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On the Complements of Graphs

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Abstract

The complement graph of Γ, denoted by $\overline{\Gamma}$, is a graph where the vertex set of $\overline{\Gamma}$ equals the vertex set of Γ and two vertices in Γ are adjacent if they are not adjacent in $\overline{\Gamma}$ and vice versa. In this article, we investigate and construct the complement graphs of certain types of graphs. Moreover, new results of complements of graphs and complements of line graphs are given.

1 Introduction

First, we introduce the basic definitions and concepts which will be used in the sequel [2,4].

Definition 1.1. A graph of $\Gamma = (V(\Gamma), E(\Gamma))$ consists of a non-empty set of vertices $V(\Gamma)$ with a prescribed set of edges $E(\Gamma)$.

Definition 1.2. The number of vertices in Γ is called the order of Γ, denoted by $|V(\Gamma)|$, while the number of the edges is called the size of Γ , denoted by $|E(\Gamma)|$.

Definition 1.3. Two vertices u and v are said to be adjacent if they are joined by an edge and two edges are said to be incident if they have a vertex in common.

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Definition 1.4. The valancy of a vertex u in Γ, denoted by $val(u)$, is the number of edges incident to it.
Definition 1.5. A graph Γ is called k-regular if $val(ui) = k$ for all $ui \in \Gamma$.
Definition 1.6. A graph Γ with n vertices is called a complete graph,, denoted by K_n if every vertex in Γ adjacent to all other vertices.
Definition 1.7. A complete bigraph $K_{m,n}$ is a graph consisting of two disjoint sets of vertices in which every vertex in the first set is adjacent to all other vertices in the second set.
Definition 1.8. A star graph $s(1, n)$ is a graph consisting of one root and n end-vertices.
Definition 1.9. A cycle graph C_n consists of a closed path with n vertices.
Definition 1.10. A wheel graph $W(1, n)$ is a graph consisting of $S(1, n) \cup C_n$.
Definition 1.11. A line graph of a graph Γ, denoted by $L(\Gamma)$, is a graph in which $V(L(\Gamma) = E(\Gamma)$ and two vertices are adjacent if they are incident in Γ.

2 Complements of Graphs

We first introduce and construct the complements of well-known graphs.

Theorem 2.1. *Let Γ be a simple graph and let $\overline{\Gamma}$ be the complement of Γ. Then $\Gamma \cup \overline{\Gamma} = K_n$.*

Proof. Let Γ be a simple graph with n vertices. All adjacent vertices in Γ will be adjacent and non-adjacent vertices will be adjacent in $\Gamma \cup \overline{\Gamma}$ consisting of $V(\Gamma) \cup V(\overline{\Gamma})$. So $V(\Gamma \cup \overline{\Gamma})$ consists of n vertices and $E(\Gamma \cup \overline{\Gamma})$ consists of $E(\Gamma) \cup E(\overline{\Gamma})$. Then every vertex in $\Gamma \cup \overline{\Gamma}$ is adjacent with all other vertices. Hence the constructed graph $\Gamma \cup \overline{\Gamma}$ is equivalent to the complete graph of K_n. □

Theorem 2.2. *([3]). The complement graph of the cage graph is a graph with n vertices and it is $(n - 4)$-regular.*

Theorem 2.3. *Let Γ be a path graph P_n. Then the complement of Γ is $E(k_n) - E(P_n)$.*

Proof. Let Γ be a path graph P_n. Then P_n consists of n vertices and every root vertex is adjacent with a predecessor and successor vertex of P_n. So. in Γ, those vertices are non-adjacent with predecessor and successor vertices and adjacent with the other vertices. Hence the constructed graph consists of the complete graph K_n after removing the edge of P_n; i.e., $E(K_n) - E(P_n)$. □

Theorem 2.4. *Let Γ be a star graph $S(1, n)$. Then $\Gamma = K_n \cup \overline{K_1}$.*

Proof. Let Γ be a star graph $S(1, n)$. Then Γ consists of $n+1$ vertices and n edges. Since the root is a common vertex of all edges in $S(1, n)$, Γ is a tree with one root and n end vertices. Thus the root is disjoint with all vertices in Γ and every vertex in Γ is adjacent with all other vertices. Consequently, the constructed graph of Γ is the complete graph K_n with an isolated vertex; i.e., $K_n \cup \overline{K_1}$. □

Theorem 2.5. *([2]). The complement graph of a complete graph K_n is a graph with n non-adjacent vertices.*

Theorem 2.6. *([2]). The complement graph of isolated n vertices graph is the complete graph K_n.*

Theorem 2.7. *Let Γ be a bigraph $K_{m,n}$ with $m, n > 2$. Then the complement of Γ consists of $K_m \cup K_n$.*

Proof. Let Γ be a bigraph $K_{m,n}$ with $m, n > 2$. Γ consists of two disjoint set of vertices and every vertex in the first set adjacent with all vertices in the second set and non-adjacent with all other vertices in its set and vice-versa. Hence every vertex in *Gamma* in the first set is adjacent with all other vertices in its set and similarly in the vertices in the second set. So the constructed graph $\overline{\Gamma}$ consists of the complete graph K_m and K_n. □

Theorem 2.8. *The complement graph of the cycle graph C_n consists of $E(K_n) - E(C_n)$.*

Proof. Let Γ be a cycle graph of C_n. Then C_n consists of n vertices and n edges in which every vertex is adjacent with a predecessor and successor vertex. They every vertex in Γ will be adjacent with other vertices except a predecessor and successor vertices. Then the resulted complement graph is the set of edges of K_n after removing the set of edges of C_n with n vertices. □

Theorem 2.9. *Let Γ be a wheel graph $W(1, n)$. The complement graph of Γ consists of $(K_1 \cup (E(K_n) - (E(C_n))$.*

Proof. Let Γ be a wheel graph $W(1, n)$. Then Γ consists of $S(1, n) \cup C_n$ since the root vertex is adjacent with all outer vertices. Thus, in the complement graph $\overline{\Gamma}$, the root will be non-adjacent with other vertices in Γ but the set of outer vertices in Γ is a cycle graph C_n. So the complement graph of Γ consists of $n+1$ vertices with isolated vertex and a set of edges of the complete graph K_n after removing the set of outer edges from $W(1, n)$. □

3 Complements of line graph

Theorem 3.1. *The complement graph of line graph of a path graph P_n is $E(K_{n-1}) - E(P_{n-1})$.*

Proof. Let Γ be a path graph P_n. Then Γ consists of n vertices and $n-1$ edges. Thus $L(\Gamma)$ consists of vertices and since every edge in P_n incident with predecessor and successor edge (except the end edge), $L(\Gamma)$ is P_{n-1}. By Theorem 2.3, the complement of the line graph of the path graph P_n is a graph with $n-1$ vertices with set of edges consisting of the edges of K_{n-1} after removing the edges of P_{n-1}; i.e., $E(K_{n-1}) - E(P_{n-1})$. □

Theorem 3.2. *The complement graph of the line graph of the star graph $S(1, n)$ is a graph with n non-adjacent vertices.*

Proof. Let Γ be a star graph $S(1, n)$. Then Γ is a tree with one root and n end vertices and so Γ consists of n edges having the root vertex as a common vertex. Hence $L(\Gamma)$ consists of n vertices and every vertex is adjacent with other vertices. Consequently, the constructed graph is K_n by Theorem 2.6. □

Theorem 3.3. *Let Γ be a cycle graph C_n. Then the complement graph of the line graph of C_n is $E(K_n) - E(C_n)$.*

Proof. Let Γ be a cycle graph with n vertices. Then Γ has n edges and every edge is incident with a predecessor and successor edge. Thus the line graph of C_n is C_n, by definition. So the complement graph of $L(\Gamma)$ is a graph with n vertices and $E(K_n) - E(C_n)$ by Theorem 2.8. □

Theorem 3.4. *([2]). Let Γ be a simple graph. Then the valancy of $e = (u, v)$ in Γ equals valancy $(u) +$ valancy$(v) - 2$.*

Theorem 3.5. *([1]) Let Γ be a graph with n vertices. Then $|E(\Gamma)| = \frac{1}{2}\sum_{i-1}^{n} valancy(u_i)$.*

Theorem 3.6. *Let Γ be a cage graph. Then the complement of Γ is a graph with $\frac{3n}{2}$ vertices and it is $\frac{3n-10}{2}$-regular.*

Proof. Let Γ be a 3-regular graph with n vertices. So $|E(\Gamma) = \frac{3n}{2}|$ by the previous Theorem. So $L(\Gamma)$ has $\frac{3n}{2}$ vertices and it is 4-regular (Theorem 3.4). Now the complement graph of $L(\Gamma)$ consists of $\frac{3n}{2}$ vertices. Since every vertex has valancy three, every vertex in the complement of $L(\Gamma)$ has valancy $\frac{3n}{2} - (4+1)$. So $L(\Gamma)$ is $\frac{3n-10}{2}$-regular . □

Theorem 3.7. *The complement graph* $\frac{3n-10}{2}$ *of the line graph of the bigraph* $K_{m,n}, L(K_{m,n})$ *is a graph with* mn *vertices and it is* $((mn+2)-((m+n))$*-regular.*

Proof. Let Γ be a bigraph $K_{m,n}$. Then Γ consists of $m+n$ vertices and mn edges since Γ consists of two disjoint set of vertices and every vertex in the first set adjacent with all vertices in the second set but not with the vertices in its set. Similarly in the second set. So $L(\Gamma)$ has mn vertices and is $(m+n-2)-$ regular. Now the complement of $L(\Gamma)$ has mn vertices since the valancy of every vertex in $L(\Gamma)$ equal $(m+n-2)$. So we have to remove $(m+n-2)$ edges from every vertex and link other vertices in the complement of $L(\Gamma)$. Hence the valancy of every vertex in the complement of $L(\Gamma)$ equals $(mn-(m+n-2)) = (mn+2)-(m+n)$. □

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