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Bright and Dark Soliton Solutions of Three-Dimensional KdV and mKdV Equations in Inviscid Liquid with Gas Bubbles

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Abstract

Mixture of liquid with gas bubbles are found in many areas of natural and physical sciences especially in engineering and industries, where nonlinear wave propagation is a common phenomenon. In this paper, we obtained the bright and dark soliton solutions of the three-dimensional Korteweg-de-Vries (KdV) equation and three-dimensional modified Korteweg-de-Vries (mKdV) equation in an inviscid bubbly liquid flow. Bright soliton solutions of these equations were derived analytically using a secant hyperbolic wave ansatz, while the dark soliton solutions of the equations were obtained explicitly using tangent hyperbolic wave ansatz. The physical parameters in the soliton solutions are obtained.

1 Introduction

Nonlinear evolution equations arise in many scientific and industrial applications such as plasma physics, diffusion process, neural physics, condensed

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matter, electro-magnetic, solid state physics, chemical reactions, optical fibres and many more [15]. Wave propagation in fluid such as dispersive fluid is modelled mathematically using nonlinear hyperbolic partial differential equations, whose solutions can be dark or bright soliton solutions [12]. Nonlinear evolution equations offer some great challenges in trying to obtain the exact solution. Interest in soliton in higher dimensional flow of a fluid media has grown rapidly in recent years. KdV-type equations are nonlinear wave equations that exhibit both nonlinearity and dispersion wave propagation phenomenon which are significant in nonlinear evolution models.

The existence of soliton in bubbly inviscid fluid was first considered theoretically by van-Wijngaarden in the late sixties [18], which represents the fact that there is a balance between nonlinearity and dispersion [14]. Soliton is a class of solution of some nonlinear evolution equations such as the Korteweg-de Vries (KdV) equation, the sine-Gordon (SG) equation, Schrdinger (NLS) equation e.t.c [13]. Dark and bright solitons are also known as topological and non-topological optical solitons in the context of nonlinear optics [4].

In recent years, many methods were developed for finding exact solutions of nonlinear evolution equations, such as inverse scattering method [7], sine-cosine method [15], simple eqation method [10], first integral method [20], extended tanh method [16, 17], Adomian decomposition method [1], F-expansion method [22, 8]. (G/G)-expansion method [2], Backlund/Darboux transformations [19], Muira transformation [9], Jacobi elliptic function method [21] to mention but a few. It is well recognized recently that the propagation of weakly nonlinear long waves in bubbly fluid flow is described by the three-dimensional KdV equation if viscosity is neglected [11]. Three dimensional KdV is integrable [11].

This paper is organized as follows: in section 2, we derived both the bright and dark solitary wave solutions of three-dimensional KdV equation. The derived bright and dark solitary wave solutions of three-dimensional mKdV equation is given In section 3. Conclusion is briefly given in the last section.

2 Three Dimensional KdV Equation

The three dimensional KdV equation derived in the model of wave propagation in bubbly fluid in three dimensional case have received great attention recently, where it was analysed qualitatively [5], with the t-dependent variable coefficients was considered in which the Auto-Backlund transformation and soliton solutions were obtained [6]. The three dimensional KdV equation is

$$(u_t + u\,u_x + u_{xxx})_x + u_{yy} + u_{zz} = 0, \tag{2.1}$$

where x, y, z, t are the independent variables. x, y, z, represent the spatial coordinates and t is the time.

2.1 Bright Soliton Solution

The bright (non-topological) soliton solution of (2.1) has a solitary wave ansatz of the form [3, 13]

$$u\,(x, y, z, t) = \lambda\,\mathrm{sech}^{\alpha}\xi, \quad \xi = m\,x + n\,y + r\,z - c\,t \tag{2.2}$$

where ξ is the frame of reference of the traveling wave. m, n, r are wave numbers, c is the speed of wave propagation and λ is the amplitude of the soliton. All the parameters are assumed to be constants. The constant α will be determined later from the equation. Differentiating (2.2) with respect to the independent variables x, y, z, t, we obtained the following derivatives

$$\begin{aligned} u_{xt} =& \lambda c\,m\,\alpha\left[(\alpha+1)\,\mathrm{sech}^{\alpha+2}\xi - \alpha\,\mathrm{sech}^{\alpha}\xi\ \right], \\ [uu_x]_x =& m^2\,\lambda^2\,\left[2\,\alpha^2\,\mathrm{sech}^{2\alpha}\xi\ - \alpha\,(2\alpha+1)\ \mathrm{sech}^{2\alpha+2}\xi\ \right], \\ u_{xxxx} =& \lambda\,m^4\left[\alpha^4 \mathrm{sech}^{\alpha}\xi\ - 2\alpha\,(\alpha+1)\left(\alpha^2+2\alpha+2\right)\mathrm{sech}^{2\alpha+2}\xi \right. \\ & \left. +\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\mathrm{sech}^{\alpha+4}\xi\ \right] \\ u_{yy} =& \lambda n^2\left[\alpha^2 \mathrm{sech}^{\alpha}\xi\ - \alpha\,(\alpha+1)\,\mathrm{sech}^{\alpha+2}\xi\ \right], \\ u_{zz} =& \lambda r^2\left[\alpha^2 \mathrm{sech}^{\alpha}\xi\ - \alpha\,(\alpha+1)\,\mathrm{sech}^{\alpha+2}\xi\ \right], \end{aligned} \tag{2.3}$$

where $\xi = m\,x + n\,y + r\,z - c\,t$. Thus, substituting the above derivatives into (2.1), yields the relation

$$\begin{aligned} &\lambda\,c\,m\,\alpha\,(\alpha+1)\,\mathrm{sech}^{\alpha+2}\xi - \lambda\,\alpha^2\,c\,m\,\mathrm{sech}^{\alpha}\xi \\ &+ 2m^2\lambda^2\,\alpha^2 \mathrm{sech}^{2\alpha}\,(\xi\ ) - m^2\lambda^2\,\alpha\,(2\alpha+1)\ \mathrm{sech}^{2\alpha+2}\xi \\ &+ \lambda m^4\alpha^4 \mathrm{sech}^{\alpha}\xi\ - 2\lambda m^4\alpha\,(\alpha+1)\left(\alpha^2+2\alpha+2\right)\mathrm{sech}^{\alpha+2}\xi \\ &+ \lambda m^4\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\ \mathrm{sech}^{\alpha+4}\xi \\ &+ \lambda n^2\alpha^2 \mathrm{sech}^{\alpha}\xi\ - \lambda n^2\,\alpha\,(\alpha+1)\ \mathrm{sech}^{\alpha+2}\xi \\ &+ \lambda r^2\,\alpha^2 \mathrm{sech}^{\alpha}\xi\ - \lambda r^2\,\alpha\,(\alpha+1)\ \mathrm{sech}^{\alpha+2}\xi\ = 0. \end{aligned} \tag{2.4}$$

Now, from (2.4), equating the exponents of $\mathrm{sech}^{2\alpha+2}\xi$ and $\mathrm{sech}^{\alpha+4}$ leads to

$$2\alpha + 2 = \alpha + 4, \implies \alpha = 2.$$

From (2.4), setting the coefficients of $\mathrm{sech}^{2\alpha+2}\xi$ and $\mathrm{sech}^{\alpha+4}\xi$ terms to zero, we obtain

$$\lambda = 12m^2. \tag{2.5}$$

Similarly, n can also be found by setting the coefficients of $\mathrm{sech}^{\alpha}\xi$ term to zero in (2.4), which is

$$-\lambda\,\alpha^2\,c\,m + \lambda\,\alpha^4 m^4 + \lambda\,\alpha^2 n^2 + \lambda\,\alpha^2 r^2 = 0,$$

also we obtain

$$n = \pm\sqrt{c\,m - 4\,m^4 - r^2}. \tag{2.6}$$

Finally, the 1-soliton solution of (2.1) is obtained as

$$u\,(x, y, z, t) = \lambda \mathrm{sech}^2\left(m\,x + n\,y + r\,z - c\,t\right)\,,$$

where n is as obtained (2.6) above, while m, r and c are given in the frame of reference defined in Eq.(2.2) above.

2.2 Dark Soliton Solution

The dark (non-topological) soliton solution of (2.1) has a solitary wave ansatz of the form [3, 13]

$$u\,(x, y, z, t) = \lambda \tanh^{\alpha}\xi, \quad \xi = m\,x + n\,y + r\,z - c\,t \tag{2.7}$$

where ξ is the frame of reference of the traveling wave. m, n, r are wave numbers, c is the speed of wave propagation and λ is the soliton amplitude. All the parameters are assumed to be constants. The constant α will be determined later from the equation. Differentiating (2.7) with respect to x, y, z and t, we have the following derivatives

$$\begin{aligned}
u_{xt} =& -\alpha c m \lambda[(\alpha-1)\tanh^{\alpha-2}\xi \;\; - 2\alpha \tanh^{\alpha}\xi \;\; + (\alpha\; + 1)\tanh^{\alpha+2}\xi\;],\\
u_{yy} =& \alpha n^2 \lambda[(\alpha-1)\tanh^{\alpha-2}\xi \;\; - 2\alpha \tanh^{\alpha}\xi \;\; + (\alpha\; + 1)\tanh^{\alpha+2}\xi\;],\\
u_{zz} =& \alpha\, r^2\, \lambda\, \left[(\alpha-1)\tanh^{\alpha-2}\xi \;\; - 2\alpha \tanh^{\alpha}\xi \;\; + (\alpha\; + 1)\tanh^{\alpha+2}\xi\; \right],\\
[u u_x]_x =& \lambda^2\, m^2\, \alpha[(2\alpha-1)\tanh^{2\alpha-2}\xi \; - \; 4\tanh^{2\alpha}\xi \;\; + (2\alpha+1)\tanh^{2\alpha+2}\xi\;],\\
u_{xxxx} =& \lambda\, m^4\, \alpha\, \Big[(\alpha-1)\,(\alpha-2)\,(\alpha-3)\tanh^{\alpha-4}\xi\\
& -4\,(\alpha-1)\left(\alpha^2 - 2\alpha + 2\right)\tanh^{\alpha-2}\xi \; + \; 2\alpha\left(3\alpha^2+5\right)\tanh^{\alpha}\xi\\
& -4\,(\alpha+1)\left(\alpha^2+2\alpha+2\right)\,\tanh^{\alpha+2}\xi \; + (\alpha+1)\,(\alpha+2)\\
& (\alpha+3)\,\tanh^{\alpha+4}\xi\; \Big]\,,
\end{aligned}$$

where $\xi = m\,x + n\,y + r\,z - c\,t$. Substituting the above derivatives into (2.1), we have the following:

$$\begin{aligned}
&- \alpha\, c\, m\, \lambda \left[(\alpha - 1) \tanh^{\alpha-2}\xi - 2\alpha \tanh^{\alpha}\xi \;\; + (\alpha \; + 1) \tanh^{\alpha+2}\xi \;\right] \\
&+ \lambda^2\, m^2\, \alpha \left[(2\alpha - 1) \tanh^{\alpha-2}\xi - 4 \tanh^{\alpha}\xi \;\; + (2\alpha + 1) \tanh^{2\alpha+2}\xi \;\right] \\
&+ \lambda\, m^4\, \alpha\, \left[(\alpha - 1)\,(\alpha - 2)\,(\alpha - 3)\, \tanh^{\alpha-4}\xi \right.\\
&-4\,(\alpha - 1)\left(\alpha^2 - 2\alpha + 2\right) \tanh^{\alpha-2}\xi \\
&+ 2\,\alpha\, \left(3\alpha^2 + 5\right)\, \tanh^{\alpha}\xi \; - 4\,(\alpha + 1)\left(\alpha^2 + 2\alpha + 2\right) \tanh^{\alpha+2}\xi \\
&\left.+ (\alpha + 1)\,(\alpha + 2)\,(\alpha + 3) \tanh^{\alpha+4}\xi \;\right] \\
&+ \alpha\, n^2\, \lambda \left[(\alpha - 1) \tanh^{\alpha-2}\xi \; - 2\,\alpha \tanh^{\alpha}\xi \;\; + (\alpha + 1)\, \tanh^{\alpha+2}\xi \;\right] \\
&+ \alpha\, r^2\, \lambda \left[(\alpha - 1)\, \tanh^{\alpha-2}\xi \; - 2\,\alpha \tanh^{\alpha}\xi \;\; + (\alpha \; + 1)\, \tanh^{\alpha+2}\xi \;\right] = 0.
\end{aligned} \tag{2.8}$$

From (2.8), equating the exponents of $\tanh^{2\alpha}\xi$ and $\tanh^{\alpha+2}\xi$ leads to

$$2\alpha = \alpha + 2, \;\; \Longrightarrow \;\; \alpha = 2. \tag{2.9}$$

Equating the coefficients of $\tanh^{2\alpha+2}\xi$ and $\tanh^{\alpha+4}\xi$ to zero from (2.8), we get

$$\lambda = -12m^2. \tag{2.10}$$

Similarly, equating the coefficients of $\tanh^{2\alpha}\xi$ and $\tanh^{\alpha+2}\xi$ to zero from (2.8), we have

$$\begin{aligned}
-\alpha c\, m\lambda\,(\alpha \; + 1) - 4\lambda^2 m^2\alpha - 4\lambda m^4\alpha\,(\alpha + 1)\left(\alpha^2 + 2\alpha + 2\right) \\
+\alpha n^2\lambda\,(\alpha \; + 1) + \alpha r^2\lambda\,(\alpha \; + 1) = 0,
\end{aligned} \tag{2.11}$$

and then we get

$$n \;=\; \pm\frac{1}{3}\sqrt{360m^4 + 12\lambda m^2 + 9mc - 9r^2 - 144m^4}. \tag{2.12}$$

From (2.8), equating the coefficients of $\tanh^{a-2}\xi$ terms leads to

$$\begin{aligned}
-\alpha c\, m\lambda\,(\alpha \; - 1) - \lambda m^4\alpha 4\,(\alpha - 1)\left(\alpha^2 - 2\alpha + 2\right) \\
+\alpha n^2\lambda\,(\alpha \; - 1) + \alpha r^2\lambda\,(\alpha \; - 1) = 0,
\end{aligned} \tag{2.13}$$

which gives

$$n \;=\; \pm\sqrt{8m^4 + m\,c - r^2}. \tag{2.14}$$

Therefore, the final 1-soliton solution to the three-dimensional KdV equation in an invicid bubbly fluid is determined as

$$u\,(x,y,z,t) = \lambda \tanh^2(m\,x + n\,y + r\,z - c\,t)\ , \tag{2.15}$$

where n is given in (2.14) and (2.12) above, while m, r are free parameters. The multiple soliton solution to this equation was already been obtained [11].

3 Modified Three Dimensional Korteweg-de-Varies Equation

The modified three dimensional KdV equation is a cubic nonlinear type of the three dimensional KdV equation, where both its bright and dark soliton solutions will be given in this work. The modified three dimensional KdV equation is

$$(u_t + u^2 u_x + u_{xxx})_x + u_{yy} + u_{zz} = 0, \tag{3.16}$$

where the wave is assumed to propagate in x, y and z directions, and t measures the time of the wave propagation.

3.1 Bright Soliton Solution

To find the bright soliton solution to Eq.(3.16), the earlier assumption given in (2.2) for the solitary wave ansatz will be used,

$$u\,(x,y,z,t) = \lambda\,\mathrm{sech}^{\alpha}\xi\ , \quad \xi = mx + ny + rz - c\,t \tag{3.17}$$

where m, n, r and c are constants. The exponent α is an unknown constant to be determined later. From the secant hyperbolic ansatz (3.17), we obtain

$$\begin{aligned}
u_{xt} &= \lambda c\,m\,\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi - \lambda\alpha^2 c\,m\,\mathrm{sech}^{\alpha}\xi\ ,\\
u_{xxxx} &= \lambda\alpha^4 m^4 \mathrm{sech}^{\alpha}\xi\ - \lambda m^4 2\alpha\,(\alpha+1)\left(\alpha^2+2\alpha+2\right)\mathrm{sech}^{2\alpha+2}\xi\\
&\quad +\lambda m^4\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\mathrm{sech}^{\alpha+4}\xi\ ,\\
u_{yy} &= \lambda\alpha^2 n^2 \mathrm{sech}^{\alpha}\xi\ - \lambda n^2\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi\ ,\\
u_{zz} &= \lambda\alpha^2 r^2 \mathrm{sech}^{\alpha}\xi\ - \lambda r^2\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi\ ,\\
[u^2 u_x]_x &= 3\lambda^3\alpha^2 m^2 \mathrm{sech}^{3\alpha}\xi\ - \lambda^3 m^2\alpha(3\alpha+1)\mathrm{sech}^{3\alpha+2}\xi.
\end{aligned}$$

Substituting the above equations into (3.16), we have

$$\begin{aligned}
&\left[\lambda c\, m\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi - \lambda\alpha^2 c\, m\mathrm{sech}^{\alpha}\xi\ \right]\\
&+\left[3\lambda^3\alpha^2 m^2\mathrm{sech}^{3\alpha}\xi\ - \lambda^3 m^2\alpha(3\alpha+1)\mathrm{sech}^{3\alpha+2}\xi\ \right]\\
&+\left[\lambda\alpha^4 m^4\mathrm{sech}^{\alpha}\xi\ - \lambda m^4 2\alpha\,(\alpha+1)\left(\alpha^2+2\alpha+2\right)\mathrm{sech}^{2\alpha+2}\xi\right.\\
&\left.+\lambda m^4\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\mathrm{sech}^{\alpha+4}\xi\ \right]\\
&+\left[\lambda\alpha^2 n^2\mathrm{sech}^{\alpha}\xi\ - \lambda n^2\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi\ \right]\\
&+\left[\lambda\alpha^2 r^2\mathrm{sech}^{\alpha}\xi\ - \lambda r^2\alpha(\alpha+1)\mathrm{sech}^{\alpha+2}\xi\ \right]=0.
\end{aligned} \tag{3.18}$$

Now, from (3.18), equating the exponents of $\mathrm{sech}^{3\alpha+2}\xi$ and $\mathrm{sech}^{\alpha+4}$ lead to

$$3\alpha+2=\alpha+4,\ \implies\ \alpha=1. \tag{3.19}$$

Setting the coefficients of $\mathrm{sech}^{3\alpha+2}\xi$ and $\mathrm{sech}^{\alpha+4}\xi$ in (3.18) to zero using (3.19), we have

$$\lambda=\pm\sqrt{6m^2}. \tag{3.20}$$

Setting the coefficients of $\mathrm{sech}^{3\alpha}\xi$ and $\mathrm{sech}^{\alpha+2}\xi$ to zero, we have

$$\lambda c\, m\alpha\,(\alpha+1)+3\lambda^3\alpha^2 m^2-\lambda n^2\alpha(\alpha+1)-\lambda r^2\alpha(\alpha+1)=0, \tag{3.21}$$

which gives

$$n=\pm\frac{1}{2}\sqrt{6\lambda^2 m^2-40m^4+4m\,c-4r^2}. \tag{3.22}$$

Setting the coefficients of $\mathrm{sech}^{\alpha}\xi$ terms to zero, we have

$$-\lambda\alpha^2 c\, m+\lambda\alpha^4 m^4+\lambda\alpha^2 n^2+\lambda\alpha^2 r^2=0, \tag{3.23}$$

which gives

$$n=\pm\sqrt{c\,m-m^4-r^2}. \tag{3.24}$$

Therefore, the bright soliton solution of three-dimensional mKdV equation (3.16) is now obtained as

$$u\,(x,y,z,t)=\pm\sqrt{6\,m^2}\,\mathrm{sech}(m\,x+n\,y+r\,z-c\,t)\ ,$$

where n is given in (3.22) and (3.24) above, while m, r, c are free parameters from the frame of reference of the traveling wave.

3.2 Dark Soliton Solutions

Here we shall find the dark solitary wave solution for the three-dimensional mKdV equation (3.16). We use an ansatz solution of the form:

$$u\,(x,y,z,t) = \lambda \tanh^{\alpha}\xi \ , \quad \xi = \ m\,x \ + \ n\,y + r\,z - \ c\,t \tag{3.25}$$

where ξ is the frame of reference of the traveling wave, and m, n, r and α are constants. The unknown parameter α is to be obtained later, and c is the velocity of the solitary wave. The exponent α is to be determined later. Differentiating (3.25) with respect to $x,\ y,\ z$, and t, we have

$$\begin{aligned}
u_{xt} &= -\alpha c\,m\lambda[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi \ + (\alpha+1)\tanh^{\alpha+2}\xi\ ],\\
u_{yy} &= \alpha n^2\lambda[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi \ \ + (\alpha+1)\tanh^{\alpha+2}\xi\ ],\\
u_{zz} &= \alpha r^2\lambda\left[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi \ \ + (\alpha+1)\tanh^{\alpha+2}\xi\ \right],\\
[u^2u_x]_x &= \frac{1}{3}(u)^3)_{xx} = \lambda^3m^2\alpha\left[(3\alpha-1)\tanh^{3\alpha-2}\xi \ - 6\alpha\tanh^{3\alpha}\xi\right.\\
&\quad \left. + (3\alpha+1)\tanh^{3\alpha+2}\xi\ \right],\\
u_{xxxx} &= \lambda m^4\alpha\left[(\alpha-1)(\alpha-2)(\alpha-3)\tanh^{\alpha-4}\xi\right.\\
&\quad -4(\alpha-1)\left(\alpha^2-2\alpha+2\right)\tanh^{\alpha-2}\xi + \ 2\alpha\left(3\alpha^2+5\right)\tanh^{\alpha}\xi\\
&\quad -4(\alpha+1)\left(\alpha^2+2\alpha+2\right)\tanh^{\alpha+2}\xi\\
&\quad \left.+ (\alpha+1)(\alpha+2)(\alpha+3)\tanh^{\alpha+4}\xi\ \right],
\end{aligned}$$

where $\xi = m\,x + n\,y + r\,z - c\,t$. Substituting the above derivatives into (3.16), we have

$$\begin{aligned}
&-\alpha c\,m\lambda\left[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi + (\alpha+1)\tanh^{\alpha+2}\xi\ \right]\\
&+ \lambda^3m^2\alpha\left[(3\alpha-1)\tanh^{3\alpha-2}\xi - 6\alpha\tanh^{3a}\xi + (3\alpha+1)\tanh^{3\alpha+2}\xi\ \right]\\
&+ \lambda m^4\alpha\left[(\alpha-1)(\alpha-2)(\alpha-3)\tanh^{\alpha-4}\xi\right.\\
&-4(\alpha-1)\left(\alpha^2-2\alpha+2\right)\tanh^{\alpha-2}\xi\\
&+ 2\alpha\left(3\alpha^2+5\right)\tanh^{\alpha}\xi - 4(\alpha+1)\left(\alpha^2+2\alpha+2\right)\tanh^{\alpha+2}\xi\\
&\left.+ (\alpha+1)(\alpha+2)(\alpha+3)\tanh^{\alpha+4}\xi\ \right]\\
&+ \alpha n^2\lambda\left[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi + (\alpha+1)\tanh^{\alpha+2}\xi\right]\\
&+ \alpha r^2\lambda\left[(\alpha-1)\tanh^{\alpha-2}\xi - 2\alpha\tanh^{\alpha}\xi + (\alpha+1)\tanh^{\alpha+2}\xi\ \right] = 0.
\end{aligned} \tag{3.26}$$

Now, from (3.26), equating the exponents of $\tanh^{3\alpha+2}\xi$ and $\tanh^{\alpha+4}$ leads to

$$3\alpha \ + \ 2 = \alpha + 4\ , \implies \alpha = 1. \tag{3.27}$$

Setting the coefficients of $\tanh^{3\alpha+2}\xi$ and $\tanh^{\alpha+4}$ to zero in (3.26) using (3.27), we have

$$\lambda = \pm\sqrt{-6\,m^2}. \tag{3.28}$$

Setting the coefficients of $\tanh^{3\alpha}\xi$ and $\tanh^{\mathrm{a}+2}\xi$ to zero using (3.27), we have

$$\begin{aligned} -c\,m\lambda\ (\alpha\ +1) - \lambda^3 m^2 6\alpha - 4\lambda m^4 (\alpha+1)\left(\alpha^2+2\alpha+2\right) \\ +\alpha n^2\lambda\,(\alpha\ +1) + r^2\lambda\,(\alpha\ +1) = 0, \end{aligned} \tag{3.29}$$

which gives

$$n = \pm\sqrt{c\,m + 2m^4 - r^2}. \tag{3.30}$$

Setting the coefficients of $\tanh^{\alpha}\xi$ terms to zero, we have

$$2\alpha^2 c\,m\,\lambda + \lambda^3 m^2\alpha\,(3\alpha-1) + 2\alpha\left(3\alpha^2+5\right)\lambda m^4\alpha - 2\alpha^2 n^2\lambda - 2\alpha^2 r^2\lambda = 0,$$

which gives

$$n = \pm\sqrt{c\,m + 2m^4 - r^2}. \tag{3.31}$$

Lastly, the dark (topological) soliton solution for the three-dimensional mKdV equation (3.16) is given as

$$u\,(x,y,z,t) = \pm\sqrt{-6m^2}\,\tanh(m\,x + n\,y + r\,z - c\,t),$$

where n is obtained in (3.30) or (3.31) above and m, r, c are free parameters from the frame of reference of the traveling wave.

4 Conclusion

In this paper, three-dimensional nonlinear evolution equations were investigated. Using the solitary hyperbolic wave ansatz method, the bright (non-topological) and dark (topological) solitary wave solutions of three-dimensional Korteweg-de-Vries and three-dimensional modified Korteweg-de-Vries equations in bubbly invicid fluid flow were obtained analytically. For the bright soliton solution, a secant ansazt was used, while for dark solitary wave solution, a tangent hyperbolic ansazt was used. We believed that these solutions may be useful in numerical analysis. The method can be applied to many linear and nonlinear evolution equations, not only with constant coefficients but also to variable coefficients and inhomogeneous equations.

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