International Journal of Mathematics and
Computer Science, **14**(2019), no. 3, 729–736

On the regularity-preserving elements in regular ordered semigroups

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(Received April 12, 2019, Revised May 19, 2019, Accepted May 24, 2019)

Abstract

A variant of an ordered semigroup $(S, \cdot, \leq)$ with respect to a is an ordered semigroup $(S, \circ, \leq)$ with multiplication $\circ$ defined by $x \circ y = xay$ for all $x, y \in S$. An element $a \in S$ is a regularity-preserving element of S if a variant of S with respect to a is regular. In this paper, we characterize the regularity-preserving elements of regular ordered semigroups.

1 Introduction and Preliminaries

Let S be a semigroup and $a \in S$. Define a new binary operation $\circ$ on S by $x \circ y = xay$ for all $x, y \in S$. It is clear that $(S, \circ)$ is a semigroup. We denote this semigroup by (S, a) and call it a variant of S. Variants of concrete semigroups of relations had been considered by Magill [5]. However, variants of abstract semigroups were first studied by Hickey in [3]. Variants

Key words and phrases: Variants, regular elements, regularity-preserving elements, ordered semigroups.
AMS (MOS) Subject Classifications: 20M17.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

of regular semigroups were studied by Khan and Lawson in [4]. Chinram [1] defined variants of rings by using the concept of variants of semigroups and characterized the regularity-preserving elements of regular rings.

A semigroup $(S,\cdot)$ with a partial order $\leq$ is said to be an *ordered semigroup* (see [2]) if it satisfies the following condition:

$$\text{for any } \; a,b,c\in S, \;\; a\leq b \quad \Rightarrow \quad ac\leq bc \; \text{ and } \; ca\leq cb.$$

Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $ab\in A$ for all $a,b\in A$. An element a of S is said to be *ordered regular* if there exists $x\in S$ such that $a\leq axa$. If every element of S is ordered regular, we call S a *regular ordered semigroup*. An element a of S is called an *idempotent* if $a\leq a^2$. We denote the set of idempotents of S by $E_{\leq}(S)$. An element $b\in S$ is called an *inverse* of a if $a\leq aba$ and $b\leq bab$. The set of all inverse elements of a will be denoted by $V_{\leq}(a)$. An element e of S is called an *identity element* of S if $ex=x=xe$ for any $x\in S$. For an ordered semigroup S with identity e, an element $x\in S$ is called *invertible* if there exist $y,z\in S$ such that $e\leq yx$ and $e\leq xz$. An element u of S is said to be a *mididentity element* of S if $xuy=xy$ for all $x,y\in S$.

For a nonempty subset A of an ordered semigroup S, we denote by $(A]$ the subset of S defined by

$$(A]:=\{s\in S\mid s\leq a \text{ for some } a\in A\}.$$

Clearly, $A\subseteq (A]$ and for subsets A and B of S, if $A\subseteq B$, then $(A]\subseteq (B]$. Note that if S is a regular ordered semigroup, then $a\in (aS]\cap(Sa]$ for all $a\in S$.

For an element a of an ordered semigroup $(S,\cdot,\leq)$, we define a binary operation $\circ$ by

$$x\circ y = xay \text{ for all } x,y\in S.$$

Then $(S,\circ,\leq)$ is again an ordered semigroup and it is called a *variant* of S. We usually write $(S,a,\leq)$ rather than $(S,\circ,\leq)$ to make the element a explicit. Note that a variant of a regular ordered semigroup need not be regular. This can be seen in the following example.

Example 1.1. Let $\leq := id_{\mathbb{Z}_3} = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2})\}$. We have that

$$\overline{0} \leq \overline{0} = \overline{0} \cdot \overline{1} \cdot \overline{0},\ \overline{1} \leq \overline{1} = \overline{1} \cdot \overline{1} \cdot \overline{1},\ \overline{2} \leq \overline{2} = \overline{2} \cdot \overline{2} \cdot \overline{2}.$$

Then $(\mathbb{Z}_3, \cdot, \leq)$ is a regular ordered semigroup. We see that $\overline{1}$ is not ordered regular in $(\mathbb{Z}_3, \overline{0}, \leq)$ because $\overline{1} \not\leq \overline{0} = \overline{1}\,\overline{0}\,\overline{x}\,\overline{0}\,\overline{1}$ for all $\overline{x} \in \mathbb{Z}_3$. Hence $(\mathbb{Z}_3, \overline{0}, \leq)$ is not a regular ordered semigroup.

An element a of an ordered semigroup $(S, \cdot, \leq)$ is said to be *regularity-preserving* if the ordered semigroup $(S, a, \leq)$ is regular. The purpose of this paper is to characterize the regularity-preserving elements of regular ordered semigroups.

2 Main Results

In this section, we denote the set of all regularity-preserving elements of an ordered semigroup S by $RP(S)$.

Theorem 2.1. *Let S be an ordered semigroup with $RP(S) \neq \emptyset$. The following statements hold.*

(i) *S is a regular ordered semigroup.*

(ii) *$RP(S)$ is a subsemigroup of S.*

Proof. (i) Since $RP(S) \neq \emptyset$, there exists $a \in S$ such that $(S, a, \leq)$ is regular. Let $x \in S$. Then $x \leq x \circ y \circ x$ in $(S, a, \leq)$ for some $y \in S$, that is, $x \leq xayax = x(aya)x$. So x is ordered regular in S. This shows that S is a regular ordered semigroup.

(ii) Let $a, b \in RP(S)$ and let $x \in S$. Then there exist $y, z, s, t \in S$ such that

$$x \leq xayax, x \leq xbzbx, a \leq absba \text{ and } b \leq batab.$$

Thus

$$\begin{aligned} x &\leq xayax \\ &\leq x(absba)ya(xbzbx) = x(ab)(sbayaxbz)bx \\ &\leq x(ab)(sbayaxbz)(batab)x = x(ab)(sbayaxbzbat)(ab)x \end{aligned}$$

from which it follows that x is ordered regular in $(S, ab, \leq)$. Consequently, $ab \in RP(S)$. Hence $RP(S)$ is a subsemigroup of S as required. ☐

Theorem 2.2. *Let S be an ordered semigroup and let $a \in RP(S)$.*

(i) $(SaS] = S$.

(ii) *If $b \in S$ is such that $a \in (bS] \cap (Sb]$, then $b \in RP(S)$.*

Proof. (i) Let $x \in S$. Since $(S, a, \leq)$ is regular, there exists $y \in S$ such that $x \leq xayax$. Then $x \in (SaS]$. This implies that $S = (SaS]$.

(ii) Let $b \in S$ such that $a \in (bS] \cap (Sb]$. Then there exist $y, z \in S$ such that $a \leq by$ and $a \leq zb$. Let $x \in S$. Then, for some $w \in S$, we have

$$x \leq xawax \leq x(by)w(zb)x = xb(ywz)bx$$

where $ywz \in S$. Thus x is ordered regular in $(S, b, \leq)$. It follows that $b \in RP(S)$. □

Theorem 2.3. *Let S be a regular ordered semigroup and let $a \in S$. Then $a \in RP(S)$ if and only if $(baS] = (bS]$ and $(Sab] = (Sb]$ for every $b \in S$.*

Proof. Suppose first that a is a regular-preserving element of S. Let $b \in S$. Since $baS \subseteq bS$ and $Sab \subseteq Sb$, it follows that $(baS] \subseteq (bS]$ and $(Sab] \subseteq (Sb]$, respectively. Since $(S, a, \leq)$ is regular, we have $b \leq baxab$ for some $x \in S$. If $y \in (bS]$, then $y \leq bz$ for some $z \in S$, so

$$y \leq bz \leq (baxab)z = (ba)(xabz),$$

which implies that $y \in (baS]$. This verifies that $(bS] \subseteq (baS]$. We can show similarly that $(Sb] \subseteq (Sab]$.

Conversely, suppose that $(baS] = (bS]$ and $(Sab] = (Sb]$ for every $b \in S$. Let $x \in S$. By assumption, $(xaS] = (xS]$ and $(Sax] = (Sx]$. Since S is a regular ordered semigroup, we have $x \in (xS]$ and $x \in (Sx]$. Then $x \leq xas$ and $x \leq tax$ for some $s, t \in S$. Since S is a regular ordered semigroup and $x \in S$, we have $x \leq xyx$ for some $y \in S$. Thus

$$x \leq xyx \leq (xas)y(tax) = xa(syt)ax.$$

It follows that x is ordered regular in $(S, a, \leq)$. Hence $a \in RP(S)$. □

Theorem 2.4. *Let S be a regular ordered semigroup and let $e \in E_{\leq}(S)$. If $e \in RP(S)$, then $V_{\leq}(f) \cap eSe \neq \emptyset$ for every $f \in E_{\leq}(S)$.*

Proof. Assume that $e \in RP(S)$. Let $f \in E_{\leq}(S)$. Since $e \in RP(S)$, it follows from Theorem 2.3 that $(feS] = (fS]$ and $(Sef] = (Sf]$. Since S is a regular ordered semigroup, we have $f \in (fS] \cap (Sf]$, so there are $x, y \in S$ such that $f \leq fex$ and $f \leq yef$. Since $f \leq f^2$, we get $f^2 \leq f^3$. Then

$$f \leq f^2 \leq f^3 = fff \leq (fex)f(yef) = f(exfye)f$$

and

$$exfye \leq exf^2ye \leq exfexfye \leq exf^2exfye \leq exfyefexfye.$$

Thus $exfye \in V_{\leq}(f) \cap eSe$. □

Example 2.1. (1) Let $\leq := \{(\overline{0},\overline{0}), (\overline{1},\overline{1}), (\overline{2},\overline{2}), (\overline{3},\overline{3}), (\overline{4},\overline{4}), (\overline{5},\overline{5})\}$. It is easy to see that $(\mathbb{Z}_6, \cdot, \leq)$ is a regular ordered semigroup. Note that $E_{\leq}(\mathbb{Z}_6) = \{\overline{0}, \overline{1}, \overline{3}, \overline{4}\}$. We see that

$$\begin{aligned}
&V_{\leq}(\overline{1}) \cap \overline{0} \cdot \mathbb{Z}_6 \cdot \overline{0} = \{\overline{1}\} \cap \{\overline{0}\} = \emptyset, \\
&V_{\leq}(\overline{1}) \cap \overline{3} \cdot \mathbb{Z}_6 \cdot \overline{3} = \{\overline{1}\} \cap \{\overline{0}, \overline{3}\} = \emptyset \text{ and} \\
&V_{\leq}(\overline{1}) \cap \overline{4} \cdot \mathbb{Z}_6 \cdot \overline{4} = \{\overline{1}\} \cap \{\overline{0}, \overline{2}, \overline{4}\} = \emptyset.
\end{aligned}$$

By Theorem 2.4, $\overline{0}, \overline{3}, \overline{4} \notin RP(\mathbb{Z}_6)$.

(2) Let $\leq := \{(\overline{0},\overline{0}), (\overline{1},\overline{1}), (\overline{2},\overline{2}), (\overline{1},\overline{0}), (\overline{2},\overline{0})\}$.
It is easy to see that $(\mathbb{Z}_3, \cdot, \leq)$ is a regular ordered semigroup. We have that

$$E_{\leq}(\mathbb{Z}_3) = \{\overline{0}, \overline{1}\}, V_{\leq}(\overline{0}) = \{\overline{0}\} \text{ and } V_{\leq}(\overline{1}) = \{\overline{0}, \overline{1}\}.$$

Moreover, $RP(\mathbb{Z}_3) = \{\overline{0}, \overline{1}, \overline{2}\}$. It is easy to see that $\overline{0} \in V_{\leq}(f) \cap e \cdot \mathbb{Z}_3 \cdot e$ for every $e, f \in E_{\leq}(\mathbb{Z}_3)$.

The next theorem, we characterize the regularity-preserving elements of a regular ordered semigroup with identity.

Theorem 2.5. *Let S be a regular ordered semigroup with identity e. Then $a \in RP(S)$ if and only if a is invertible.*

Proof. Assume that $a \in RP(S)$. Since $(S, a, \leq)$ is regular, there exists $x \in S$ such that

$$e \leq eaxae = axa.$$

Thus a is invertible. Conversely, assume that a is invertible. Then there exist $x, y \in S$ such that $e \leq ax$ and $e \leq ya$. Let $b \in S$. Since $(S, \cdot, \leq)$ is regular, $b \leq bzb$ for some $z \in S$. So

$$b \leq bzb = bezeb \leq baxzyab.$$

Therefore b is regular in $(S, a, \leq)$. This implies that $a \in RP(S)$. □

Example 2.2. (1) Consider a regular ordered semigroup $(\mathbb{Z}_6, \cdot, \leq)$ where $\leq := \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2}), (\overline{3}, \overline{3}), (\overline{4}, \overline{4}), (\overline{5}, \overline{5})\}$. It is easy to see that the set of invertible of $(\mathbb{Z}_6, \cdot, \leq)$ is $\{\overline{1}, \overline{5}\}$. By Theorem 2.5, $RP(\mathbb{Z}_6) = \{\overline{1}, \overline{5}\}$.

(2) Consider a regular ordered semigroup $(\mathbb{Z}_3, \cdot, \leq)$ where

$$\leq := \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2}), (\overline{1}, \overline{0}), (\overline{2}, \overline{0})\}.$$

From Example 2.1, we have $RP(\mathbb{Z}_3) = \{\overline{0}, \overline{1}, \overline{2}\}$. By Theorem 2.5, the set of invertible of $(\mathbb{Z}_3, \cdot, \leq)$ is $\{\overline{0}, \overline{1}, \overline{2}\}$.

Finally, the regularity-preserving elements of a regular ordered semigroup with mididentity are characterized.

Theorem 2.6. *Let S be a regular ordered semigroup with a mididentity and let $a \in S$. Define $M := \{b \in S \mid xy \leq xby \text{ for all } x, y \in S\}$. Then $a \in RP(S)$ if and only if $(aS] \cap (Sa] \cap M \neq \emptyset$.*

Proof. Let u be a mididentity of S. Assume that $a \in RP(S)$. Then u is ordered regular in $(S, a, \leq)$, so $u \leq uazau$ for some $z \in S$. Let $x, y \in S$. Then $aza \in S$ and

$$xy = xuy \leq x(uazau)y = (xua)z(auy) = xazay$$

from which it follows that $aza \in (aS] \cap (Sa] \cap M$.

Conversely, suppose that $(aS] \cap (Sa] \cap M \neq \emptyset$. Let $b \in (aS] \cap (Sa] \cap M$. Then $b \leq as$ and $b \leq ta$ for some $s, t \in S$. Let $x \in S$. Then there exists $y \in S$ such that $x \leq xyx$. Hence

$$x \leq xyx \leq xbyx \leq xbybx \leq x(as)y(ta)x = xa(syt)ax,$$

so x is ordered regular in $(S, a \leq)$. Thus $a \in RP(S)$ as required. □

Example 2.3. (1) Consider a regular ordered semigroup $(\mathbb{Z}_6, \cdot, \leq)$ where $\leq := \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2}), (\overline{3}, \overline{3}), (\overline{4}, \overline{4}), (\overline{5}, \overline{5})\}$. We have that $\overline{1}$ is a mididentity of $\mathbb{Z}_6$ and $M = \{\overline{1}\}$. By Example 2.2, we have $RP(\mathbb{Z}_6) = \{\overline{1}, \overline{5}\}$. By Theorem 2.6, $(\overline{1} \cdot \mathbb{Z}_6] \cap (\mathbb{Z}_6 \cdot \overline{1}] \cap M \neq \emptyset$ and $(\overline{5} \cdot \mathbb{Z}_6] \cap (\mathbb{Z}_6 \cdot \overline{5}] \cap M \neq \emptyset$.

(2) Consider a regular ordered semigroup $(\mathbb{Z}_3, \cdot, \leq)$ where

$$\leq := \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{2}, \overline{2}), (\overline{1}, \overline{0}), (\overline{2}, \overline{0})\}.$$

We have that $\overline{1}$ is a mididentity of $\mathbb{Z}_3, M = \{\overline{0}, \overline{1}\}$ and $RP(\mathbb{Z}_3) = \{\overline{0}, \overline{1}, \overline{2}\}$. By Theorem 2.6, $(\overline{0} \cdot \mathbb{Z}_3] \cap (\mathbb{Z}_3 \cdot \overline{0}] \cap M \neq \emptyset$, $(\overline{1} \cdot \mathbb{Z}_3] \cap (\mathbb{Z}_3 \cdot \overline{1}] \cap M \neq \emptyset$ and $(\overline{2} \cdot \mathbb{Z}_3] \cap (\mathbb{Z}_3 \cdot \overline{2}] \cap M \neq \emptyset$.

3 Discussion

It is known that every semigroup can be considered to be an ordered semigroup by using $\leq := id_S$ where id_S is an identity relation on S. In this paper, we generalize some theorems in [4]. All results in this paper can be used in case semigroups and ordered semigroups.

4 Acknowledgments

We would like to thank the reviewers for their comments improving this paper.

This paper was supported by Algebra and Applications Research Unit, Faculty of Science, Prince of Songkla University.

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