International Journal of Mathematics and
Computer Science, **14**(2019), no. 3, 573–582

Subordination inequalities of a new Salagean-difference operator

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(Received March 4, 2019, Accepted April 5, 2019)

Abstract

We study some interesting inequalities involving subordination and superordination of a class of univalent functions $T_m(\kappa, \alpha)$ given by a new differential-difference operator in the open unit disk.

1 Introduction

Let Λ be the class of analytic function formulated by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z : |z| < 1\}.$$

Key words and phrases: Univalent function, differential operator, unit disk, analytic function, bounded turning, subordination, superordination.
AMS (MOS) Subject Classifications: 30C45.
ISSN 1814-0432, 2019 , http://ijmcs.future-in-tech.net

We symbolize by $T(\alpha)$ the subclass of Λ for which $\Re\{f'(z)\} > \alpha$ in U. For a function $f \in \Lambda$, we present the following difference operator

$$\begin{aligned}
&D_\kappa^0 f(z) = f(z)\\
&D_\kappa^1 f(z) = zf'(z) + \frac{\kappa}{2}\left(f(z) - f(-z) - 2z\right), \quad \kappa \in \mathbb{R}\\
&\vdots\\
&D_\kappa^m f(z) = D_\kappa(D_\kappa^{m-1} f(z))\\
&\qquad\qquad = z + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m\, a_n z^n.
\end{aligned} \tag{1.1}$$

It is clear that when $\kappa = 0$, we have the Salagean differential operator [1]. We call D_κ^m the Salagean-difference operator. Moreover, D_κ^m is a modified Dunkl operator of complex variables [2] and for recent work [3, 4]. Dunkl operator describes a major generalization of partial derivatives and realizes the commutative law in $\mathbb{R}^n$. In geometry, it attains the reflexive relation, which is plotting the space into itself as a set of fixed points.

Example 1.1. *(see Figs 1 and 2)*

- *Let* $f(z) = z/(1-z)$. *Then*
$$D_1^1 f(z) = z + 2z^2 + 4z^3 + 4z^4 + 6z^5 + 6z^6 + ...$$
- *Let* $f(z) = z/(1-z)^2$. *Then*
$$D_1^1 f(z) = z + 4z^2 + 12z^3 + 16z^4 + 30z^5 + 36z^6 + ...$$

We proceed to define a generalized class of bounded turning utilizing the the Salagean-difference operator. Let $T_m(\kappa, \alpha)$ denote the class of functions $f \in \Lambda$ which achieve the condition

$$\Re\{(D_\kappa^m f(z))'\} > \alpha, \quad 0 \le \alpha \le 1,\ z \in U,\ m = 0, 1, 2, \ldots .$$

Clearly, $T_0(\kappa, \alpha) = T(\alpha)$ (the bounded turning class of order α.) The Hadamard product or convolution of two power series is denoted by $(*)$ achieving

$$\begin{aligned}
f(z) * h(z) &= \Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) * \Big(z + \sum_{n=2}^{\infty} \eta_n z^n\Big)\\
&= z + \sum_{n=2}^{\infty} a_n \eta_n z^n.
\end{aligned} \tag{1.2}$$

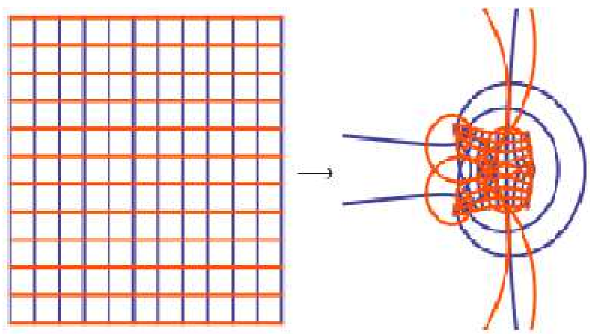

Figure 1: $D_1^1(z/(1-z))$

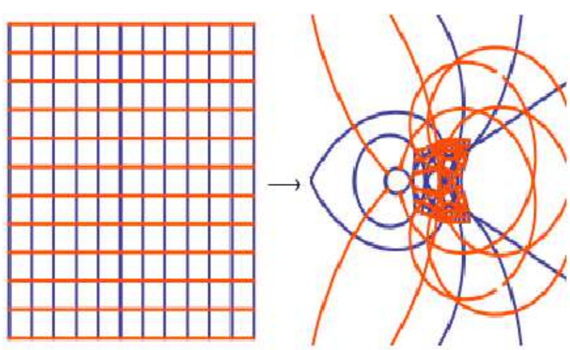

Figure 2: $D_1^1(z/(1-z)^2)$

There are different techniques of studying the class of bounded turning functions, such as using partial sums or applying Jack Lemma [5]-[7]. The aim of this effort is to present several important inequalities of the class $T_m(\kappa, \alpha)$. For this purpose, we need the following auxiliary preliminaries.

For analytic functions f, h in U, we recall that the function f is subordinate to h, if there exists a Schwarz function $\omega \in U$ such that $\omega(0) = 0, |\omega(z)| < 1$, $z \in U$ satisfying $f(z) = h(\omega(z))$ for all $z \in U$ (see [8]). This subordination is represented by $f \prec h$. If the function h is univalent in U, then $f(z) \prec h(z)$ is equivalent to $f(0) = h(0)$ and $f(U) \subset h(U)$.

Moreover, the concept of subordination

$$\sum_{n=0}^{\infty} a_n z^n \prec \sum_{n=0}^{\infty} \eta_n z^n,$$

implies the following inequality

$$\sum_{n=0}^{\infty} |a_n|^2 \leq \sum_{n=2}^{\infty} |\eta_n|^2.$$

Especially: If an analytic function $f(z) \in U$ (bounded by 1), then $f \prec z$, and the relation (see [9])

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1.$$

For finding the main outcome, we will utilize the method of differential subordinations which established by Miller and Mocanu [8]. Namely, $\phi : \mathbb{C}^2 \to \mathbb{C}$ is analytic in U, h is univalent in U and p, p' are analytic in U then $p(z)$ is said to achieve a first order differential subordination if

$$\phi(p, zp') \prec h(z).$$

Lemma 1.1. *[10] Let $f \in \Lambda$ and $\nu > 0$. If*

$$\Re\Big(f'(z) - \frac{f(z)}{z}\Big) > -\frac{\nu}{2}, \quad z \in U$$

then

- $\Re\Big(\dfrac{f(z)}{z}\Big) > 1 - \nu;$

- $\Re\Big(f'(z)\Big) > 1 - \frac{3}{2}\nu;$
- $\Re\Big(f'(z)\Big) > 0, \quad \nu \leq \frac{2}{3}.$

All the above inequalities are sharp.

Lemma 1.2. *[11] For all $z \in U$ the sum*

$$\Re\Big(\sum_{n=2} \frac{z^{n-1}}{n+1}\Big) > -\frac{1}{3}.$$

Lemma 1.3. *[12] Let $h(z)$ be analytic in U with $h(0) = 0$. If $|h(z)|$ approaches its maximality at a point $z_0 \in U$ when $|z| = r$, then*

$$z_0 h'(z_0) = \epsilon\, h(z_0),$$

where $\epsilon \geq 1$ is a real number.

2 Results

In this section, we illustrate our results.

Theorem 2.4. $T_{m+1}(\kappa, \alpha) \subset T_m(\kappa, \alpha)$, $0 \leq \kappa \leq 1/2$.

Proof. Our aim is to apply Lemma 1.1. Let $f \in T_{m+1}(\kappa, \alpha)$ then we have

$$\Re\{1 + \sum_{n=2}^{\infty} n[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > \alpha.$$

A computation implies the inequality

$$\Re\{1 + \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} n[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > \frac{1}{2}.$$

Or

$$\Re\{\frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} n[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > -\frac{1}{2}. \qquad (2.3)$$

By employing the definition of the convolution, we have the construction

$$
\begin{aligned}
(D_\kappa^m f(z))' - \frac{D_\kappa^m f(z)}{z} &= \sum_{n=2}^{\infty}(n-1)[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^m\, a_n z^{n-1} \\
&= \Big(\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\Big) \\
&* \Big(2(1-\alpha)\sum_{n=2}^{\infty}(\frac{n-1}{n})\frac{z^{n-1}}{n+\kappa(1+(-1)^{n+1})}\Big)
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
(\frac{n}{n-1})[n+\frac{\kappa}{2}(1+(-1)^{n+1})] &\le (\frac{n}{n-1})[n+2\kappa] \\
&\le (\frac{n}{n-1})(n+1) \\
&\le (n+1)^2, \quad 0 \le \kappa \le 1/2.
\end{aligned}
$$

By applying Lemma 1.2 on the second term of the above convolution and using the fact

$$
\sum_{n=1}^{\infty}(-1)^{n-1}/(n+1)^2 = (1/12)(12-\pi^2) = 0.177
$$

or

$$
\sum_{n=2}^{\infty}(-1)^{n-1}/(n+1)^2 = -0.073 > -1/3,
$$

we obtain

$$
\begin{aligned}
&\Re\Big(2(1-\alpha)\sum_{n=2}^{\infty}(\frac{n-1}{n})\frac{z^{n-1}}{n+\frac{\kappa}{2}(1+(-1)^{n+1})}\Big) \\
&\ge \Re\Big(2(1-\alpha)\sum_{n=2}\frac{z^{n-1}}{(n+1)^2}\Big) \\
&\ge 2(1-\alpha)\sum_{n=2}\frac{(-1)^{n-1}}{(n+1)^2}, \quad \Re z = -1 \\
&> -\frac{2(1-\alpha)}{3}.
\end{aligned}
\tag{2.4}
$$

Combining (2.3) and (2.4), we have for $z \in U$

$$\Re\Big((D_\kappa^m f(z))' - \frac{D_\kappa^m f(z)}{z}\Big) > \frac{2/3(1-\alpha)}{2}$$
$$> -\frac{2/3(1-\alpha)}{2}.$$

Hence, by letting $\nu := 2/3(1-\alpha)$, Lemma 1.1 implies that

$$\Re(D_\kappa^m f(z))') > 1 - \frac{3}{2}\nu = \alpha,$$

consequently, $f \in T_m(\kappa, \alpha)$. □

Theorem 2.5. *Let* $z \in U,\ f \in \Lambda, 1 < \tau < 2$. *If*

$$\Re\{\frac{z(D_\kappa^m)\,(f)''(z)}{(D_\kappa^m)\,(f)'(z)}\} > \frac{\tau}{2},$$

then

$$(D_\kappa^m f)'(z) \prec (1-z)^\tau$$

and $(D_\kappa^m)\,(f)(z)$ *is bounded turning function.*

Proof. Consider a function $\chi(z)$, $z \in U$ as follows:

$$(D_\kappa^m f)'(z) = (1-\chi(z))^\tau, \quad z \in U,$$

where, $\chi(z)$ is analytic with $\chi(0) = 0$. We must show that $|\chi(z)| < 1$. By the definition of χ, we get

$$\frac{z(D_\kappa^m)\,(f)''(z)}{(D_\kappa^m)\,(f)'(z)} = \tau\frac{-z\chi'(z)}{1-\chi(z)}.$$

Thus, we arrive at

$$\Re\{\frac{z(D_\kappa^m)\,(f)''(z)}{(D_\kappa^m)\,(f)'(z)}\} = \tau\Re\{\frac{-z\chi'(z)}{1-\chi(z)}\}$$
$$> \frac{\tau}{2}, \quad \tau \in (1,2).$$

In view of Lemma 1.3, there exists a complex number $z_0 \in U$ such that $\chi(z_0) = e^{i\theta}$ and

$$z_0\,\chi'(z_0) = \epsilon\chi(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

But
$$\Re\Big(\frac{1}{1-\chi(z_0)}\Big)=\Re\Big(\frac{1}{1-e^{i\theta}}\Big)=\frac{1}{2}$$
then, we obtain
$$\begin{aligned}\Re\{\frac{z(D_\kappa^m f)''(z_0)}{(D_\kappa^m f)'(z_0)}\} &= \tau\Re\{\frac{-\epsilon\chi(z_0)}{1-\chi(z_0)}\}\\ &= \tau\Re\{\frac{-\epsilon e^{i\theta}}{1-e^{i\theta}}\}\\ &\le \frac{\tau}{2}, \quad \epsilon=1,\end{aligned}$$
which contradicts the condition of the theorem. Therefore, there is no $z_0\in U$ with $|\chi(z_0)|=1$, which implies that $|\chi(z)|<1$. Furthermore, we get
$$(D_\kappa^m)\,(f)'(z)\prec(1-z)^\tau,$$
which means that $\Re[(D_\kappa^m f)'(z)]>0$. This completes the proof. □

Theorem 2.6. *Let $\tau>1/2$ such that*
$$\Re\{\frac{z(D_\kappa^m f)'(z)}{(D_\kappa^m f)(z)}\}>\frac{2\tau-1}{2\tau},$$
then
$$\frac{(D_\kappa^m f)(z)}{z}\prec(1-z)^{1/\tau}, \quad f\in\Lambda.$$

Proof. Suppose that there is a function $w(z)$, $z\in U$ defining as follows:
$$\frac{(D_\kappa^m)\,f)(z)}{z}=(1-w(z))^{1/\tau}, \quad z\in U,$$
where, $w(z)$ is analytic with $w(0)=0$. We shall prove that $|w(z)|<1$. From the definition of w, we attain
$$\frac{z(D_\kappa^m f)'(z)}{(D_\kappa^m)\,(f)(z)}=1-\frac{zw'(z)}{\tau(1-w(z))}.$$
This implies that
$$\begin{aligned}\Re\Big\{\frac{z(D_\kappa^m f)'(z)}{(D_\kappa^m)\,(f)(z)}\Big\} &= \Re\Big\{1-\frac{zw'(z)}{\tau(1-w(z))}\Big\}\\ &> \frac{2\tau-1}{2\tau}, \quad \tau>1/2.\end{aligned}$$

By Lemma 1.3, there exists a complex number $z_0 \in U$ such that $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = \epsilon w(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

This yields that

$$\begin{aligned}
\Re\Big\{\frac{z_0(D_\kappa^m f)'(z_0)}{(D_\kappa^m f)(z_0)}\Big\} &= \Re\Big\{1 - \frac{z_0 w'(z_0)}{\tau(1-w(z_0))}\Big\} \\
&= \Re\Big\{1 - \frac{\epsilon w(z_0)}{\tau(1-w(z_0))}\Big\} \\
&= 1 - \Re\Big\{\frac{\epsilon e^{i\theta}}{\tau(1+e^{i\theta})}\Big\} \\
&= \frac{2\tau-1}{2\tau},
\end{aligned}$$

and this is a contradiction with the assumption of the theorem. Thus, there is no $z_0 \in U$ with $|w(z_0)| = 1$, which yields that $|w(z)| < 1$. This completes the proof. □

Note that as an application of the operator D_κ^m is in theory of computer science to generalize the results in [13]. Moreover, this operator can be considered in the work [14] to give new results in the geometric function theory.

Acknowledgment

The work here is partially supported by the Universiti Kebangsaan Malaysia grant: GUP (Geran Universiti Penyelidikan)-2017-064.

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