International Journal of Mathematics and
Computer Science, **15**(2020), no. 3, 845–855

Some Matrix and Norm Inequalities for Positive Definite Matrices

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(Received February 19, 2020, Accepted April 23, 2020)

Abstract

For $n \times n$ complex matrices, we show many inequalities under the unitarily invariant norm.

1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all complex matrices of order $n \times n$. For a matrix $T \in \mathbb{M}_n(\mathbb{C})$, the singular values of T are the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. These singular values are denoted by $s_1(T), ..., s_n(T)$.

The positive semidefinite matrix $T \in \mathbb{M}_n(\mathbb{C})$, written as $T \geq 0$, is a Hermitian matrix with $x^*Tx \geq 0$ for all $x \in \mathbb{C}^n$. If $T \in \mathbb{M}_n(\mathbb{C})$ is a Hermitian matrix with $x^*Tx > 0$ for all nonzero $x \in \mathbb{C}^n$, then T is called a positive definite matrix, written as $T > 0$.

Key words and phrases: Positive definite matrix, Hermitian matrix, geometric mean, singular value, unitarily invariant norm, inequality.
AMS (MOS) Subject Classifications: 15A18, 15A42, 15A45, 15A60, 47A64, 15B48.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

The Ky Fan $k-$norms $||\cdot||_{(k)}$ $(k = 1, ..., n)$ are the norms defined on $\mathbb{M}_n(\mathbb{C})$ by $||T||_{(k)} = \sum_{j=1}^{k} s_j(T)$, $k = 1, ..., n$. For each $k = 1, ..., n$, we have

$$||T||_{(k)} = \max \left| \sum_{j=1}^{k} y_j^* T x_j \right|, \tag{1.1}$$

where the maximum is taken over all choices of orthonormal $k-$tuples $x_1, ..., x_k$ and $y_1, ..., y_k$. Also, replacing each y_j by $z_j y_j$ for some suitable complex number z_j with $|z_j| = 1$ for which $\bar{z}_j y_j^* T x_j = \left| y_j^* T x_j \right|$, gives that the $k-$tuple $z_1 y_1, ..., z_k y_k$ is still orthonormal, and so the following identity is equivalent to the identity (1.1):

$$||T||_{(k)} = \max \sum_{j=1}^{k} \left| y_j^* T x_j \right|, \tag{1.2}$$

where the maximum is taken over all choices of orthonormal $k-$tuples $x_1, ..., x_k$ and $y_1, ..., y_k$.

On $\mathbb{M}_n(\mathbb{C})$, a unitarily invariant norm $|||\cdot|||$ is a norm that satisfies the invariance property; that is, for every $T \in \mathbb{M}_n(\mathbb{C})$ and every unitary matrices $V, W \in \mathbb{M}_n(\mathbb{C})$, we have

$$|||VTW||| = |||T||| .$$

The direct sum of the matrices T and R, denoted by $T \oplus R$, is the matrix

$$T \oplus R = \begin{bmatrix} T & 0 \\ 0 & R \end{bmatrix}.$$

For every unitarily invariant norm, the following two inequalities are equivalent

$$|||T \oplus T||| \geq |||R \oplus R||| \tag{1.3}$$

$$|||T||| \geq |||R||| . \tag{1.4}$$

Also,

$$|||T \oplus R||| = |||T^* \oplus R||| = |||R \oplus T||| = \left|\left|\left| \begin{bmatrix} 0 & R \\ T^* & 0 \end{bmatrix} \right|\right|\right|, \tag{1.5}$$

for every unitarily invariant norm. The Ky Fan $k-$norms are unitarily invariant norms. For typical examples and further properties of unitarily invariant norms, the reader is referred to [4], [9], or [10].

In 1975, Pusz and Woronowicz [8] gave the definition of the geometric mean of two positive definite matrices as

$$T\#R = T^{1/2}(T^{-1/2}RT^{-1/2})^{1/2}T^{1/2}, \tag{1.6}$$

for positive definite matrices $T, R \in \mathbb{M}_n(\mathbb{C})$. In 1980, a detailed study was done by Kubo and Ando [6].

The geometric mean for positive definite matrices plays an important role in operator theory, physics, engineering and statistics (see, [1], [6], and [7]).

One of the most important properties of the geometric mean is

$$T\#R = R\#T. \tag{1.7}$$

For more about the geometric mean and its properties, we refer the reader to [2] and Chapter 4 of [5].

Also, since any two positive definite matrices can be joined by a unique geodesic, we have

$$T\#_\mu R = T^{1/2}(T^{-1/2}RT^{-1/2})^{\mu}T^{1/2}, \tag{1.8}$$

for positive definite matrices $T, R \in \mathbb{M}_n(\mathbb{C})$, where $\mu \in [0,1]$ (see for example, [3, p. 464]).

A well known inequality is the sum of reciprocal fractions (see for example, [3, p. 31]) asserts that for every positive real numbers x and y, we have

$$\frac{x}{y} + \frac{y}{x} \geq 2, \tag{1.9}$$

with equality if and only if $x = y$.

In this paper, we use the inequality (1.9) to give lower bounds for some matrix and norm inequalities. In Section 2, we given inequalities that depend mainly on the geometric mean of two positive definite matrices. In Section 3, we give matrix and norm inequalities for a certain type of positive definite matrices.

2 Inequalities containing the geometric mean of two positive definite matrices

In this section, we give matrix and norm inequalities containing the geometric mean of two positive definite matrices. First, we need the following lemma (see for example, [4, p. 62]).

Lemma 2.1. *Let $T, R \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices. Then*

$$s_j(T+R) \geq s_k(T) + s_{j-k+n}(R),$$

for $j, k = 1, ..., n$ with $k \geq j$.

The next lemma is a direct consequence of the Weyl's Monotonicity Theorem (see for example, [4, p. 63]).

Lemma 2.2. *Let $T \in \mathbb{M}_n(\mathbb{C})$ such that $T > 0$. Then, for every $X \in \mathbb{M}_n(\mathbb{C})$, we have*

$$s_j(X^*TX) \geq s_j^2(X)\, s_n(T),$$

for $j = 1, ..., n$.

Our first result is:

Theorem 2.3. *Let $T, R, X, Y \in \mathbb{M}_n(\mathbb{C})$ such that $T, R > 0$. Then*

$$s_j(X^*T\#_\mu RX + Y^*R\#_\mu TY) \geq 2c_0 \min(s_k^2(X), s_{j-k+n}^2(Y)) \tag{2.1}$$

for $j, k = 1, ..., n$ with $k \geq j$, where $c_0 = \min\left(\frac{s_n^{\mu+1}(T)}{s_1^{\mu}(T)}, \frac{s_n^{\mu+1}(R)}{s_1^{\mu}(R)}\right)$.

Proof. Since $T\#_\mu R > 0$ and $R\#_\mu T > 0$, we have

$$\begin{aligned}
& s_j(X^*T\#_\mu RX+Y^*R\#_\mu TY) \\
\geq\ & s_k(X^*T\#_\mu RX) + s_{j-k+n}(Y^*R\#_\mu TY) \qquad \text{(by Lemma 2.1)} \\
\geq\ & \left(s_k^2(X)s_n(T\#_\mu R) + s_{j-k+n}^2(Y)s_n(R\#_\mu T)\right) \qquad \text{(by Lemma 2.2)} \\
\geq\ & \min(s_k^2(X), s_{j-k+n}^2(Y))\left(s_n(T\#_\mu R) + s_n(R\#_\mu T)\right) \\
\geq\ & \min(s_k^2(X), s_{j-k+n}^2(Y)) \begin{pmatrix} s_n(T)\, s_n\left((T^{-1/2}RT^{-1/2})^\mu\right) \\ +s_n(R)\, s_n\left((R^{-1/2}TR^{-1/2})^\mu\right) \end{pmatrix} \\
& \text{(by Lemma 2.2)} \\
=\ & \min(s_k^2(X), s_{j-k+n}^2(Y)) \begin{pmatrix} s_n(T)\, s_n^\mu\left(T^{-1/2}RT^{-1/2}\right) \\ +s_n(R)\, s_n^\mu\left(R^{-1/2}TR^{-1/2}\right) \end{pmatrix} \\
\geq\ & \min(s_k^2(X), s_{j-k+n}^2(Y))\left(s_n(T)\, s_n^\mu(T^{-1})s_n^\mu(R) + s_n(R)\, s_n^\mu(R^{-1})s_n^\mu(T)\right) \\
& \text{(by Lemma 2.2)} \\
=\ & c_0 \min(s_k^2(X), s_{j-k+n}^2(Y))\left(\frac{s_n^\mu(R)}{s_n^\mu(T)} + \frac{s_n^\mu(T)}{s_n^\mu(R)}\right) \\
\geq\ & 2c_0 \min(s_k^2(X), s_{j-k+n}^2(Y)) \qquad \text{(by The inequality (1.9))}
\end{aligned}$$

for $j = 1, ..., n$ with $k \geq j$. □

An application of Theorem 2.3 can be seen in the following corollary.

Corollary 2.4. *Let* $T, R, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $T, R > 0$. *Then*

$$X^*T\#_\mu RX + Y^*R\#_\mu TY \geq 2c_0 \min(s_n^2(X), s_n^2(Y))I_n \tag{2.2}$$

and

$$T\#_\mu R + R\#_\mu T \geq 2c_0 I_n \tag{2.3}$$

where $c_0 = \min\left(\frac{s_n^{\mu+1}(T)}{s_1^\mu(T)}, \frac{s_n^{\mu+1}(R)}{s_1^\mu(R)}\right)$.

Proof. Since $X^*T\#_\mu RX + Y^*R\#_\mu TY$ is positive semidefinite, we have

$$\begin{aligned} X^*T\#_\mu RX + Y^*R\#_\mu TY & \\ &\geq s_n(X^*T\#_\mu RX + Y^*R\#_\mu TY)I_n \\ &\geq 2c_0 \min(s_n^2(X), s_n^2(Y))I_n \qquad \text{(by the inequality (2.1))} \end{aligned}$$

which proves the inequality (2.2). The inequality (2.3) follows directly from the inequality (2.2) by taking $X = Y = I_n$. ☐

Corollary 2.5. *Let* $T, R \in \mathbb{M}_n(\mathbb{C})$ *such that* $T, R > 0$. *Then*

$$T\#R \geq c_0 I_n$$

where $c_0 = \min\left(\frac{s_n^{3/2}(T)}{s_1^{1/2}(T)}, \frac{s_n^{3/2}(R)}{s_1^{1/2}(R)}\right)$.

Proof. The proof follows by taking $\mu = 1/2$ in the inequality (2.3) and using the property mentioned in the inequality (1.7). ☐

Now, we have the following result:

Theorem 2.6. *Let* $T, R, X \in \mathbb{M}_n(\mathbb{C})$ *such that* $T, R > 0$ *and* X *is Hermitian. Then*

$$|||T\#_\mu RX + XR\#_\mu T||| \geq 2c_0\, |||X|||\,, \tag{2.4}$$

for every unitarily invariant norm. In particular,

$$|||T\#_\mu R + R\#_\mu T||| \geq 2c_0, \tag{2.5}$$

for every unitarily invariant norm, where $c_0 = \min\left(\frac{s_n^{\mu+1}(T)}{s_1^\mu(T)}, \frac{s_n^{\mu+1}(R)}{s_1^\mu(R)}\right)$.

Proof. Since X is Hermitian, there exist an orthonormal basis $\{e_j\}_{j=1}^n$ of $\mathbb{C}^n$ consisting of eigenvectors corresponding to the eigenvalues $\{\lambda_j(X)\}_{j=1}^n$ arranged in such a way that $|\lambda_1(X)| \geq \cdots \geq |\lambda_n(X)|$. Since $s_j(X) = |\lambda_j(X)|$ for $j = 1, ..., n$, we have

$$
\begin{aligned}
&||T\#_\mu RX + XR\#_\mu T||_{(k)} \\
\geq\ & \sum_{j=1}^{k} \left| e_j^* \left(T\#_\mu RX + XR\#_\mu T\right) e_j \right| \qquad \text{(by the identity (1.2))} \\
=\ & \sum_{j=1}^{k} \left| e_j^* T\#_\mu RXe_j + e_j^* XR\#_\mu Te_j \right| \\
=\ & \sum_{j=1}^{k} \left| e_j^* T\#_\mu RXe_j + (Xe_j)^* R\#_\mu Te_j \right| \\
=\ & \sum_{j=1}^{k} \left| \lambda(X) e_j^* (T\#_\mu R + R\#_\mu T) e_j \right| \\
=\ & \sum_{j=1}^{k} |\lambda_j(X)| \ \left(e_j^* (T\#_\mu R + R\#_\mu T) e_j \right) \\
=\ & \sum_{j=1}^{k} s_j(X) \ \left(e_j^* (T\#_\mu R + R\#_\mu T) e_j \right) \\
\geq\ & 2c_0 \sum_{j=1}^{k} s_j(X) \qquad \text{(by the inequality (2.3))} \\
=\ & 2c_0 \, ||X||_{(k)} ,
\end{aligned}
$$

for $k = 1, ..., n$. Now, the inequality (2.4) follows by the Fan Dominance Theorem (see for example [4, p. 93]). □

An application of Theorem 2.6 can be seen in the following corollary.

Corollary 2.7. *Let $T, R, X \in \mathbb{M}_n(\mathbb{C})$ such that $T, R > 0$. Then*

$$|||(T\#_\mu RX + XR\#_\mu T) \oplus (XT\#_\mu R + R\#_\mu TX)||| \geq 2c_0 \, |||X \oplus X||| ,$$

for every unitarily invariant norm, where $c_0 = \min \left(\frac{s_n^{\mu+1}(T)}{s_1^{\mu}(T)}, \frac{s_n^{\mu+1}(R)}{s_1^{\mu}(R)} \right)$.

Proof. Let

$$\tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}, V = \begin{bmatrix} T\#_\mu R & 0 \\ 0 & T\#_\mu R \end{bmatrix},$$

and

$$W = \begin{bmatrix} R\#_\mu T & 0 \\ 0 & R\#_\mu T \end{bmatrix}.$$

Then $\tilde{X}$ is Hermitian and V and W are positive definite. Also, $s_n(V) = s_n(T\#_\mu R)$ and $s_n(W) = s_n(R\#_\mu T)$.

Now,

$$\begin{aligned}
&|||(T\#_\mu RX + XR\#_\mu T) \oplus (XT\#_\mu R + R\#_\mu TX)||| \\
&= |||(XT\#_\mu R + R\#_\mu TX)^* \oplus (T\#_\mu RX + XR\#_\mu T)||| \\
&\qquad \text{(by the identities (1.5))} \\
&= |||(T\#_\mu RX^* + X^*R\#_\mu T) \oplus (T\#_\mu RX + XR\#_\mu T)||| \\
&= \left|\left|\left| \begin{bmatrix} T\#_\mu RX^* + X^*R\#_\mu T & 0 \\ 0 & T\#_\mu RX + XR\#_\mu T \end{bmatrix} \right|\right|\right| \\
&= \left|\left|\left| \begin{bmatrix} 0 & T\#_\mu RX + XR\#_\mu T \\ T\#_\mu RX^* + X^*R\#_\mu T & 0 \end{bmatrix} \right|\right|\right| \\
&\qquad \text{(by the identities (1.5))} \\
&= \left|\left|\left| V\tilde{X} + \tilde{X}W \right|\right|\right| \\
&\geq 2c_0 \left|\left|\left| \tilde{X} \right|\right|\right| \qquad \text{(by Theorem 2.6)} \\
&= 2c_0 \left|\left|\left| \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right|\right|\right| \\
&= 2c_0 |||X \oplus X|||,
\end{aligned}$$

as required. □

3 Inequalities for a certain type of positive definite matrices

In this section, we give matrix and norm inequalities for a certain type of positive definite matrices. We start this section by the following result.

Theorem 3.1. *Let $T, R, X, Y \in \mathbb{M}_n(\mathbb{C})$ such that $T, R > 0$. Then*

$$s_j(X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y) \geq 2c_0 \min(s_k^2(X), s_{j-k+n}^2(Y)) \tag{3.1}$$

for $j, k = 1, ..., n$ with $k \geq j$, where $c_0 = \min\left(\frac{s_n(T)}{s_1(T)}, \frac{s_n(R)}{s_1(R)}\right)$.

Proof. Since $T > 0$ and $R > 0$, we have

$$s_j(X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y)$$

$$\begin{aligned}
&\geq s_k(X^*T^{1/2}R^{-1}T^{1/2}X) + s_{j-k+n}(Y^*R^{1/2}T^{-1}R^{1/2}Y) && \text{(by Lemma 2.1)}\\
&\geq s_k^2(X)s_n\left(T^{1/2}R^{-1}T^{1/2}\right) + s_{j-k+n}^2(Y)s_n\left(R^{1/2}T^{-1}R^{1/2}\right) && \text{(by Lemma 2.2)}\\
&\geq \min(s_k^2(X), s_{j-k+n}^2(Y))(s_n\left(T^{1/2}R^{-1}T^{1/2}\right) + s_n\left(R^{1/2}T^{-1}R^{1/2}\right))\\
&\geq \min(s_k^2(X), s_{j-k+n}^2(Y))(s_n(T)s_n\left(R^{-1}\right) + s_n(R)s_n\left(T^{-1}\right)) && \text{(by Lemma 2.2)}\\
&\geq c_0 \min(s_k^2(X), s_{j-k+n}^2(Y))\left(\frac{s_n(T)}{s_n(R)} + \frac{s_n(R)}{s_n(T)}\right)\\
&\geq 2c_0 \min(s_k^2(X), s_{j-k+n}^2(Y)) && \text{(by The inequality (1.9))}
\end{aligned}$$

for $j = 1, ..., n$ with $k \geq j$. □

An application of Theorem 3.1 can be seen in the following result.

Corollary 3.2. *Let $T, R, X, Y \in \mathbb{M}_n(\mathbb{C})$ such that $T, R > 0$. Then*

$$X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y \geq 2c_0 \min(s_n^2(X), s_n^2(Y))I_n$$

and

$$T^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2} \geq 2c_0 I_n \tag{3.2}$$

where $c_0 = \min\left(\frac{s_n(T)}{s_1(T)}, \frac{s_n(R)}{s_1(R)}\right)$.

Proof. Since $X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y$ is positive semidefinite, we have

$$X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y$$

$$\begin{aligned}
&\geq s_n(X^*T^{1/2}R^{-1}T^{1/2}X + Y^*R^{1/2}T^{-1}R^{1/2}Y)I_n\\
&\geq c_0 \min(s_n^2(X), s_n^2(Y))I_n && \text{(by the inequality (3.1))}.
\end{aligned}$$

□

A generalization of the inequality (1.9) in the setting of unitarily invariant norms can be seen in the following result.

Theorem 3.3. *Let $T, R, X \in \mathbb{M}_n(\mathbb{C})$ such that $T, R > 0$ and X is Hermitian. Then*

$$\left|\left|\left|T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2}\right|\right|\right| \geq 2c_0 \left|\left|\left|X\right|\right|\right|, \tag{3.3}$$

for every unitarily invariant norm, and

$$\left|\left|\left|T^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2}\right|\right|\right| \geq 2c_0, \tag{3.4}$$

for every unitarily invariant norm, where $c_0 = \min\left(\frac{s_n(T)}{s_1(T)}, \frac{s_n(R)}{s_1(R)}\right)$.

Proof. Since X is Hermitian, there exist an orthonormal basis $\{e_j\}_{j=1}^n$ of $\mathbb{C}^n$ consisting of eigenvectors corresponding to the eigenvalues $\{\lambda_j(X)\}_{j=1}^n$ arranged in such a way that $|\lambda_1(X)| \geq \cdots \geq |\lambda_n(X)|$. Since $s_j(X) = |\lambda_j(X)|$ for $j = 1, ..., n$, we have

$$\begin{aligned}
&\left|\left|T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2}\right|\right|_{(k)} \\
\geq\ & \sum_{j=1}^{k} \left|e_j^*\left(T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2}\right)e_j\right| \qquad \text{(by the identity (1.2))} \\
=\ & \sum_{j=1}^{k} \left|e_j^*T^{1/2}R^{-1}T^{1/2}Xe_j + e_j^*XR^{1/2}T^{-1}R^{1/2}e_j\right| \\
=\ & \sum_{j=1}^{k} \left|e_j^*T^{1/2}R^{-1}T^{1/2}Xe_j + (Xe_j)^*R^{1/2}T^{-1}R^{1/2}e_j\right| \\
=\ & \sum_{j=1}^{k} \left|\lambda(X)e_j^*(T^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2})e_j\right| \\
=\ & \sum_{j=1}^{k} |\lambda_j(X)| \ \left(e_j^*(T^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2})e_j\right) \\
=\ & \sum_{j=1}^{k} s_j(X) \ \left(e_j^*(T^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2})e_j\right) \\
\geq\ & 2c_0\sum_{j=1}^{k} s_j(X) \qquad \text{(by the inequality (3.2))} \\
=\ & 2c_0 \left|\left|X\right|\right|_{(k)},
\end{aligned}$$

for $k = 1, ..., n$. Now the inequality (3.3) follows by the Fan Dominance Theorem (see for example, [4, p. 93]). □

An application of Theorem 3.3 can be seen in the following result.

Corollary 3.4. *Let* $T, R, X \in \mathbb{M}_n(\mathbb{C})$ *such that* $T, R > 0$. *Then*

$$\left|\left|\left|(T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2}) \oplus (XT^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2}X)\right|\right|\right| \geq 2c_0 \left|\left|\left|X \oplus X\right|\right|\right|,$$

for every unitarily invariant norm, where $c_0 = \min\left(\frac{s_n(T)}{s_1(T)}, \frac{s_n(R)}{s_1(R)}\right)$.

Proof. Let

$$\tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}, V = \begin{bmatrix} T^{1/2}R^{-1}T^{1/2} & 0 \\ 0 & T^{1/2}R^{-1}T^{1/2} \end{bmatrix},$$

and

$$W = \begin{bmatrix} R^{1/2}T^{-1}R^{1/2} & 0 \\ 0 & R^{1/2}T^{-1}R^{1/2} \end{bmatrix}.$$

Then $\tilde{X}$ is Hermitian and V and W are positive definite.

Also, $s_n(V) = s_n\left(T^{1/2}R^{-1}T^{1/2}\right)$ and $s_n(W) = s_n\left(R^{1/2}T^{-1}R^{1/2}\right)$.

Now,

$$\begin{aligned}
&\left|\left|\left|(T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2}) \oplus (XT^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2}X)\right|\right|\right| \\
&= \left|\left|\left|(XT^{1/2}R^{-1}T^{1/2} + R^{1/2}T^{-1}R^{1/2}X)^* \oplus (T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2})\right|\right|\right| \\
&\qquad \text{(by the identities (1.5))} \\
&= \left|\left|\left|(T^{1/2}R^{-1}T^{1/2}X^* + X^*R^{1/2}T^{-1}R^{1/2}) \oplus (T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2})\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} T^{1/2}R^{-1}T^{1/2}X^* + X^*R^{1/2}T^{-1}R^{1/2} & 0 \\ 0 & T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2} \end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} 0 & T^{1/2}R^{-1}T^{1/2}X + XR^{1/2}T^{-1}R^{1/2} \\ T^{1/2}R^{-1}T^{1/2}X^* + X^*R^{1/2}T^{-1}R^{1/2} & 0 \end{bmatrix}\right|\right|\right| \\
&\qquad \text{(by the identities (1.5))} \\
&= \left|\left|\left|V\tilde{X} + \tilde{X}W\right|\right|\right| \\
&\geq 2c_0 \left|\left|\left|\tilde{X}\right|\right|\right| \qquad \text{(by Theorem 3.3)} \\
&= 2c_0 \left|\left|\left|\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right|\right|\right| = 2c_0 \left|\left|\left|X \oplus X\right|\right|\right|,
\end{aligned}$$

as required. □

Acknowledgement. The authors are grateful to the Middle East University, Amman, Jordan for the financial support granted to cover the publication fee of this research article.

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