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A remark on the qualitative conditions of nonlinear IDEs

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Abstract

In this paper, the stability (S), asymptotic stability (AS), uniform stability (US) of null solution and boundedness of solutions of non-linear Volterra integro-differential equations (IDEs) are investigated by means of the Lyapunov-Krasovskii method (LKM). Four new theorems are provided on the mentioned concepts for the considered (IDEs). Compared with past related results in the literature, our conditions are simple, weaker and convenient for testing and applications. The obtained results also improve and extend the past related results. Finally, two numerical examples are provided to illustrate the applications of our results.

1 Introduction

Up till now, the qualitative behavior of linear and nonlinear (IDEs) have been discussed extensively in the literature [1-23, 25-65]. One of the important techniques or methods for investigating the qualitative behavior of linear and nonlinear (IDEs) is known as the Lyapunov-Krasovskii functional (LKF) method [24]. To utilize this useful method to nonlinear (IDEs), we

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need to construct or define a suitable (LKF) or (LKFs). Depending on convenient (LKFs), proper conditions for the qualitative behavior of nonlinear (IDEs) are brought out for problems under study. In this paper, exploiting this approach, we investigate the qualitative behavior of nonlinear (IDEs). Our objective is two-fold. First, we prove some results that already exist in the literature but under more suitable weaker conditions. Secondly, we establish some additional new results. For this, we construct an appropriate new (LKF).
For now, let us now summarize a few recent results on the qualitative behavior of nonlinear (IDEs).

In 2018, Alahmadi, et al. [1] considered the following (IDE) given by

$$y'(t) = A(t)y(t) + f(y(t)) + \int_0^t C(t,s)h(y(s))ds + p(t). \tag{1.1}$$

Alahmadi, et al. [1] studied the qualitative properties of the nonlinear Volterra (IDE) (1.1) such as boundedness and stability of solutions to the nonlinear Volterra (IDE) (1.1). They applied a Lyapunov functional coupled with the Laplace transform to proceed with the proofs of the results therein which they then illustrated by examples.

Afterwards, using the same method, El Hajji [9] obtained the results on the (S), (AS) and (B) of solutions of (IDE) (1.1) were obtained.

Finally, Tunç and Tunç [51] analyzed some qualitative properties of solutions of non-linear Volterra (IDEs) with constant delay and without delay, which are more general than that (IDE) (1.1), by means of the Razumikhin method. A suitable Lyapunov function was defined and then utilized to that (IDEs) such that results of El Hajji [9] can be fulfilled under weaker conditions and the authors also gave some new results on the qualitative properties of those (IDEs).

Now we introduce the results of Alahmadi, et al. [1].
Consider (IDE) (1.1) and assume that

$$p(t) \neq 0.$$

Theorem 1 ([1]). We assume the following assumptions hold:

(A1) Let $\lambda_1, \lambda_2, M \in \mathbb{R}$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $M > 0$ such that the functions

f, h and p satisfy:

$$|f(y)| \leq \lambda_1 |y|, \quad \forall y \in \mathbb{R},$$
$$|h(y)| \leq \lambda_2 |y|, \quad \forall y \in \mathbb{R},$$
$$|p(t)| \leq M, \quad \forall t \geq 0.$$

(A2) There exists a positive and differentiable scalar function ϕ such that

$$\phi(t) \geq 0,\ \frac{d}{dt}\phi(t) \leq 0,\ \forall t \geq 0,\ \int_0^\infty \phi(t) < \infty \ \textit{and}$$
$$\lambda_2 |C(t,s)| + \lambda_3 \frac{d}{dt}\phi(t-s) \leq 0, \quad \forall 0 \leq s \leq t < \infty, \quad t \in \mathbb{R}.$$

(A3) There exists a negative scalar function $A(t)$ defined on $[0,\infty)$ and a positive constant α such that

$$A(t) + \lambda_1 + \lambda_3 \phi(0) \leq -\alpha, \quad \forall 0 \leq s \leq t < \infty.$$

(A4) There exists a uniformly continuous scalar function $\beta(t)$ on $[0,\infty)$ and a continuous scalar function $H(t)$ such that

$$H(t) = \beta(t) + \lambda_3 \int_0^t \phi(t-s)\beta(s)ds,$$
$$\frac{d}{dt}H(t) = -\alpha\beta(t),\ \alpha > 0,\ \beta(0) = 1,$$

which imply that

$$\beta(t) + \int_0^t \{\lambda_3 \phi(t-s) + \alpha\}\beta(s)ds = 1,$$
$$\beta(t) > 0 \ \textit{on}\ [0,\infty),\ \beta(t) \in L^1[0,\infty)$$

and

$$\lim_{t\to\infty} \beta(t) = 0.$$

Then all solutions of (IDE) (1) are bounded by $V(0) + \frac{M}{\alpha}$. Moreover, $\lim_{t\to\infty} |y(t)| = \frac{M}{\alpha}$.

It is worth noting that the following (LKF) was defined and used by Alahmadi, et al. [1]:

$$V(t) = |y| + \lambda_3 \int_0^t \phi(t-s)|y(s)|ds,\ \forall t \geq 0,$$

where (IDE) (1.1) was used with $y(0) = y_0$, $f(0) = h(0) = 0$ by setting $p(t) = 0$.

Theorem 2 ([1]). Under the following assumptions:

(C1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 0$ and $\lambda_2 > 0$ such that the functions f and g satisfy

$$|f(y)| \leq \lambda_1 |y|, \ \forall y \in \mathbb{R},$$
$$|h(y)| \leq \lambda_2 |y|, \ \forall y \in \mathbb{R},$$

(C2) The scalar function $A(t)$ is negative definite; that is, $A(t) \leq 0$, such that

$$|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \geq 0, \ \forall 0 \leq s \leq t < \infty,$$

Then the zero solution of (IDE) (1.1) is (S).

Theorem 3 ([7]). In addition to assumptions $(C1)$ and $(C2)$, suppose that there are $t_2 \geq 0$ and $\alpha > 0$ such that:

(C3) $|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \geq \alpha$, $\forall 0 \leq s \leq t < \infty$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 0$ and $\lambda_2 > 0$,

(C4) Both $|A(s)|$ and $\int_s^t |C(u,s)| ds$ are bounded.

Then the zero solution of (IDE) (1.1) is (AS).

Again, it is worth noting that the following (LKF) was defined and used by Alahmadi, et al. [1]:

$$H(t, y(.)) = |y| + \int_0^t \big(|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(u,s)| du \big) |y(s)| ds,$$

$\forall 0 \leq s \leq t < \infty$.

Our paper was motivated by the works of Alahmadi, et al. [1], El-Hajji [9], Tunç and Tunç [51] as well as the books and papers in the bibliography of this paper.
As a result, we consider a non-linear (IDE) having the form:

$$\frac{dy}{dt} = -a(t)g(y) - f(y) + \int_0^t C(t,s,y(s))h(s,y(s))ds + p(t,y), \qquad (1.2)$$

where $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $y \in \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$, $a(t) \in C(\mathbb{R}^+, (0, \infty))$, $g(0) = 0$, $f(0) = 0$, g, $f \in C(\mathbb{R}, \mathbb{R})$, $h(s, 0) = 0$, $h, p \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $C(t, s, y) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ with $0 \leq s \leq t < \infty$. We assume that the functions g, f, $C(.)$, h and p are such that the existence and uniqueness of solutions of (IDE) (1.2) are satisfied.

We first investigate the (S), (US), (AS), (I) and the boundedness of solutions at infinity of (IDE) (1.2) when $p(t, x) = 0$. Secondly, we study the boundedness of solutions of (IDE) (1.2) when $p(t, x) \neq 0$. We see that, corresponding to the term y, the kernel $C(t, s)$, the functions $h(y)$ and $p(t)$ in (IDE) (1.1), (see Alahmadi, et al. [1] and El-Hajji [9]). We take the nonlinear function $g(y)$, the nonlinear kernel $C(t, s, y)$, the nonlinear functions $h(s, y)$ and $p(t, y)$, respectively, in (IDE) (1.2). These choices clearly show the first difference of this work from that of Alahmadi, et al. [1], El-Hajji [9] and Tunç and Tunç [51] by using (LKF)instead of the Laplace transform. For brevity, x denotes $x(t)$.

2 Qualitative analysis

We now present the qualitative results of solutions of (IDE) (2).

Consider (IDE) (2) and, through the proofs of Theorems 4-6, we suppose that $p(t, x) = 0$.

Theorem 4. We assume that the following hypotheses hold:

(H1) Let $f_0,\ g_0,\ h_0 \in \mathbb{R}$, $f_0 > 0,\ g_0 > 0,\ h_0 > 0$ such that the functions f, g and h satisfy:

$$\begin{aligned} a(t) &> 0,\ \forall t \in \mathbb{R}^+ \\ g(0) = 0,\ g(y)y &\geq g_0 y^2,\ (y \neq 0),\ \forall y \in \mathbb{R}, \\ f(0) = 0,\ f(y)y &\geq f_0 y^2,\ (y \neq 0),\ \forall y \in \mathbb{R}, \\ h(s, 0) = 0,\ |h(s, y(s))| &\leq h_0 |y(s)|,\ (y \neq 0),\ \forall s \in \mathbb{R}^+,\ \forall y \in \mathbb{R}, \end{aligned}$$

(H2) Let $c_0 \in \mathbb{R}$, $c_0 > 0$, such that

$$|C(t, s, y(s))| \leq c_0 |K(t, s)|,\ 0 \leq s \leq t < \infty.$$

(H3) Let $\theta_0 \in \mathbb{R}$, $\theta_0 > 0$, such that

$$g_0 a(t) + f_0 - c_0 h_0 \int_0^t |K(t, s)| ds - c_0 h_0 \int_t^\infty |K(u, t)| du \geq \theta_0$$

with

$$\int_0^t |K(t,s)|ds < \infty,\ 0 \le s \le t < \infty,$$
$$\int_t^\infty |K(u,t)|du < \infty,\ 0 \le t \le u < \infty.$$

Then, the null solution of (IDE) (1.2) is (AS).

Proof. We define a new (LKF) $W = W(t,y)$ by

$$W(t,y) = \frac{1}{2}y^2 + \gamma \int_0^t \int_t^\infty |K(u,s)|du y^2(s)ds,$$

where $\gamma \in \mathbb{R},\ \gamma > 0$ to be chosen later.
We have

$$W(t,0) = 0$$

and

$$\frac{1}{2}y_0 y^2 \le W(t,y) \le \frac{1}{2}(\bar{y}_0)y^2, \tag{2.3}$$

where $0 < y_0 < 1,\ 1 \le \bar{y}_0,\ y_0, \bar{y}_0 \in \mathbb{R}$.

Differentiating the (LKF) along the solutions of non-linear (IDE) (1.2), we have:

$$\begin{aligned}\frac{d}{dt}W(t,y) &= y\frac{dy}{dt} \\ &= -a(t)g(y)y - f(y)y + y\int_0^t C(t,s,y(s))h(s,y(s))ds \\ &+ \gamma y^2 \int_t^\infty |K(u,t)|du - \gamma \int_0^t |K(t,s)|y^2(s)ds.\end{aligned}$$

Utilizing hypotheses $(H1) - (H2)$ and the simple inequality $2|uv| \le u^2 + v^2$, we get:

$$\begin{aligned}
\frac{d}{dt}W(t,y) &\le -g_0a(t)y^2 - f_0y^2 + |y|\int_0^t |C(t,s,y(s))||h(s,y(s))|ds \\
&+ \gamma y^2\int_t^\infty |K(u,t)|du - \gamma\int_0^t |K(t,s)|y^2(s)ds \\
&\le -g_0a(t)y^2 - f_0y^2 + c_0|y|\int_0^t |K(t,s)||h(s,y(s))|ds \\
&+ \gamma y^2\int_t^\infty |K(u,t)|du - \gamma\int_0^t |K(t,s)|y^2(s)ds \\
&\le -g_0a(t)y^2 - f_0y^2 + c_0h_0|y|\int_0^t |K(t,s)||y(s)|ds \\
&+ \gamma y^2\int_t^\infty |K(u,t)|du - \gamma\int_0^t |K(t,s)|y^2(s)ds \\
&\le -g_0a(t)y^2 - f_0y^2 + c_0h_0\int_0^t |K(t,s)|(y^2(t)+y^2(s))ds \\
&+ \gamma y^2\int_t^\infty |K(u,t)|du - \gamma\int_0^t |K(t,s)|y^2(s)ds \\
&= -g_0a(t)y^2 - f_0y^2 + c_0h_0y^2\int_0^t |K(t,s)|ds + c_0h_0\int_0^t |K(t,s)|y^2(s)ds \\
&+ \gamma y^2\int_t^\infty |K(u,t)|du - \gamma\int_0^t |K(t,s)|y^2(s)ds.
\end{aligned}$$

Let $\gamma = c_0h_0$. In view of hypothesis $(H3)$, we get:

$$\begin{aligned}
\frac{d}{dt}W(t,y) &\le -g_0a(t)y^2 - f_0y^2 + c_0h_0y^2\int_0^t |K(t,s)|ds + c_0h_0y^2\int_t^\infty |K(u,t)|du \\
&= -[g_0a(t) + f_0 - c_0h_0\int_0^t |K(t,s)|ds - c_0h_0\int_t^\infty |K(u,t)|du]y^2 \\
&\ge -\theta_0y^2 \le 0.
\end{aligned}$$

This inequality implies that the null solution of the non-linear (IDE) (1.2) is (AS).

Corollary 1. If hypotheses $(H1)-(H3)$ are fulfilled, then the null solution of the non-linear (IDE) (1.2) is (S) and (US).

Theorem 5. Besides hypotheses $(H1)$ and $(H2)$, suppose the following is satisfied:

(H4)

$$g_0 a(t) + f_0 - c_0 h_0 \int_0^t |K(t,s)|ds - c_0 h_0 \int_t^\infty |K(u,t)|du \geq 0$$

with

$$\int_0^t |K(t,s)|ds < \infty, \quad \int_t^\infty |K(u,t)|du < \infty.$$

Then, the solutions of (IDE) (1.2) are (B) as $t \to \infty$.

Proof. By hypotheses $(H1)$, $(H2)$ and $(H4)$, we have

$$\frac{1}{2}(y_0)y^2 \leq W(t,y)$$

and

$$\dot{W}(t,y(t)) \leq 0.$$

As far as the boundedness of solutions at infinity, these estimates and an integration of $\dot{W}(t,y(t)) \leq 0$ yield

$$\frac{1}{2}(y_0)y^2 \leq W(t,y) \leq W(t_0,y(t_0)).$$

Then,

$$|y(t)| \leq \sqrt{2y_0^{-1}W(t_0,y(t_0))} \equiv \textit{ a positive constant}$$

provided that $W(t_0,y(t_0)) \neq 0$. Consequently, by taking the limit of the prior inequality as $t \to \infty$, we conclude that the solutions of (IDE) (1.2) are bounded at infinity. $p(t,y) = 0$.

Theorem 6. Suppose hypotheses $(H1) - (H3)$ are fulfilled. Then, the solutions of (IDE) (1.2) are square integrable.

Proof. As before, by using the (LKF) $W(t,x)$ and hypotheses $(H1) - (H3)$, we get the following inequality:

$$\frac{d}{dt}W(t,y) \leq -\theta_0 y^2.$$

An integration yields

$$\theta_0 \int_{t_0}^{t} x^2(s)ds \leq W(t_0, x(t_0)) - W(t, x(t)) \leq W(t_0, x(t_0)).$$

As a consequence of this inequality, one can conclude that

$$\int_{t_0}^{\infty} x^2(s)ds \leq \theta_0^{-1} W(t_0, x(t_0)) < \infty.$$

This completes the proof.

Consider (IDE) (1.2) and let $p(t,x) \neq 0$.

Theorem 7. Assume hypotheses $(H1)$, $(H2)$, $(H4)$ and the following hypothesis is satisfied:

(H5) $|p(t,y)| \leq \frac{1}{2}|r(t)||y|, \ \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}, \ r(t) \in L^1[0,\infty)$.

Then, the solutions of (IDE) (2) are bounded.

Proof. Utilizing the mathematical operations in Theorem 4 and the hypothesis $|p(t,y)| \leq 2^{-1}|r(t)||y|$, we have

$$\frac{d}{dt}W(t,y) \leq yp(t,y) \leq \frac{1}{2}|r(t)|y^2 = |r(t)|W(t,y).$$

An integration results in

$$W(t, x(t)) \leq W(t_0, x(t_0)) \exp\Big[\int_{t_0}^{t} |r(s)|ds\Big].$$

From this fact and the inequality (2.3), we get

$$\frac{1}{2}y^2 \leq W(t,y) \leq W(t_0, y(t_0)) \exp\Big[\int_{t_0}^{\infty} |r(s)|ds\Big].$$

This inequality finishes the proof. Note the advantage that we did not use Gronwall inequality.

Example 1. Consider the following scalar nonlinear Volterra (IDE):

$$\begin{aligned}\frac{dy}{dt} &= -(10 + \frac{1}{2}\exp(-t))(2y + y^3) + 2y + \frac{y}{1+y^2} \\ &\quad + \int_0^t \exp(-t + s - |y(s)|) \sin y(s) ds. \qquad (2.4)\end{aligned}$$

A comparison of (IDE) (2.4) and (IDE) (1.2) implies the existence of the following relations:

$$
\begin{aligned}
&a(t) = 10 + \frac{1}{2}\exp(-t) \geq 10 > 0, \ \forall t \geq 0,\\
&g(y) = 2y + y^3, \ g(0) = 0,\\
&g(y)y = 2y^2 + y^4 \geq 2y^2 = g_0 y^2, \ g_0 = 2, \ (y \neq 0), \ \forall y \in \mathbb{R},\\
&f(y) = 2y + \frac{y}{1+y^2}, \ f(0) = 0,\\
&f(y)y = 2y^2 + \frac{y^2}{1+y^2} \geq 2y^2 = f_0 y^2, \ f_0 = 2, \ (y \neq 0), \ \forall y \in \mathbb{R},\\
&h(s, y(s)) = \sin y(s), \ h(s, 0) = 0,\\
&|h(s, y(s))| = |\sin y(s)| \leq h_0 |y(s)|, \ h_0 = 1, \ (y \neq 0), \ \forall y \in \mathbb{R},\\
&C(t, s, y(s)) = \exp(-t + s - |y(s)|),\\
&|C(t, s, y(s))| = \exp(-t + s - |y(s)|),\\
&\qquad\qquad\qquad \leq \exp(-t + s) = c_0 |K(t, s)|,\\
&|K(t, s)| = \exp(-t + s), \ c_0 = 1,\\
&\int_0^t |K(t, s)| ds = \int_0^t \exp(-t + s) ds = 1 - \exp(-t) \leq 1 < \infty, \ 0 \leq s \leq t < \infty,\\
&\int_t^\infty |K(u, t)| du = \int_t^\infty \exp(-u + t) du = 1 < \infty, \ 0 \leq t \leq u < \infty,\\
&g_0 a(t) + f_0 - c_0 h_0 \int_0^t |K(t, s)| ds - c_0 h_0 \int_t^\infty |K(u, t)| du\\
&= 20 + \exp(-t) + 2 - 1 + \exp(-t) - 1 = 20 + 2\exp(-t) \geq 20 = \theta_0.
\end{aligned}
$$

As a consequence of this discussion, the hypotheses of Theorems 4-6 are satisfied. From this point, the (S), (AS), (US) of the null solution and the (I) of solutions, the (B) of solutions at infinity of nonlinear Volterra (IDE) (2.4) are proved.

Figure 1 shows the solution of (IDE) (2.4) and displays the behaviors of the paths of the solution for different initial values.

Example 2. Now, consider the following scalar nonlinear Volterra (IDE):

$$
\begin{aligned}
\frac{dy}{dt} &= -(10 + \frac{1}{2}\exp(-t))(2y + y^3) + 2y + \frac{y}{1+y^2}\\
&\quad + \int_0^t \exp(-t + s - |y(s)|) \sin y(s) ds + \frac{1}{2}\frac{y\exp(-t)}{1+\exp(-y^2)}. \qquad (2.5)
\end{aligned}
$$

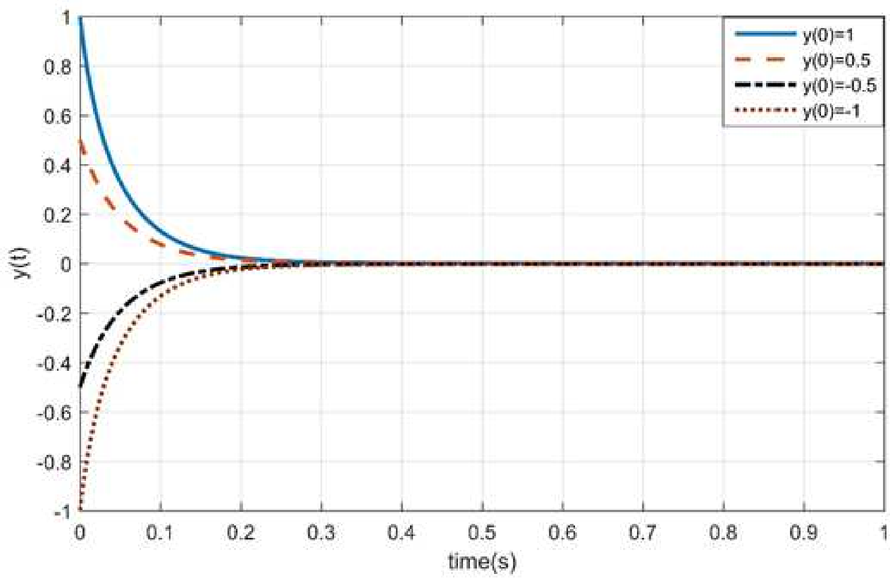

Figure 1: Behaviors of the paths of the solution $y(t)$ of (IDE) (2.4) for different initial values.

From Example 1, we verified that hypotheses $(H1) - (H4)$ can hold for (IDE) (2.5). We only need to prove that hypothesis $(H5)$ is satisfied.

A comparison of (IDE) (2.5) and (IDE) (1.2) implies the existence of the following relations:

$$p(t,y) = \frac{1}{2}\frac{y\exp(-t)}{1+\exp(-y^2)}, \ \forall t \geq 0, \ \forall y \in \mathbb{R},$$

$$|p(t,y)| = \frac{1}{2}\frac{|y|\exp(-t)}{1+\exp(-y^2)} \leq \frac{1}{2}\exp(-t)|y|,$$

$$|r(t)| = \exp(-t), \quad \int_0^\infty |r(t)|dt = \int_0^\infty \exp(-t)dt = 1 < \infty, \ thus, \ r(t) \in L^1[0,\infty).$$

As a consequence of these operations, it follows that hypothesis $(H5)$ is satisfied. Consequently, the solutions of (IDE) (2.5) are bounded.

Figure 2 shows the solution of (IDE) (2.5) and displays the behaviors of the orbits of the solution $y(t)$ for different initial values.

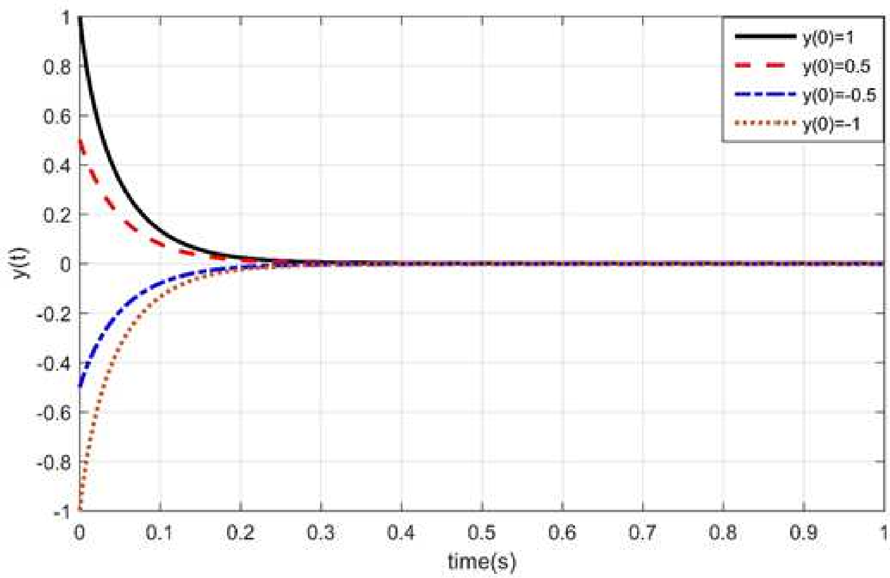

Figure 2: Behaviors of the paths of the solution $y(t)$ of (IDE) (2.5) for different initial values.

3 Conclusion

In this paper, we studied the (S), (AS), (US), (I) and (B) of solutions of nonlinear Volterra (IDEs). We have proved four theorems on the (S), (AS), (US), (I) and (B) of solutions by the help of (LKM). Two nonlinear examples of Volterra (IDEs) were addressed to test the practicability of the given results and the convenience of the proposed (LKF). Compared with the qualitative results in the literature on (IDEs), our results improve and extend those in Alahmadi, et al. [1], El Hajji [9] and Tunç and Tunç [51]. We obtained the results of Alahmadi et al. [1] and El Hajji [9] under weaker conditions.

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