e^9}$, $\|\varphi_q\left(\frac{1}{\sigma}\right)\|_{L^1} = \frac{1}{5e^2}$, $\|\varphi_q\left(\frac{1}{\sigma}\right)\|_\infty = 1$. Therefore,

$$\Lambda = max\left\{(2+1)\left(\frac{1}{5e^2}\right), (1+1)\left(\frac{1}{5e^2}\right) + 1\right\} = \max\{0.0812, 1.0541\} = 1.0541.$$

$$\Lambda(\|x_1\|_Z^3 + \|x_2\|_Z^3 + \|x_3\|_Z^3) = 1.0541 \left[\frac{1}{e^3} + \frac{1}{8e^6} + \frac{8}{25e^9}\right] = 0.0529 < 1$$

Therefore, (3.28) holds. Also (H_2) and (H_3) hold. Since all the conditions of Theorem 2.7 hold, (3.38)-(3.39) has at least one solution.

Acknowledgment. The authors would like to thank Covenant University for support during the study period.

References

[1] R. P. Agarwal, D. O'Regan, Infnite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, Stud. Appl. Math., **111,** no. 3, (2003), 339–358.

[2] H. Feng, H. Lian, W. Ge, A symmetric solution of a multipoint boundary value problems with one-dimensional p-Laplacian at resonance, Nonlinear Anal., **69,** (2008), 3964–3972.

[3] W. Ge, J. Ren, An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian, Nonlinear Anal., **58,** (2004), 477–488.

[4] W. Ge, Boundary value problems for ordinary nonlinear differential equations, Science Press, Beijing, (2007), (in Chinese).

[5] S. A. Iyase, O. F. Imaga, Higher order boundary value problems with integral boundary conditions at resonance on the half-line, J. Nig. Math. Soc., **38**, no. 2, (2019), 168–183.

[6] S. A. Iyase, O. F. Imaga, On a singular second-order multipoint boundary value problem at resonance, Int. J. Diff. Eq., (2017).

[7] W. Jiang, Y. Zhang, J. Qiu, The existence of solutions for p-Laplacian boundary value problems at resonance on the half-line, Bound. Val. Probl., (2009).

[8] N. Kosmatov, Multi-point boundary value problems on an unbounded domain at resonance, Nonlinear Anal., **68,** (2008), 2158–2171.

[9] X. Lin, Z. Du, F. Meng, A note on a third-order multi-point boundary value problem at resonance, Math. Nachr., **284,** (2011), 1690–1700.

[10] X. Lin, Q. Zhang, Existence of Solution for a p-Laplacian Multi-point Boundary Value Problem at Resonance, (2017).

[11] R. Ma, Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian, Comput. Math. Appl., **42,** (2001), 755–765.

[12] J. Mawhin, Topological degree methods in nonlinear boundary value problems, NSFCMBS, Regional Conference Series in Mathematics, AMS, Providence, RI, 1979.

[13] A. Yang, C. Miao, W. Ge, Solvability for a second-order non-local boundary value problems with a p-Laplacian at resonance on a half-line. Electronic J. Qual. Theor. Diff. Eq., **19,** (2009), 1–15.

[14] A. J. Yang, W. Ge, Existence of symmetric solutions for a fourth-order multi-point boundary value problem with a p-Laplacian at resonance, J. Appl. Math. Comput., **29,** (2009), 301–309.