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On the Faithfulness of the Representations of the Extraspecial 2-Groups E_m^{-1}

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Abstract

We consider a family of representations of the braid groups B_n corresponding to a specific solution to the Yang-Baxter equation. The images of the pure braid group P_n, a normal subgroup of B_n, under these representations are extraspecial 2-groups and the images of the braid group B_n are extensions of extraspecial 2-groups. We determine conditions under which any representation of the extraspecial 2-group, E_m^{-1}, is faithful. We then show that the irreducible representations of E_m^{-1}, constructed by Franko, Rowell and Wang, are faithful if and only if $m = 2k$ or $m = 2k - 1$ (k odd); where as it is not faithful if $m = 2k - 1$ (k even).

1 Introduction

Let B_n be the Artin's braid group on n strands. The kernel of the surjective group homomorphism from B_n to the symmetric group S_n that sends each generator σ_i of B_n to $(i, i+1)$ is the pure braid group P_n. The representations of B_n are of great importance to mathematicians [2] and physicists [9]. In addition, researchers gave a great value for representations of the pure braid group P_n (see [1]) and for the (nearly) extra-special 2-groups E_m^v for

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their close association with P_n and B_n [5]. The authors in [5] construct a representation π_n corresponding to the flipped R-matrix
$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ that satisfies the Yang-Baxter equation (YBE):

$$(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R), \text{ where}$$

I_2 is the 2×2 identity matrix. All solutions to the YBE of the form $R : U \otimes U \to U \otimes U$ with U 2-dimensional have been listed in [8]. Dye found all unitary solutions of this form to the braid relations based on this list [4]. The images of the pure braid group under π_n are (nearly) extra-special 2-groups E_{n-1}^{-1}. The images of the braid group B_n are extensions of the (nearly) extra-special 2-groups E_{n-1}^{-1} by the symmetric group S_n (see [5]). The question of faithfulness of the braid group has been the subject of research for a long time. It has been shown that the Burau representation is faithful for $n \leq 3$ and is not faithful for $n \geq 5$ [2]. However, the faithfulness of the Burau representation in the case $n = 4$ is still unknown in spite of some results [3]. In section 4, we study the faithfulness of the representations of E_m^{-1} for all dimensions. Using the generators of E_m^{-1} and their relations, the normal form of an element in E_m^{-1} and simple rules in linear algebra, we find a sufficient condition that guarantees the faithfulness of any representation of E_m^{-1} according to the parity of m (Theorem 4.2). Also, by Theorem 4.2 we show that all irreducible representations of E_{2k}^{-1} given in [5] are faithful and those irreducible representations of E_{2k-1}^{-1} in [5] are faithful iff k is odd (Corollary 4.5). Moreover, we construct new representations of the pure braid group P_n. These representations are faithful if and only if π_n is faithful (Corollary 4.7).

2 Preliminaries

Definition 2.1. *[2] The braid group on n strings, B_n, is the abstract group with presentation*
$B_n = \{\sigma_1, ..., \sigma_{n-1}; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$ *for* $i = 1, 2, ..., n-2,$ $\sigma_i\sigma_j = \sigma_j\sigma_i$ *if* $|i - j| \geq 2\}$.

The generators $\sigma_1, ..., \sigma_{n-1}$ are called the standard generators of B_n.

Definition 2.2. *[2] The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \longrightarrow S_n$, defined by $\sigma_i \longrightarrow (i, i+1), 1 \leq i \leq n-1$. It has the following generators:*

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, 1 \leq i, j \leq n$$

.

Proposition 2.3. *(Shur's Lemma) Suppose that F is $n \times n$ matrix such that $F\alpha(g) = \alpha(g)F$ for every $g \in G$, where α is an irreducible representation of the group G. Then $F = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the $n \times n$ identity matrix.*

As well-known, any invertible solution of YBE gives rise to representations of B_n for any n. Consider the representation π_n of B_n corresponding to the matrix R.

Proposition 2.4. *[5] The representation $\pi_n : B_n \to GL_{2^n}(\mathbb{C})$ is unitary and defined by:*

$$\pi_n(\sigma_i) = I_2^{\otimes(i-1)} \otimes R \otimes I_2^{\otimes(n-i-1)},$$

where σ_i is the ith braid generator.

To exploit the relationship between B_n and P_n, the authors in [5] restrict π_n to P_n and introduce the following definition.

Definition 2.5. *[5] $H_n = \pi_n(P_n)$ and $G_n = \pi_n(B_n)$.*

We use the convention "left into right" for tensor products of matrices; that is, if

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, then $A \otimes B = \begin{pmatrix} b_1A & b_2A \\ b_3A & b_4A \end{pmatrix}$.

Some matrices will be needed, so we define them here:

a) I_m is $m \times m$ identity matrix.

b) $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

c) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

d) $g_i = \pi_n((\sigma_i)^2) = I_2^{\otimes(i-1)} \otimes R^2 \otimes I_2^{\otimes(n-i-1)}$ (we ignore the dependence of g_i on n).

e) $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.

Lemma 2.6. *H_n is generated by $g_1, ..., g_{n-1}$.*

3 Definitions, Properties and Irreducible Representations of E_m^v

Definition 3.1. *The group E_m^v is the abstract group generated by*

$$x_1, ..., x_m$$

with relations:

$$x_i^2 = v, \qquad 1 \le i \le m \tag{1}$$

$$x_i x_j = x_j x_i, \qquad |i-j| \ge 2 \tag{2}$$

$$x_{i+1} x_i = -x_i x_{i+1}, \qquad 1 \le i \le m, \tag{3}$$

where -1 is an order two central element, and $v = \pm 1$.

E_m^v is a finite group of order 2^{m+1}. These groups have important connections with Clifford algebras. The case $v = -1$ appears in Exercise 3.9 in the text by Fulton and Harris [6] and other cases appeared in [7].

3.1 Properties of E_m^v

Any element in E_m^v can be written in the normal form $\pm x_1^{\alpha_1} ... x_m^{\alpha_m}$, where $\alpha_i \in \mathbb{Z}_2$. This form is unique by the following lemma.

Lemma 3.2. *[5] Denote by $Z(E^v_m)$ the center of E^v_m. We have*

(a) $Z(E^v_m) = \begin{cases} \{\pm 1\} & m \text{ even}, \\ \{\pm 1, \pm x_1 x_3 ... x_m\} & m \text{ odd} \end{cases}$,

(b) $E^v_m / \{\pm 1\} \cong (Z_2)^m$,

(c) Any $x \in E^v_m / Z(E^v_m)$ *is conjugate to* $-x$,

(d) Any nontrivial normal subgroup of E^v_m *intersects* $Z(E^v_m)$ *nontrivially,*

(e) For $m = 2k-1$ *odd,* $Z(E^v_{2k-1}) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & if\ v = 1 \text{ or } k \text{ even} \\ \mathbb{Z}_4 & if\ v = -1 \text{ and } k \text{ odd} \end{cases}$,

(f) The normal form $\pm x_1^{\alpha_1} ... x_m^{\alpha_m}$ *is unique.*

Definition 3.3. *[7] A group G of order 2^{m+1} is an extraspecial 2-group if*

(1) the center $Z(G)$ and the commutator subgroup G' coincide and are isomorphic to $\mathbb{Z}_2$.

(2) $G/Z(G) \cong (Z_2)^m$.

We easily see that the commutator subgroup of E^v_m is $\{\pm 1\}$ using anti-commutation relations. For $m = 2k$ the other conditions were verified in Lemma 3.2. As a result, we have the following proposition:

Proposition 3.4. *[5] E^v_m is nearly extraspecial 2-groups for any m.*

Proof. E^v_{2k} is an extraspecial 2-group using Lemma 3.2 and Definition 3.3. In addition, E^v_{2k} is contained in E^v_{2k+1} and so E^v_m is nearly extraspecial 2-groups for any m (they include extraspecial 2-groups). ■

Remark 3.5. *The cases where the center of E^v_m is isomorphic to $\mathbb{Z}_4$ are almost extraspecial, but when center is $\mathbb{Z}_2 \times \mathbb{Z}_2$ they are not [7].*

3.2 Irreducible Representations of E^{-1}_m

There are two cases corresponding to the parity of m. As in [5], there are $2^{2k} + 1$ inequivalent irreducible representations of E^{-1}_{2k}.
Let $Irr(E^{-1}_{2k}) = \{V_1, ..., V_{2^{2k}+1}\}$ denote a set of inequivalent irreducible representations of E^{-1}_{2k}.

There are 2^{2k} representations of dimension one (say $V_2, ..., V_{2^{2k}+1}$) and only one non-trivial irreducible representation of dimension 2^k ($dimV_1 = 2^k$). The representation (ρ_1, V_1) is defined as follows:

Proposition 3.6. *[5] The irreducible representation $\rho_1 : E_{2k}^{-1} \to GL_{2^k}(\mathbb{C})$ is defined as follows:*

$$\begin{gathered}
\rho_1(x_1) = \sqrt{-1}(\sigma_z \otimes I_2^{\otimes(k-1)}), \\
\rho_1(x_2) = s \otimes I_2^{\otimes(k-1)}, \\
\vdots \\
\rho_1(x_{2i}) = I_2^{\otimes(i-1)} \otimes s \otimes I_2^{\otimes(k-i)}, \\
\rho_1(x_{2i+1}) = \sqrt{-1}(I_2^{\otimes(i-1)} \otimes \sigma_z \otimes \sigma_z \otimes I_2^{\otimes(k-i-1)}), \\
\vdots \\
\rho_1(x_{2k}) = I_2^{\otimes(k-1)} \otimes s,
\end{gathered}$$

where s and σ_z are defined in section 2.

Now, for E_{2k-1}^{-1} there are $2^{2k-1} + 2$ inequivalent irreducible representations. Let $Irr(E_{2k-1}^{-1}) = \{W_1, W_2, ..., W_{2^{2k-1}+2}\}$ denote a set of inequivalent irreducible representations of E_{2k-1}^{-1}. We have 2^{2k-1} distinct representations of dimension one (say $W_3, ..., W_{2^{2k-1}+2}$) and only two non-trivial irreducible representations of dimension 2^{k-1} (say W_1, W_2). The explicit realizations of (λ_1, W_1) and (λ_2, W_2) are in the following proposition.

Proposition 3.7. *[5] Consider the two irreducible representations λ_1 and $\lambda_2 : E_{2k-1}^{-1} \to GL_{2^{k-1}}(\mathbb{C})$ defined as follows:*

$$\begin{gathered}
\lambda_1(x_1) = \lambda_2(x_1) = \sqrt{-1}(\sigma_z \otimes I_2^{k-2}), \\
\lambda_1(x_2) = \lambda_2(x_2) = s \otimes I_2^{k-2}, \\
\vdots \\
\lambda_1(x_{2i}) = \lambda_2(x_{2i}) = I_2^{i-1} \otimes s \otimes I_2^{k-i-1}, \\
\lambda_1(x_{2i+1}) = \lambda_2(x_{2i+1}) = \sqrt{-1}(I_2^{i-1} \otimes \sigma_z \otimes \sigma_z \otimes I_2^{k-i-2}),
\end{gathered}$$

$$\vdots$$

$$\lambda_1(x_{2k-2}) = \lambda_2(x_{2k-2}) = I_2^{k-2} \otimes s,$$
$$\lambda_1(x_{2k-1}) = -\lambda_2(x_{2k-1}) = \sqrt{-1}(I_2^{k-2} \otimes \sigma_z),$$

where s and σ_z are defined in section 2.

Note that the only difference between λ_1 and λ_2 on the generators is that the image of x_{2k-1} differs in sign.

4 Faithfulness of the representations of E_m^{-1}

In this section, we find a sufficient condition that guarantees the faithfulness of any representation of $E_m^{-1} =< x_1, x_2, ..., x_m >$ for different parity of m and we prove that ρ_1 (Proposition 3.6) is a faithful representation of E_{2k}^{-1}. Also, λ_1, and λ_2, defined in Proposition 3.7 are faithful representations of E_{2k-1}^{-1} when k is odd and not faithful when k is even. Moreover, we construct new representations of the pure braid group P_n, where these representations are faithful if and only if π_n is faithful.

Lemma 4.1. *Consider the nearly extraspecial 2-groups E_m^{-1}. Let ϕ be a representation of dimension n of E_m^{-1} such that $\phi(-1) = -I_n$. The following are true:*

a) The elements $\pm x_i$ does not belong to kerϕ.

b) Let $g = \pm x_{i_1} x_{i_2} ... x_{i_l} \in E_m^{-1}$, where $i_r \in \mathbb{N}^*$, $1 \leq i_r \leq m$, and $r \in \{1, 2, ..., l\}$. If there exist two consecutive terms x_p and x_q in g such that $i_q - i_p \geq 3$ ($1 \leq p < q \leq l$), then g does not belong to kerϕ.

Proof. a) Suppose that $\phi(x_i) = I_n$. This implies that $\phi(x_i^2) = \phi(x_i)$ and so $\phi(-1) = \phi(x_i)$. Thus $\phi(x_i) = -I_n$, a contradiction. Therefore, x_i does not belong to kerϕ. Similarly, $-x_i$ does not belong to kerϕ.

b) Suppose that $g = x_{i_1} x_{i_2} ... x_{i_l} \in ker\phi$. Then $\phi(g) = I_n$. Without loss of generality, take $i_1 < i_2 < ... < i_l$. We have $\phi(g) = \phi(x_{i_1} x_{i_2} ... x_{i_p} x_{i_q} ... x_{i_l}) = I_n$. This implies that $\phi(x_{i_1} x_{i_2} ... x_{i_p} x_{i_q} ... x_{i_l} x_{i_q - 1}) = \phi(x_{i_q - 1})$..............(1)
But $|i_q - 1 - i_r| \geq 2$ for all $1 \leq r \leq l$ and $r \neq q$. By the commutation/anti-commutation relations in E_m^{-1}, we see that $x_{i_q-1} x_{i_r} = x_{i_r} x_{i_q-1}$ and $x_{i_q} x_{i_q-1} = -x_{i_q-1} x_{i_q}$.
Now (1) implies that $\phi(-x_{i_q-1} x_{i_1} x_{i_2} ... x_{i_p} x_{i_q} ... x_{i_l}) = \phi(x_{i_q-1})$.

So $\phi(-1)\phi(x_{i_q-1})\phi(g) = \phi(x_{i_q-1})$. Thus $\phi(g) = -I_n$, a contradiction. Therefore, $g = x_{i_1}x_{i_2}...x_{i_l} \notin \ker\phi$.
Similarly, we show that $g = -x_{i_1}x_{i_2}...x_{i_l} \notin ker\phi$. ■

Theorem 4.2. *Consider the nearly extraspecial 2-groups E_m^{-1}. Let ϕ be a representation of dimension n of E_m^{-1} such that $\phi(-1) = -I_n$. The following are true:*

1. *If m is even, then ϕ is faithful.*

2. *When m is odd such that $m = 2s+1$, we have two cases:*
   a) If s is even, then ϕ is faithful.
   b) If s is odd, then ϕ is faithful if and only if $\pm x_1x_3...x_{2s+1} \notin ker\phi$.

Proof. Let $g \in E_m^{-1}$. Then $g = \pm x_1^{\alpha_1}x_2^{\alpha_2}...x_m^{\alpha_m}$ (Lemma 3.2), where $\alpha_i \in \{0,1\}$.
It is easy to see that g is either $\pm 1, \pm x_i$, or $\pm x_{i_1}...x_{i_l}$, where $1 \leq i_1 < ... < i_l \leq m$. It is clear that $-1 \notin ker\phi$ and $\pm x_i \notin ker\phi$ (Lemma 4.1). Let $g = \pm x_{i_1}...x_{i_l}$. Let us show that $g \notin ker\phi$. Consider the two possibilities whether or not there are two consecutive terms with two consecutive index integers in g say x_{i_q} and x_{i_q+1} $(i_1 \leq i_q \leq i_l - 1)$.

Case 1: Suppose there exist two consecutive terms with two consecutive index integers in g, say x_{i_q} and x_{i_q+1}. We take x_{i_q}, x_{i_q+1} to be the first two consecutive terms with two consecutive index integers in g; that is, if there exist r_1 and r_2 such that $1 \leq r_1 < r_2 \leq q$, then $i_{r_2} - i_{r_1} \geq 2$. Take $g = x_{i_1}...x_{i_l}$. The trace of the matrix $\phi(g)$ equals $tr(\phi(x_{i_1}...x_{i_q}x_{i_q+1}...x_{i_l}))$ which is equal to $tr(\phi(-x_{i_1}...x_{i_q+1}...x_{i_l}x_{i_q}))$ (by the commutation/anti-commutation relations in E_m^{-1}). This implies that $tr(\phi(g)) = -tr(\phi(x_{i_1}...x_{i_q+1}...x_{i_l})\phi(x_{i_q}))$ which is equivalent to $tr(\phi(g)) = -tr(\phi(x_{i_q})\phi(x_{i_1}...x_{i_q+1}...x_{i_l}))$. Hence $tr(\phi(g)) = -tr(\phi(x_{i_1}...x_{i_q}x_{i_q+1}...x_{i_l}))$, so $tr(\phi(g)) = -tr(\phi(g))$. Thus $tr(\phi(g)) = 0$. Therefore, g does not belong to $\ker\phi$. A similar proof shows that $g = -x_{i_1}...x_{i_l}$ does not belong to $\ker\phi$.

Case 2: Now suppose that g belongs to $\ker\phi$ and there are no two consecutive terms with two consecutive index integers in g. Using Lemma 4.1 and Case 1, we see that $g = \pm x_{i_1}x_{i_1+2}...x_{i_1+2s}$, where the natural number $i_1 + 2s$ is less than or equal m. First, we have two possibilities for the values of i_1, either $i_1 = 1$ or not. Suppose that $i_1 \neq 1$; that is, $i_1 \in \{2, ..., m-2\}$. Take $g = x_{i_1}x_{i_1+2}...x_{i_1+2s}$. We have $\phi(g) \in \ker\phi$; that is, $\phi(g) = I_n$ which is equivalent to $\phi(x_{i_1}x_{i_1+2}...x_{i_1+2s}x_{i_1-1}) = \phi(x_{i_1-1})$. Hence $\phi(-x_{i_1-1}x_{i_1}x_{i_1+2}...x_{i_1+2s}) =$

$\phi(x_{i_1-1})$. It follows that $-\phi(x_{i_1-1})\phi(g) = \phi(x_{i_1-1})$. Thus $\phi(g) = -I_n$. Similarly, we get $\phi(g) = -I_n$ when $g = -x_{i_1}x_{i_1+2}...x_{i_1+2s}$. This gives a contradiction. This means that $i_1 = 1$; that is, $g = \pm x_1x_3...x_{2s+1}$. Here there are two possibilities either $2s+1 = m$ or not. Suppose $2s+1 \neq m$; that is, $1 < 2s+1 < m$. Take $g = x_1x_3...x_{2s+1}$. Then we get $\phi(x_1x_3...x_{2s+1}x_{2s+2}) = \phi(x_{2s+2})$. It follows that $\phi(-x_{2s+2}x_1x_3...x_{2s+1}) = \phi(x_{2s+2})$. Hence $-\phi(x_{2s+2})\phi(g) = \phi(x_{2s+2})$. Thus $\phi(g) = -I_n$. Similarly, $\phi(g) = -I_n$ when $g = -x_1x_3...x_{2s+1}$. This gives a contradiction and so $m = 2s+1$. Now if we assume m is even, then $g \notin \ker\phi$ and so ϕ is faithful. Thus (1) follows.
Now to prove (2), we assume that $m = 2s+1$. Let $g = \pm x_1x_3...x_m$. We have $\phi(g) = I_n$ which is equivalent to $\phi(x_m...x_3x_1)\phi(x_1x_3...x_m) = \phi(x_m...x_3x_1)$. This implies that $\phi((-1)^{\frac{m+1}{2}}) = \phi(g)$. Hence $\phi((-1)^{s+1}) = \phi(g)$. Now we have two possibilities according to the parity of s. If s is even, then $\phi((-1)^{s+1}) = \phi(g)$ is equivalent to $-I_n = \phi(g)$. Similarly, $\phi(g) = -I_n$ when $g = -x_1x_3...x_m$, a contradiction. Therefore, when $m = 2s+1$ with s is even, we can not find any element $g \in E_m^{-1}$ such that $\phi(g) = I_n$. Hence if $m = 2s+1$ and s even, then ϕ is faithful. Thus the proof of $(2a)$ is achieved. Otherwise, we assume that s is odd. But, in this case, we have, by our assumption, that $\phi(\pm x_1x_3...x_{2s+1}) \neq I_n$. Therefore, when $m = 2s+1$ (s odd) such that $\phi(\pm x_1x_3...x_{2s+1}) \neq I_n$, we can not find any element $g \in E_m^{-1}$ such that $\phi(g) = I_n$. Hence if $m = 2s+1$, s is odd and $\phi(\pm x_1x_3...x_{2s+1}) \neq I_n$, then ϕ is faithful. Thus the proof of $(2b)$ is achieved.

■

Lemma 4.3. *Let ϕ be an irreducible representation of E_{2k-1}^{-1} of dimension n and $z = x_1x_3...x_{2k-1}$. If k is even, then $\phi(z)$ or $\phi(-z)$ are equal to I_n.*

Proof. By Lemma 3.2, z is a non-trivial element which belongs to the center of E_{2k-1}^{-1} and is of order 2 if k is even. Hence, by Proposition 2.3, $\phi(z) = \lambda I_n$. Also, $\phi(z^2) = I_n$. It follows that $\lambda = \pm 1$. Hence if $\lambda = 1$, then $\phi(z) = I_n$. Otherwise, $\phi(-z) = I_n$. ■

Theorem 4.4. *Consider the nearly extraspecial 2-groups E_m^{-1}. Let ϕ be an irreducible representation of dimension n of E_m^{-1} such that $\phi(-1) = -I_n$. Hence, the following are true:*

1. *If m is even, then ϕ is faithful.*

2. *When m is odd such that $m = 2k-1$, we have two cases:*
   a) If k is odd, then ϕ is faithful.
   b) If k is even, then ϕ is not faithful.

Proof. It is easy to see that (1) is satisfied by Theorem 4.2. By ($2a$) in Theorem 4.2 when $m = 2k-1$ and k is odd, ϕ is faithful. Otherwise, by ($2b$) in Theorem 4.2 and Lemma 4.3, if k is even, then ϕ is not faithful. ■

Since the representations defined in Propositions 3.6 and 3.7 are irreducible, Theorem 4.4 is true for ρ_1, λ_1 and λ_2. More precisely, we get the following corollary.

Corollary 4.5. *Consider the representations ρ_1, λ_1, and λ_2 (Propositions 3.6 and 3.7). The following are true:*

1. *ρ_1 is faithful representation of E_{2k}^{-1} for all natural number k.*
2. *If k is odd, then λ_1 and λ_2 are two faithful representations of E_{2k-1}^{-1}.*
3. *If k is even, then λ_1 and λ_2 are not faithful representations of E_{2k-1}^{-1}.*

Proof. By Theorem 4.4 it is enough to show that $\rho_1(-1) = -I_{2^k}$ and $\lambda_1(-1) = \lambda_2(-1) = -I_{2^{k-1}}$.

$$\rho_1(-1) = \rho_1(x_1^2) = (\sqrt{-1}\sigma_z \otimes I_2^{\otimes(k-1)})^2 = -(\sigma_z \otimes I_2^{\otimes(k-1)})^2.$$

We know that $I_2^{\otimes(k-1)} = I_{2^{k-1}}$ and $\sigma_z^2 = I_2$. So

$$\rho_1(-1) = -\begin{pmatrix} \sigma_z^2 & 0 & . & . & . & . & 0 \\ 0 & \sigma_z^2 & 0 & . & . & . & 0 \\ . & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 & \sigma_z^2 \end{pmatrix} = -\begin{pmatrix} I_2 & 0 & . & . & . & . & 0 \\ 0 & I_2 & 0 & . & . & . & 0 \\ . & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 & I_2 \end{pmatrix},$$

where I_2 appears 2^{k-1} times. This implies that

$$\rho_1(-1) = -I_2 \otimes \begin{pmatrix} 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 & 1 \end{pmatrix} = -I_2 \otimes I_{2^{k-1}}.$$

Hence

$$\rho_1(-1) = -I_{2^k}.$$

In a similar way, we show that the image of -1 under λ_1 and λ_2 is equal to $-I_{2^{k-1}}$. ■

We recall Definition 2.5 when $H_n = \pi_n(P_n)$ and $G_n = \pi_n(B_n)$. We have

Theorem 4.6. *[5]*

1. *$H_n \cong E_{n-1}^{-1}$ and the image B_n under the representation π_n is an extension of E_{n-1}^{-1} by S_n.*

2. *The representation π_n of B_n decomposes as*

$$(\mathbb{C}^2)^{\otimes n} \cong \begin{cases} (\mathbb{C}^2)^{\otimes \frac{n+1}{2}} \otimes V_1 & n \text{ odd} \\ (\mathbb{C}^2)^{\otimes \frac{n}{2}} \otimes (W_1 \oplus W_2) & n \text{ even} \end{cases}$$

Corollary 4.7. *Consider the representations ρ_1 of E_{2k}^{-1} ($k \in \mathbb{N}^*$), λ_1, and λ_2 of E_{2k-1}^{-1} (k is odd). Let ρ_1' be the representation of H_{2k+1} induced by ρ_1, λ_1', and λ_2' be the representations of H_{2k} induced by λ_1, and λ_2 respectively. Hence, $\rho_1' \circ \pi_n$ ($n = 2k+1$) and $\lambda_1' \circ \pi_n$, $\lambda_2' \circ \pi_n$ ($n = 2k$), are representations of the pure braid group P_n, which are faithful if and only if π_n is.*

Proof. We have $H_n \cong E_{n-1}^{-1}$. Any faithful representation of E_{n-1}^{-1} induces a faithful representation of H_n. But the representations ρ_1 of E_{2k}^{-1} ($k \in \mathbb{N}^*$), λ_1, and λ_2 of E_{2k-1}^{-1} (k is odd) are faithful (Corollary 4.5). This implies that the representations ρ_1' of H_{2k+1}, λ_1', and λ_2' of H_{2k} are faithful. It follows that the representations $\rho_1' \circ \pi_n$ ($n = 2k+1$) and $\lambda_1' \circ \pi_n$, $\lambda_2' \circ \pi_n$ ($n = 2k$) are faithful if and only if π_n is faithful. ■

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