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Solving a System of MPEs by Modified Power Series Method

Prapart Pue–on

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha sarakham 44150, Thailand

email: prapart.p@msu.ac.th

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Abstract

This research deals with a system of multi-pantograph equations with proportional delays. The power series method is modified for solving the system. The obtained series solution rapidly converges to the exact solution. The method can be applied without discretization, perturbation, and linearization. Illustrative examples show the validity and efficiency of the proposed method.

1 Introduction

Many natural phenomena are formulated by delay differential equations. Delay differential equations are similar to ordinary differential equations, but their evolution at a certain time instant depends on past values. Pantograph differential equation is a special type of functional differential equation with proportional delay. The equation is widely applicable in mathematics, physics, biology, and engineering such as number theory, probability theory and algebraic structure, nonlinear dynamic system, electronic system, astrophysics, population models, quantum mechanics, and cell growth, etc.

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In recent years, the multi-pantograph differential equations (MPDEs) were solved by many authors numerically and analytically. For instance, Cakir and Arslan [1] showed the numerical solutions of MPDEs by using Adomian decomposition method and differential transform method, Yu [11] used the variational iteration method for solving the multi-pantograph delay equation, Komashynska et al.[3] applied the residual power series method to a system of MPDEs, Feng [2] employed homotopy perturbation method to MPDEs with variables coefficients, Yübaşi [10] developed the matrix and collocation methods for the approximate solutions of the system of MPDEs, Sezer et al. [7] constructed an approximate solution of a system of MPDEs by the technique based on a Taylor matrix method, Koroma et al. [4] adapted a combination of Laplace transform and the decomposition method to find numerical solution of a system of MPDEs, Widatalla [9] used a combination of Laplace transform and Adomian decomposition method to approximate solution of a system of pantograph equations, and reference therein.

The present paper deals with a system multi-pantograph equations with proportional delays in form

$$\begin{aligned} u_1'(t) =& \alpha_1 u_1(t) + f_1(t, u_i(t), u_i(q_j t)), \\ u_2'(t) =& \alpha_2 u_2(t) + f_2(t, u_i(t), u_i(q_j t)), \\ &\vdots \\ u_n'(t) =& \alpha_n u_n(t) + f_n(t, u_i(t), u_i(q_j t)), \end{aligned} \tag{1.1}$$

with initial contions $u_i(0) = u_{i0}$, $i = 1, 2, \ldots, n$, $0 < q_j < 1$. The existence and uniqueness of solution of MPEs was proved by Liu and Li [5].

The motivation of this paper is to extend the application of the modified power series method suggested in [6] to system (1.1). The purpose of this research is to give an approximate analytic solution of system (1.1) by adapting the modified power series method.

This paper is organized as follows. Section 2 describes the algorithm of the modified power series method whereas section 3 shows the three illustrative examples. The conclusion is provided in the last section.

2 Modified Power Series Method

In this section, the modified power series method is reviewed. For the modified power series method, the approximate solution of (1.1) is given by

$$u_{i,N}(t) = \sum_{k=1}^{N} a_{ik} t^k \tag{2.1}$$

where $u_{i,N} \to u_i$ as $n \to \infty$. The strategy of the modified power series method [6] is adjusted as follows.

Step 1. Rewrite system (1.1) such that only the nonhomogeneous term is on the right-hand side of the equation.

Step 2. On the left-hand side of the nonhomogeneous differential equation, substitute

$$\begin{aligned} u_{i,N}(t) &= \sum_{k=1}^{N} a_{ik} t^k \\ u_{i,N}(qt) &= \sum_{k=1}^{N} a_{ik} (qt)^k,\ 0 < q < 1 \end{aligned}$$

and the derivatives of $u_{i,N}(t)$. However, if the nonhomogeneous term or a coefficient of u_i or its derivative is not a polynomial but analytic at $t = 0$, then replace it by its Taylor series expansion of degree N about $t = 0$.

Step 3. Collect the power of t on the left-hand side of the equation resulting from Step 2 and set the coefficient of each power of t on the left-hand side equal to the corresponding coefficient on the right-hand side of the equation. Then by equating the coefficients of the corresponding powers of t from t^0 up to t^{N-1}. The system of Nn algebraic equations is obtained.

Step 4. Solve the system of algebraic equations from Step 3 by using forward substitution method. And with respect to initial condition u_{i0}, the coefficients $a_{ik}, k = 0, 1, \ldots, N$ are generated.

Step 5. Substitute the coefficients a_{ik} determined in Step 4 into Equation (2.1) to obtain an approximate solution of degree N to the system of multi-pantograph equations.

3 Illustrative Examples

This section gives examples that illustrate the validity and applicability of the above strategy. The result reveals the accuracy and efficiency of the method. Throughout this article, all computations have been performed by using the Maple software package.

Example 3.1. We consider the following system of multi-pantograph equations

$$\begin{aligned} u'(t) &= u(t) - v(t) + u(\frac{t}{2}) + e^{-t} - e^{\frac{t}{2}}, \\ v'(t) &= -u(t) - v(t) - v(\frac{t}{2}) + e^{t} + e^{-\frac{t}{2}} \end{aligned}$$

with the initial conditions $u(0) = 1, v(0) = 1$[8]. The exact solution of the problem is $u(t) = e^t$, $v(t) = e^{-t}$. Suppose the approximate solution of this problem is in the form

$$u_N(t) = \sum_{k=0}^{N} a_k t^k \quad \text{and} \quad v_N(t) = \sum_{k=0}^{N} b_k t^k. \tag{3.1}$$

By applying the proposed method for N = 5, 7, 9, 11, here, the obtained system of algebraic equations from Step 3 of above method when N = 11,

$$a_1 - 2a_0 + b_0 = 0,\ b_1 + 2b_0 + a_0 = 2,\ 2a_2 - \frac{3}{2}a_1 + b_1 = -\frac{3}{2},\ 2b_2 + \frac{3}{2}b_1 + a_1 = \frac{1}{2},$$
$$3a_3 - \frac{5}{4}a_2 + b_2 = \frac{3}{8},\ 3b_3 + \frac{5}{4}b_2 + a_2 = \frac{5}{8},\ 4a_4 - \frac{9}{8}a_3 + b_3 = -\frac{3}{16},$$
$$4b_4 + \frac{9}{8}b_3 + a_3 = \frac{7}{48},\ 5a_5 - \frac{17}{16}a_4 + b_4 = \frac{5}{128},\ 5b_5 + \frac{17}{16}b_4 + a_4 = \frac{17}{384},$$
$$6a_6 - \frac{33}{32}a_5 + b_5 = -\frac{11}{1280},\ 6b_6 + \frac{33}{32}b_5 + a_5 = \frac{31}{3840},\ 7a_7 - \frac{65}{64}a_6 + b_6 = \frac{7}{5120},$$
$$7b_7 + \frac{65}{64}b_6 + a_6 = \frac{13}{9216},\ 8a_8 - \frac{129}{128}a_7 + b_7 = -\frac{43}{215040},\ 8b_8 + \frac{129}{128}b_7 + a_7 = \frac{127}{645120},$$
$$9a_9 - \frac{257}{256}a_8 + b_8 = \frac{17}{688128},\ 9b_9 + \frac{257}{256}b_8 + a_8 = \frac{257}{10321920},$$
$$10a_{10} - \frac{513}{512}a_9 + b_9 = -\frac{19}{6881280},\ 10b_{10} + \frac{513}{512}b_9 + a_9 = \frac{73}{26542080}.$$
$$11a_{11} - \frac{1025}{1024}a_{10} + b_{10} = \frac{341}{1238630400},\ 11b_{11} + \frac{1025}{1024}b_{10} + a_{10} = \frac{41}{148635648}.$$

After solving the above system with respect to $a_0 = 1, b_0 = 1$, one obtains

$$a_k = \frac{1}{k!},\ b_k = \frac{(-1)^k}{k!},\ k = 1, \dots, 11.$$

Hence, the approximate solutions of this problem for N = 5, 7, 9, 11 are

$$u_N = \sum_{k=0}^{N} \frac{1}{k!} t^k \quad \text{and} \quad v_N = \sum_{k=0}^{N} \frac{(-1)^k}{k!} t^k.$$

The comparison of approximate solution u_N, v_N for N = 5, 7, 9, 11 and exact solution are shown in Table 1. The graphs of approximated and exact solution are plotted in Figure 1. It is obvious that $\lim_{N\to\infty} u_N(t) = e^t$, $\lim_{N\to\infty} v_N(t) = e^{-t}$ which is the exact solution of Example 3.1.

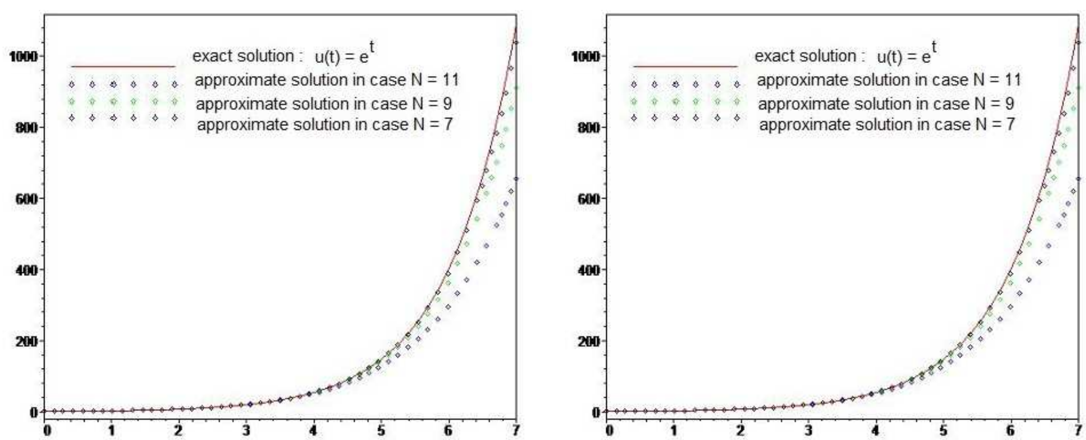

Figure 1: shows the comparison of the exact solution and approximate solutions in case N= 7, 9, 11 for Example 3.1.

t	u_5	u_7	u_9	u_{11}	$u(t) = e^t$
0.2	1.221402667	1.221402759	1.221402759	1.221402759	1.221402758
0.4	1.491818667	1.491824681	1.491824698	1.491824698	1.491824698
0.6	1.822048000	1.822118354	1.822118799	1.822118801	1.822118800
0.8	2.225130667	2.225536366	2.225540897	2.225540929	2.225540928
1.0	2.716666667	2.718253969	2.718281527	2.718281828	2.718281828
t	v_5	v_7	v_9	v_{11}	$v(t) = e^{-t}$
0.2	0.818730667	0.818730753	0.818730753	0.818730753	0.818730753
0.4	0.670314667	0.670320031	0.670320046	0.670320046	0.670320046
0.6	0.548752000	0.548811246	0.548811635	0.548811636	0.548811636
0.8	0.449002667	0.449325145	0.449328937	0.4493289640	0.449328964
1.0	0.366666667	0.367857143	0.367879189	0.367879439	0.367879441

Table 1: shows the comparison of the obtained approximate solution and the exact solution for Example 3.1

Example 3.2. Consider the system of multi-pantograph equations [8]

$$\begin{aligned} u'(t) &= -u(t) - e^{-t}\cos(\frac{t}{2})v(\frac{t}{2}) + 2e^{-\frac{3}{4}t}\cos(\frac{t}{2})\sin(\frac{t}{4})u(\frac{t}{4}), \\ v'(t) &= e^t u^2(\frac{t}{2}) - v^2(\frac{t}{2}) \end{aligned}$$

subject to the initial conditions $u(0) = 1, v(0) = 0$. The Exact solution is $u(t) = e^{-t}\cos t$, $v(t) = \sin t$. Suppose the approximate solution of this problem is in the form

$$u_N(t) = \sum_{k=0}^{N} a_k t^k \quad \text{and} \quad v_N(t) = \sum_{k=0}^{N} b_k t^k. \tag{3.2}$$

After applying the above algorithm for N = 5, 7, 9, 11, the obtained system of algebraic equations when N = 11 are,

$$b_0 + a_0 + a_1 = 0,\ b_1 + b_0^2 - a_0^2 = 0,\ \frac{1}{2}a_0 + a_1 + 2a_2 - b_0 + \frac{1}{2}b_1 = 0,$$

$$-a_0^2 + 2b_2 - a_0a_1 + b_0b_1 = 0,\ -\frac{3}{8}a_0 + \frac{1}{8}a_1 + 3a_3 + \frac{1}{4}b_2 + a_2 - \frac{1}{2}b_1 + \frac{3}{8}b_0 = 0,$$

$$-\frac{1}{2}a_0a_2 - \frac{1}{2}a_0^2 + \frac{1}{4}b_1^2 + 3b_3 - a_0a_1 - \frac{1}{4}a_1^2 + \frac{1}{2}b_0b_2 = 0,$$

$$\frac{7}{96}a_0 - \frac{3}{32}a_1 + \frac{1}{32}a_2 + 4a_4 + a_3 - \frac{1}{24}b_0 - \frac{1}{4}b_2 + \frac{3}{16}b_1 + \frac{1}{8}b_3 = 0,$$

$$-\frac{1}{4}a_1^2 - \frac{1}{6}a_0^2 - \frac{1}{4}a_1a_2 + \frac{1}{4}b_1b_2 + 4b_4 - \frac{1}{2}a_0a_2 - \frac{1}{4}a_0a_3 - \frac{1}{2}a_0a_1 + \frac{1}{4}b_0b_3 = 0,$$

$$\frac{7}{384}a_1 + \frac{1}{64}a_0 - \frac{3}{128}a_2 + \frac{1}{128}a_3 + a_4 - \frac{1}{48}b_1 - \frac{1}{8}b_3 + \frac{3}{32}b_2 - \frac{7}{384}b_0 + \frac{1}{16}b_4 + 5a_5 = 0,$$

$$-\frac{1}{8}a_1^2 - \frac{1}{24}a_0^2 - \frac{1}{6}a_0a_1 - \frac{1}{8}a_1a_3 + \frac{1}{16}b_2^2 - \frac{1}{4}a_0a_3 - \frac{1}{4}a_1a_2 - \frac{1}{8}a_0a_4 - \frac{1}{4}a_0a_2 + 5b_5 + \frac{1}{8}b_1b_3 - \frac{1}{16}a_2^2 + \frac{1}{8}b_0b_4 = 0,$$

$$a_5 + 6a_6 + \frac{1}{32}b_5 - \frac{161}{15360} - \frac{3}{512}a_3 + \frac{7}{1536}a_2 + \frac{1}{256}a_1 + \frac{1}{512}a_4 + \frac{3}{64}b_3 - \frac{1}{96}b_2 - \frac{7}{768}b_1 - \frac{1}{16}b_4 = 0,$$

$$-\frac{1}{16}a_2a_3 - \frac{1}{120}a_0^2 - \frac{1}{24}a_0a_1 - \frac{1}{12}a_0a_2 - \frac{1}{24}a_1^2 - \frac{1}{8}a_1a_2 - \frac{1}{8}a_0a_3 - \frac{1}{8}a_1a_3 - \frac{1}{8}a_0a_4 - \frac{1}{16}a_2^2 + \frac{1}{16}b_2b_3 + \frac{1}{16}b_1b_4 - \frac{1}{16}a_0a_5 + 6b_6 + \frac{1}{16}b_0b_5 - \frac{1}{16}a_1a_4 = 0,$$

$$a_6 + 7a_7 + \frac{1}{64}b_6 + \frac{47}{20480} - \frac{3}{2048}a_4 + \frac{7}{6144}a_3 + \frac{1}{1024}a_2 - \frac{161}{61440}a_1 + \frac{1}{2048}a_5 + \frac{3}{128}b_4 - \frac{1}{192}b_3 - \frac{7}{1536}b_2 + \frac{19}{3840}b_1 - \frac{1}{32}b_5 = 0,$$

$$-\frac{1}{720}a_0^2 - \frac{1}{32}a_2a_4 - \frac{1}{120}a_0a_1 - \frac{1}{48}a_0a_2 - \frac{1}{96}a_1^2 - \frac{1}{24}a_1a_2 - \frac{1}{24}a_0a_3 - \frac{1}{16}a_1a_3 - \frac{1}{16}a_0a_4 - \frac{1}{32}a_2^2 - \frac{1}{16}a_1a_4 - \frac{1}{16}a_2a_3 - \frac{1}{16}a_0a_5 - \frac{1}{32}a_1a_5 - \frac{1}{64}a_3^2 + \frac{1}{32}b_2b_4 + \frac{1}{32}b_1b_5 + \frac{1}{64}b_3^2 - \frac{1}{32}a_0a_6 + 7b_7 + \frac{1}{32}b_0b_6 = 0,$$

$$-\frac{1247}{5160960}+a_7+8a_8+\frac{1}{128}b_7-\frac{3}{8192}a_5+\frac{7}{24576}a_4+\frac{1}{4096}a_3-\frac{161}{245760}a_2$$
$$+\frac{47}{81920}a_1+\frac{1}{8192}a_6+\frac{3}{256}b_5-\frac{1}{384}b_4-\frac{7}{3072}b_3+\frac{19}{7680}b_2-\frac{13}{10240}b_1-\frac{1}{64}b_6=0,$$
$$-\frac{1}{5040}a_0^2-\frac{1}{64}a_0a_7-\frac{1}{32}a_1a_5-\frac{1}{32}a_2a_3-\frac{1}{32}a_2a_4-\frac{1}{32}a_1a_4-\frac{1}{96}a_1a_2-\frac{1}{720}a_0a_1$$
$$-\frac{1}{48}a_1a_3-\frac{1}{240}a_0a_2-\frac{1}{96}a_0a_3-\frac{1}{48}a_0a_4-\frac{1}{32}a_0a_5-\frac{1}{32}a_0a_6-\frac{1}{64}a_3^2-\frac{1}{480}a_1^2$$
$$-\frac{1}{96}a_2^2+8b_8+\frac{1}{64}b_0b_7-\frac{1}{64}a_3a_4+\frac{1}{64}b_2b_5+\frac{1}{64}b_3b_4+\frac{1}{64}b_1b_6-\frac{1}{64}a_1a_6-\frac{1}{64}a_2a_5=0,$$
$$\frac{1}{491520}+a_8+9a_9+\frac{1}{256}b_8-\frac{1247}{20643840}a_1+\frac{139}{645120}b_1-\frac{3}{32768}a_6$$
$$+\frac{7}{98304}a_5+\frac{1}{16384}a_4-\frac{161}{983040}a_3+\frac{47}{327680}a_2+\frac{1}{32768}a_7+\frac{3}{512}b_6-\frac{1}{768}b_5$$
$$-\frac{7}{6144}b_4+\frac{19}{15360}b_3-\frac{13}{20480}b_2-\frac{1}{128}b_7=0,$$
$$-\frac{1}{128}a_0a_8-\frac{1}{128}a_3a_5-\frac{1}{128}a_1a_7-\frac{1}{128}a_2a_6-\frac{1}{40320}a_0^2-\frac{1}{64}a_0a_7-\frac{1}{64}a_1a_5-\frac{1}{96}a_2a_3$$
$$-\frac{1}{64}a_2a_4-\frac{1}{96}a_1a_4-\frac{1}{480}a_1a_2-\frac{1}{5040}a_0a_1-\frac{1}{192}a_1a_3+\frac{1}{256}b_4^2-\frac{1}{1440}a_0a_2-\frac{1}{480}a_0a_3$$
$$-\frac{1}{192}a_0a_4-\frac{1}{96}a_0a_5-\frac{1}{64}a_0a_6-\frac{1}{256}a_4^2+\frac{1}{128}b_0b_8+\frac{1}{128}b_1b_7+\frac{1}{128}b_2b_6+\frac{1}{128}b_3b_5$$
$$-\frac{1}{128}a_3^2-\frac{1}{2880}a_1^2-\frac{1}{384}a_2^2+9b_9-\frac{1}{64}a_3a_4-\frac{1}{64}a_1a_6-\frac{1}{64}a_2a_5=0,$$
$$\frac{9601}{2972712960}+a_9+10a_{10}+\frac{1}{512}b_9-\frac{1247}{82575360}a_2+\frac{1}{1966080}a_1+\frac{139}{1290240}b_2$$
$$-\frac{3}{131072}a_7+\frac{7}{393216}a_6+\frac{1}{65536}a_5-\frac{161}{3932160}a_4+\frac{47}{1310720}a_3+\frac{1}{131072}a_8+\frac{3}{1024}b_7$$
$$-\frac{527}{20643840}b_1-\frac{1}{1536}b_6-\frac{7}{12288}b_5+\frac{19}{30720}b_4-\frac{19}{30720}b_4-\frac{13}{40960}b_3-\frac{1}{256}b_8=0,$$
$$-\frac{1}{128}a_0a_8-\frac{1}{128}a_3a_5-\frac{1}{128}a_1a_7-\frac{1}{128}a_2a_6-\frac{1}{362880}a_0^2-\frac{1}{128}a_0a_7-\frac{1}{192}a_1a_5$$
$$-\frac{1}{384}a_2a_3-\frac{1}{192}a_2a_4-\frac{1}{256}a_2a_7-\frac{1}{384}a_1a_4-\frac{1}{2880}a_1a_2-\frac{1}{40320}a_0a_1-\frac{1}{960}a_1a_3$$
$$+\frac{1}{256}b_1b_8-\frac{1}{10080}a_0a_2-\frac{1}{2880}a_0a_3+\frac{1}{256}b_3b_6-\frac{1}{960}a_0a_4-\frac{1}{384}a_0a_5-\frac{1}{192}a_0a_6$$
$$-\frac{1}{256}a_4a_5-\frac{1}{256}a_4^2-\frac{1}{256}a_1a_8-\frac{1}{384}a_3^2+\frac{1}{256}b_0b_9-\frac{1}{256}a_3a_6-\frac{1}{20160}a_1^2-\frac{1}{256}a_0a_9$$
$$-\frac{1}{1920}a_2^2+10b_{10}+\frac{1}{256}b_4b_5+\frac{1}{256}b_2b_7-\frac{1}{128}a_3a_4-\frac{1}{128}a_1a_6-\frac{1}{128}a_2a_5=0,$$

$$-\frac{9881}{19818086400}+a_{10}+11a_{11}+\frac{1}{1024}b_{10}-\frac{1247}{330301440}a_3+\frac{1}{7864320}a_2+\frac{9601}{11890851840}a_1+$$

$$\frac{139}{2580480}b_3-\frac{3}{524288}a_8+\frac{7}{1572864}a_7+\frac{1}{262144}a_6-\frac{161}{15728640}a_5+\frac{47}{5242880}a_4+\frac{1}{524288}a_9-$$

$$\frac{527}{41287680}b_2+\frac{359}{185794560}b_1+\frac{3}{2048}b_8-\frac{1}{3072}b_7-\frac{7}{24576}b_6+\frac{19}{61440}b_5-\frac{13}{81920}b_4-\frac{1}{512}b_9=0,$$

$$-\frac{1}{256}a_0a_8-\frac{1}{256}a_3a_5-\frac{1}{256}a_1a_7-\frac{1}{256}a_2a_6-\frac{1}{3628800}a_0^2-\frac{1}{384}a_0a_7-\frac{1}{768}a_1a_5-\frac{1}{1920}a_2a_3-$$

$$\frac{1}{768}a_2a_4-\frac{1}{512}a_1a_9-\frac{1}{256}a_2a_7-\frac{1}{1920}a_1a_4-\frac{1}{20160}a_1a_2-\frac{1}{326880}a_0a_1-\frac{1}{5760}a_1a_3-\frac{1}{512}a_2a_8+$$

$$\frac{1}{1024}b_5^2+\frac{1}{512}b_1b_9-\frac{1}{80640}a_0a_2-\frac{1}{20160}a_0a_3-\frac{1}{5760}a_0a_4-\frac{1}{1920}a_0a_5-\frac{1}{768}a_0a_6-\frac{1}{256}a_4a_5-$$

$$\frac{1}{512}a_4a_6-\frac{1}{512}a_4^2-\frac{1}{256}a_1a_8-\frac{1}{1536}a_3^2-\frac{1}{1024}a_5^2-\frac{1}{256}a_3a_6-\frac{1}{512}a_3a_7-\frac{1}{161280}a_1^2-\frac{1}{256}a_0a_9-$$

$$\frac{1}{512}a_0a_{10}-\frac{1}{11520}a_2^2+11b_{11}+\frac{1}{512}b_0b_{10}+\frac{1}{512}b_4b_6+\frac{1}{512}b_2b_8+\frac{1}{512}b_3b_7-\frac{1}{384}a_3a_4-$$

$$\frac{1}{384}a_1a_6-\frac{1}{384}a_2a_5=0,$$

Because of the initial conditions $a_0=1$, $b_0=0$, solving the system leads to

$$a_1=-1,\ b_1=1,\ a_2=0,\ b_2=0,\ a_3=\frac{1}{3},\ b_3=-\frac{1}{6},\ a_4=-\frac{1}{6},\ b_4=0,\ a_5=\frac{1}{30},$$
$$b_5=\frac{1}{120},\ a_6=0,\ b_6=0,\ a_7=-\frac{1}{630},\ b_7=-\frac{1}{5040},\ a_8=-\frac{1}{2520},\ b_8=0,$$
$$a_9=-\frac{1}{22680},\ b_9=\frac{1}{362880},\ a_{10}=0,\ b_{10}=0,\ a_{11}=\frac{1}{1247400},\ b_{11}=-\frac{1}{39916800}.$$

Hence, the approximate solution of this problem for N = 11 is

$$\begin{aligned}u_{11}(t) &= 1-t+\frac{1}{3}t^3-\frac{1}{6}t^4+\frac{1}{30}t^5-\frac{1}{630}t^7+\frac{1}{2520}t^8-\frac{1}{22680}t^9+\frac{1}{1247400}t^{11},\\ v_{11}(t) &= t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7+\frac{1}{362880}t^9-\frac{1}{39916800}t^{11}.\end{aligned}$$

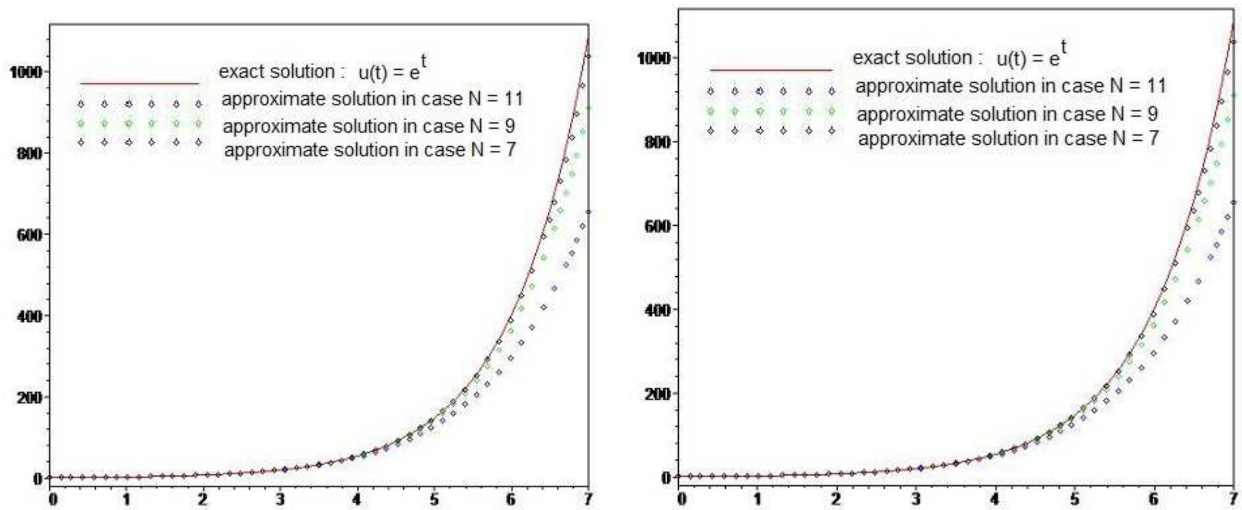

Figure 2: shows the comparison of the exact solution and approximate solutions in case N= 7, 9, 11 for Example 3.2.

The comparison of approximate solution u_N, v_N for N=5, 7, 9, 11 and the exact solution are presented in Table 2. The graphs of approximated and exact solution are plotted in Figure 2. It is obvious that $\lim_{N\to\infty} u_N(t) = e^{-t}\cos t$, $\lim_{N\to\infty} v_N(t) = \sin t$ which is the exact solution of Example 3.2.

t	u_5	u_7	u_9	u_{11}	$u(t) = e^{-t}\cos t$
0.2	0.8024106667	0.8024106464	0.8024106474	0.8024106474	0.8024106473
0.4	0.6174079999	0.6174053993	0.6174056478	0.6174056478	0.617405647
0.6	0.4529920000	0.4529475657	0.4529537865	0.4529537894	0.4529537891
0.8	0.3133226667	0.3129897854	0.3130504438	0.3130505127	0.3130505040
1.0	0.1999999999	0.1984126983	0.1987654320	0.1987662337	0.1987661104
t	v_5	v_7	v_9	v_{11}	$v(t) = \sin t$
0.2	0.1986693334	0.1986693309	0.1986693309	0.1986693309	0.1986693308
0.4	0.3894186666	0.3894183415	0.3894183422	0.3894183422	0.3894183423
0.6	0.5646480000	0.5646424457	0.5646424735	0.5646424734	0.5646424734
0.8	0.7173973334	0.7173557232	0.7173560931	0.7173560909	0.7173560909
1.0	0.8416666666	0.8414682539	0.8414710096	0.8414709845	0.8414709848

Table 2: Comparison of the obtained approximate solution and the exact solution for Example 3.2

Example 3.3. Consider the three dimensional system of multi-pantograph equations [8]

$$\begin{aligned} u'(t) &= 2v(\frac{t}{2}) + w - t\cos(\frac{t}{2}) \\ v'(t) &= 1 - t\sin t - 2w^2(\frac{t}{2}) \\ w'(t) &= v - u - t\cos t \end{aligned}$$

subject to the initial conditions $u(0) = -1, v(0) = 0, w(0) = 0$. The Exact solution of the problem is $u(t) = -\cos t$, $v(t) = t\cos t, w(t) = \sin t$.

By Applying the proposed method for N = 5, 7, 9, 11, the system of algebraic equation when $N = 11$ are

$$
\begin{gathered}
a_1 - 2b_0 - c_0 = 0,\ b_1 + 2c_0^2 = 1,\ a_0 - b_0 + c_1 = 0,\ -b_1 - c_1 + 2a_2 + 1 = 0,\\
2c_0c_1 + 2b_2 = 0,\ a_1 - b_1 + 2c_2 = -1,\ 3a_3 - \frac{1}{2}b_2 - c_2 = 0,\ c_0c_2 + 3b_3 + \frac{1}{2}c_1^2 = -1,\\
a_2 + 3c_3 - b_2 = 0,\ -\frac{1}{8} - \frac{1}{4}b_3 - c_3 + 4a_4 = 0,\ \frac{1}{2}c_0c_3 + 4b_4 + \frac{1}{2}c_1c_2 = 0,\\
a_3 + 4c_4 - \frac{1}{2} - b_3 = 0,\ -\frac{1}{8}b_4 - c_4 + 5a_5 = 0,\ \frac{1}{4}c_0c_4 - \frac{1}{6} + 5b_5 + \frac{1}{8}c_2^2 + \frac{1}{4}c_1c_3 = 0,\\
a_4 + 5a_5 - b_4 = 0,\ \frac{1}{384} - \frac{1}{16}b_5 - c_5 + 6a_6 = 0,\ \frac{1}{8}c_0c_5 + 6b_6 + \frac{1}{8}c_1c_4 + \frac{1}{8}c_2c_3 = 0,\\
a_5 + 6c_6 + \frac{1}{24} - b_5 = 0,\ -\frac{1}{32}b_6 - c_6 + 7a_7 = 0,\\
\frac{1}{16}c_0c_6 + \frac{1}{120} + 7b_7 + \frac{1}{32}c_3^2 + \frac{1}{16}c_2c_4 + \frac{1}{16}c_1c_5 = 0,\\
a_6 + 7c_7 - b_6 = 0,\ -\frac{1}{46080} - \frac{1}{64}b_7 - c_7 + 8a_8 = 0,\\
8b_8 + \frac{1}{32}c_3c_4 + \frac{1}{32}c_2c_5 + \frac{1}{32}c_1c_6 + \frac{1}{32}c_0c_7 = 0\\
a_7 + 8c_8 - \frac{1}{720} - b_7 = 0,\ -\frac{1}{128}b_8 - c_8 + 9a_9 = 0,\\
\frac{1}{64}c_0c_8 - \frac{1}{5040} + 9b_9 + \frac{1}{128}c_4^2 + \frac{1}{64}c_1c_7 + \frac{1}{64}c_3c_5 + \frac{1}{64}c_2c_6 = 0,\\
a_8 + 9c_9 - b_8 = 0,\ \frac{1}{10321920} - \frac{1}{256}b_9 - c_9 + 10a_{10} = 0,\\
\frac{1}{128}c_0c_9 + 10b_{10} + \frac{1}{128}c_3c_6 + \frac{1}{128}c_2c_7 + \frac{1}{128}c_4c_5 + \frac{1}{128}c_1c_8 = 0\\
a_9 + 10c_{10} + \frac{1}{40320} - b_9 = 0,\ -\frac{1}{512}b_{10} - c_{10} + 11a_{11} = 0,\\
\frac{1}{256}c_0c_{10} + \frac{1}{362880} + 11b_{11} + \frac{1}{152}c_5^2 + \frac{1}{256}c_3c_7 + \frac{1}{256}c_4c_6 + \frac{1}{256}c_2c_8 + \frac{1}{256}c_1c_9 = 0.
\end{gathered}
$$

Solving the above system with the initial conditions, $a_0 = -1$, $b_0 = 0$, $c_0 = 0$

yields

$$a_1 = 0,\ b_1 = 1,\ c_1 = 1,\ a_2 = \frac{1}{2},\ b_2 = 0,\ c_2 = 0,\ a_3 = 0,\ b_3 = -\frac{1}{2},\ c_3 = -\frac{1}{6},$$
$$a_4 = -\frac{1}{24},\ b_4 = 0,\ c_4 = 0,\ a_5 = 0,\ b_5 = \frac{1}{24},\ c_5 = \frac{1}{120},\ a_6 = \frac{1}{720},\ b_6 = 0,\ c_6 = 0$$
$$a_7 = 0,\ b_7 = -\frac{1}{720},\ c_7 = -\frac{1}{5040},\ a_8 = -\frac{1}{40320},\ b_8 = 0,\ c_8 = 0,$$
$$a_9 = 0,\ b_9 = \frac{1}{40320},\ c_9 = \frac{1}{362880},\ a_{10} = \frac{1}{3628800},\ b_{10} = 0,\ c_{10} = 0,$$
$$a_{11} = 0,\ b_{11} = -\frac{1}{3628800},\ c_{11} = -\frac{1}{39916800}.$$

Therefore, the approximate solution of this problem for N=5, 7, 9, 11 are

$$\begin{aligned} u_{11}(t) &= -1 + \frac{1}{2}t^2 - \frac{1}{24}t^4 + \frac{1}{720}t^6 - \frac{1}{40320}t^8 + \frac{1}{3628800}t^{10}, \\ v_{11}(t) &= t - \frac{1}{2}t^3 + \frac{1}{24}t^5 - \frac{1}{720}t^7 + \frac{1}{40320}t^9 - \frac{1}{3628800}t^{11}, \\ w_{11}(t) &= t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9 - \frac{1}{39916800}t^{11}. \end{aligned}$$

The comparison of approximate solution u_N, v_N, w_N for N=5, 7, 9, 11 and the exact solution are shown in Table 3. The graphs of approximated and exact solution are plotted in Figure 3. It is obvious that $\lim_{N\to\infty} u_N(t) = -\cos t$, $\lim_{N\to\infty} v_N(t) = t\cos t$, $\lim_{N\to\infty} w_N(t) = \sin t$ which is the exact solution of the problem.

t	u_5	u_7	u_9	u_{11}	$u(t) = -\cos t$
0.2	-0.9800666667	-0.9800665778	-0.9800665779	-0.9800665779	-0.9800665778
0.4	-0.9210666667	-0.9210609778	-0.9210609941	-0.9210609941	-0.9210609940
0.6	-0.8254000000	-0.8253352000	-0.8253356166	-0.8253356149	-0.8253356149
0.8	-0.6970666667	-0.6967025778	-0.6967067388	-0.6967067092	-0.6967067093
1.0	-0.5416666667	-0.5402777778	-0.5403025794	-0.5403023038	-0.5403023059
t	v_5	v_7	v_9	v_{11}	$v(t) = t\cos t$
0.2	-0.1960133333	-0.1960133155	-0.1960133155	-0.1960133155	-0.1960133156
0.4	-0.3684266667	-0.3684243911	-0.3684243976	-0.3684243976	-0.3684243976
0.6	-0.4952400000	-0.4952011200	-0.4952013699	-0.4952013689	-0.4952013689
0.8	-0.5576533333	-0.5573620622	-0.5573653910	-0.5573653673	-0.5573653674
1.0	-0.5416666667	-0.5402777778	-0.5403025794	-0.5403023038	-0.5403023059
t	w_5	w_7	w_9	w_{11}	$w(t) = \sin t$
0.2	0.1986693334	0.1986693309	0.1986693309	0.1986693309	0.1986693308
0.4	0.3894186666	0.3894183415	0.3894183422	0.3894183422	0.3894183423
0.6	0.5646480000	0.5646424457	0.5646424735	0.5646424734	0.5646424734
0.8	0.7173973334	0.7173557232	0.7173560931	0.7173560909	0.7173560909
1.0	0.8416666666	0.8414682539	0.8414710096	0.8414709845	0.8414709848

Table 3: Comparison of the obtained approximate solution and the exact solution for Example 3.3

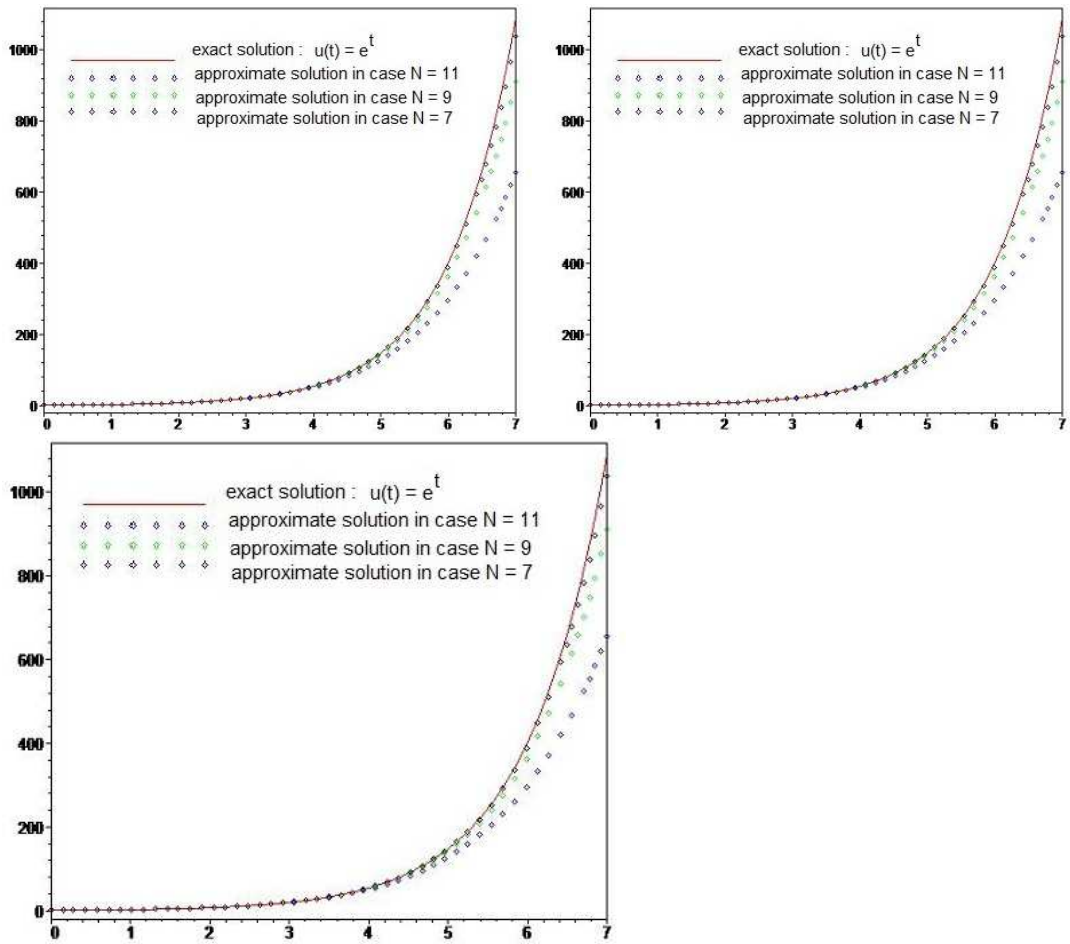

Figure 3: shows the comparison of the exact solution and approximate solutions in case N= 7, 9, 11 for Example 3.3.

4 Conclusion

In this work, we successfully applied a simple technique of the modified power series method in solving a system of MPDEs. The results reveal the efficiency of the method. The main advantage of this technique is that it does not require any discretization, perturbation, and linearization.

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