International Journal of Mathematics and
Computer Science, **14**(2019), no. 3, 563–572

Generalized Orders and Approximation Errors of Entire Harmonic Functions in $\mathbb{R}^n, n \geq 3$

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(Received February 21, 2019, Accepted April 4, 2019)

Abstract

Coefficient characterizations of generalized order and lower order of an entire harmonic function represented by Fourier-Laplace series have been obtained in terms of ratio of harmonic polynomial approximation errors in sup norm. Similar results also have been obtained in terms of ratio of L^2-approximation errors.

1 Introduction

It has been noticed that time dependent problems in $\mathbb{R}^3$ leads to the study of entire harmonic functions in $\mathbb{R}^4$. Also, the harmonic functions play an

Key words and phrases: Approximation errors, entire harmonic functions, generalized orders and ball of radius R.
AMS (MOS) Subject Classifications: 30E10, 41A15.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

important role in physics, mechanics and theoretical mathematical research to describe different stationary processes. Thus, it is significant to study generalized orders of harmonic functions in an n-dimensional space.

Several authors had obtained the characterization of growth parameters of an entire function $f(z)$ in terms of the sequences of polynomial approximation and interpolation errors taken over different domains in the complex plane. Similar characterizations had been investigated for entire harmonic functions in $\mathbb{R}^n, n \geq 3$ in terms of harmonic polynomial approximation errors.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}, a_k \neq 0$, be an entire function. Following Seremeta [5], we define the generalized order and generalized lower order as:

$$\rho = \limsup_{r\to\infty} \frac{\alpha(\log M(r;f))}{\beta(\log r)}, \tag{1.1}$$

$$\lambda = \liminf_{r\to\infty} \frac{\alpha(\log M(r;f))}{\beta(\log r)}, \tag{1.2}$$

where $M(r;f) = \max_{|z|=r} |f(z)|$
and $\alpha \in \Lambda$ and $\beta \in L^0$. Let L^0 denote the class of functions h satisfying the following conditions:
(i) h(x) is defined on $[a,\infty)$ such that h is differentiable, monotonically strictly increasing, tends to ∞ as $x \to \infty$ and

$$\lim_{x\to\infty} \frac{h[(1+\delta(x))x]}{h(x)} = 1, h(x) > 0, x \in [a,\infty), \tag{1.3}$$

for every $\delta(x) \to 0$ as $x \to \infty$. By Λ we denote the class of functions $h \in L^0$ and satisfies

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1, 0 < c < \infty, \tag{1.4}$$

provided that convergence in (1.4) is uniform with respect to $c, 0 < c_1 \leq c \leq c_2 < \infty$.
Following results have been proved by Bajpai et. al.,[1]:
Theorem A. Let $\alpha \in \Lambda, \beta \in L^0$ and let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be entire, $a_k \neq 0$. Set $F(x;c) = \beta^{-1}[c(\alpha(x))]$. If $dF(x;c)/d\log x = O(1)$ as $x \to \infty$ for all $c, 0 < c < \infty$ and $\varphi(k) = \{\log|\frac{a_k}{a_{k+1}}|\}/(\lambda_{k+1} - \lambda_k)$ is non-decreasing, then

$$\rho = \limsup_{k\to\infty} \frac{\alpha(\lambda_k)}{\beta\{\frac{1}{(\lambda_k - \lambda_{k-1})}\log|\frac{a_{k-1}}{a_k}|\}}. \tag{1.5}$$

Theorem B. Let $f(z)=\sum_{k=0}^{\infty}a_k z^{\lambda_k}, a_k\neq 0$, be an entire function of growth λ. If $\alpha\in\Delta$ and $\beta\in L^0$ and

$$\lim_{x\to\infty}\frac{\alpha[x\varphi(x)]}{\alpha[x]}=1, if \lim_{x\to\infty}\frac{\varphi(x)}{x}=0 \tag{1.6}$$

for φ defined on $[a,\infty)$ increasing indefinitely, then we have

$$\begin{aligned}\lambda &= \max_{\{n_k\}}\{\liminf_{k\to\infty}\frac{\alpha[\lambda_{n_{k-1}}]}{\beta[\frac{1}{\lambda_{n_k}}\log|a_{n_k}|^{-1}]}\}\\ &= \max_{\{n_k\}}\{\liminf_{k\to\infty}\frac{\alpha[\lambda_{n_k-1}]}{\beta[\frac{1}{(\lambda_{n_k}-\lambda_{n_k-1})}\log|\frac{a_{n_k-1}}{a_{n_k}}|]}\},\end{aligned}$$

where $\{\lambda_{n_k}\}$ is subspace of integers $\{\lambda_k\}$ and $a_{n_k}\in\{a_n\}$.

Lemma C. Let $f(z)=\sum_{n=0}^{\infty}a_n z^{\lambda_n}, a_n\neq 0$, be an entire function of growth λ defined by (1.2), where α satisfies in addition (1.6), then we have

$$\lambda\geq\liminf_{n\to\infty}\frac{\alpha[\lambda_{n-1}]}{\beta[\frac{1}{\lambda_n}\log|a_n|^{-1}]} \tag{1.7}$$

and

$$\lambda\geq\liminf_{n\to\infty}\frac{\alpha[\lambda_{n-1}]}{\beta[\frac{1}{(\lambda_n-\lambda_{n-1})}\log|\frac{a_{n-1}}{a_n}|]}. \tag{1.8}$$

For (1.8), we define β negatively in the complement of $[a,\infty)$.

Lemma D. Let $f(z)=\sum_{k=0}^{\infty}a_k z^{\lambda_k}, a_k\neq 0$, be such an entire function of growth λ that $\{|\frac{a_k}{a_{k+1}}|^{\frac{1}{(\lambda_{k+1}-\lambda_k)}}\}$ forms a non-decreasing function for $k>k_0$, then we have

$$\lambda\leq\liminf_{k\to\infty}\frac{\alpha[\lambda_{k-1}]}{\beta[\frac{1}{\lambda_k}\log|a_k|^{-1}]}$$

and

$$\lambda\leq\liminf_{k\to\infty}\frac{\alpha[\lambda_{k-1}]}{\beta[\frac{1}{(\lambda_k-\lambda_{k-1})}\log|\frac{a_{k-1}}{a_k}|]}.$$

2 Entire Harmonic Functions

Let $B_R^n=\{y\in\mathbb{R}^n:|y|\leq R\}$ be a ball of radius R in $\mathbb{R}^n, n\geq 3$ centered at the origin of coordinates and $\overline{B_R^n}$ be its closure. Let $H_R, 0<R<\infty$, be

denote the class of harmonic functions in B_R^n and continuous on $\overline{B_R^n}$.
It is known [7] that $u \in H_R$ can be expanded in a Fourier-Laplace series

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x;u)r^k, \forall r, 0 < r < R, \tag{2.9}$$

where $Y^{(k)}$ denote the spherical harmonics or Laplace spherical functions of degree k in the unit sphere $S^n = \{y \in \mathbb{R}^n : |y| = 1\}$ in $\mathbb{R}^n, n \geq 3$ [3, pp.157-174; 4, 6],

$$\begin{aligned} Y^{(k)}(x;u) =& a_1^{(k)} Y_1^{(k)}(x) + a_2^{(k)} Y_2^{(k)}(x) + \cdots + a_{\gamma_k}^{(k)} Y_{\gamma_k}^{(k)}(x), \\ a_j^{(k)} &= (u, Y_j^{(k)}), j = \overline{1, \gamma_k}, x \in S^n, \end{aligned}$$

$(u, Y_j^{(k)})$ is the scalar product in $L^2(S^n)$ and $\gamma_k = \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}$ is the number of linearly independent spherical harmonics of degree k and scalar product in $L^2(S^n)$ is defined as

$$(f,g) = \frac{1}{w_n} \int_{S^n} f(x)g(x)ds,$$

where ds is an element of area of the sphere S^n and $w_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ be the area of surface.
Veselovs'ka [8] defined the generalized order and lower order of an entire harmonic function $u \in \mathbb{R}^n$ for $\alpha \in L^0$ and $\beta \in \Delta$ by the formulas:

$$\rho_{\alpha,\beta}(u) = \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta(r)},$$

$$\lambda_{\alpha,\beta}(u) = \liminf_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta(r)},$$

where $M(r;u) = \max_{x\in S^n} |u(rx)|$.
Veselovs'ka [8] obtained coefficient characterizations of $\rho_{\alpha,\beta}(u)$ and $\lambda_{\alpha,\beta}(u)$ in terms of approximation error in sup norm. In this paper, we will study generalized order $\rho_{\alpha,\beta}(u)$ and generalized lower order $\lambda_{\alpha,\beta}(u)$ in terms of ratio of approximation errors in sup norm and L^2-norm.
The approximation error of function $u \in H_R$ by harmonic polynomials $P \in \Pi_k$ is defined as

$$E_R^k(u) = \inf_{P\in\Pi_k} \{\max_{y\in\overline{B_R}} |u(y) - P(y)|\}, \tag{2.10}$$

where Π_k be a set of harmonic polynomials of degree not exceeding k.
Now we state the following lemmas which will be used in the sequel.
Lemma 2.1 [8]. If $u \in H_R$, then for all $k \in \mathbb{N}$ inequality

$$\max_{\xi \in S^n} |Y^{(k)}(\xi; u)| R^k \leq \frac{4(k+2\nu)^{2\nu}}{(2\nu)!} E_R^{k-1}(u)$$

holds, where $\nu = \frac{n-2}{2}$.
Lemma 2.2 [8]. For an entire harmonic function $u \in \mathbb{R}^n, n \geq 3$, the following estimation holds

$$E_R^k(u) \leq \sqrt{\frac{2}{(2\nu)!}}(2\nu+1)(k+2\nu)^{2\nu} M(r;u)(\frac{R}{r})^k \forall k \in \mathbb{Z}_+, r > eR.$$

3 Main Results

Theorem 3.1. Let $u \in H_R$ be continued to an entire harmonic function in $\mathbb{R}^n$. For $\alpha(x) \in \Delta, \beta(x) \in L^o$ Set $F(x;c) = \beta^{-1}[c\alpha(x)]$. If $\frac{dF(x,c)}{d\log x} = O(1)$ as $x \to \infty$ for all $c, 0 < c < \infty$ and $\varphi(k) = \{\log|\frac{E_R^k(u)}{E_k^{k+1}(u)}|\}/(\lambda_{k+1} - \lambda_k)$ is non-decreasing function of k, then

$$\rho_{\alpha,\beta}(u) = \limsup_{k\to\infty} \frac{\alpha(\lambda_k)}{\beta\{\frac{1}{(\lambda_k - \lambda_{k-1})}\log|\frac{RE_R^{k-1}(u)}{E_R^k(u)}|\}}. \tag{3.11}$$

Proof. Consider the entire function of one complex variable

$$f_1(z) = \sum_{k=0}^{\infty} \frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}} E_R^k(u)(\frac{z}{R})^{\lambda_k},$$
$$f_2(z) = \sum_{k=1}^{\infty} \frac{4}{(2\nu)!}(k+2\nu)^{2\nu} E_R^{k-1}(u)(\frac{z}{R})^{\lambda_k}.$$

Now using lemmas 2.1 and 2.2 for $r > eR$, we get

$$\mu(r; f_1) \leq M(r; u) \leq |Y^k(\xi; u)| + M(r; f_2). \tag{3.12}$$

where $\mu(r; f_1)$ is the maximum term in power series expansion of the function $f_1(z)$ and $M(r; f_2$ is the maximum term of the modulus of the function $f_2(z)$. Thus from (3.2) we obtain

$$\rho_{\alpha,\beta}(f_1) \leq \rho_{\alpha,\beta}(u) \leq \rho_{\alpha,\beta}(f_2). \tag{3.13}$$

Now using Theorem A for the functions $f_1(z)$ and $f_2(z)$ we get

$$\rho_{\alpha,\beta}(f_1) = \rho_{\alpha,\beta}(f_2) = \limsup_{k\to\infty} \frac{\alpha(\lambda_k)}{\beta\{\frac{1}{(\lambda_k-\lambda_{k-1})}\log|\frac{RE_R^{k-1}(u)}{E_R^k(u)}|\}}.$$

In view of (3.3) we obtain the required result (3.1).

Theorem 3.2. Let $u \in H_R$ be continued to an entire harmonic function in $\mathbb{R}^n$. For $\alpha(x) \in \Delta, \beta(x) \in L^o$ and

$$\lim_{\xi\to\infty} \frac{\alpha[\xi\varphi(\xi)]}{\alpha[\xi]} = 1, if \lim_{\xi\to\infty} \frac{\varphi(\xi)}{\xi} = 0$$

for φ defined on $[a,\infty)$ increases indefinitely, then

$$\lambda_{\alpha,\beta}(u) = \max_{\{k_m\}}\{\liminf_{m\to\infty} \frac{\alpha[\lambda_{k_{m-1}}]}{\beta[\frac{1}{\lambda_{k_m}}\log[R^{-\lambda_{k_m}}E_R^{k_m}(u)]^{-1}]}\} \tag{3.14}$$

and

$$\lambda_{\alpha,\beta}(u) = \max_{\{k_m\}}\{\liminf_{m\to\infty} \frac{\alpha[\lambda_{k_{m-1}}]}{\beta[\frac{1}{(\lambda_{k_m}-\lambda_{k_{m-1}})}\log|\frac{RE_R^{k_m-1}(u)}{E_R^{k_m}(u)}|]}\}, \tag{3.15}$$

where $\{\lambda_{k_m}\}$ is subsequence of integers $\{\lambda_k\}$ and $E_R^{k_m}(u) \in \{E_R^k(u)\}$.

Proof. It is known from the construction of Newton's polygon that order ρ and lower order λ defined bu (1.1) and (1.2) are the same for the functions $f_1(z), f_2(z)$ and their auxiliary functions

$$f_1^*(z) = \sum_{k=0}^{\infty} \frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}} E_R^{k_m}(u)(\frac{z}{R})^{\lambda_{k_m}},$$

$$f_2^*(z) = \sum_{k=1}^{\infty} \frac{4}{(2\nu)!}(k+2\nu)^{2\nu} E_R^{k_m-1}(u)(\frac{z}{R})^{\lambda_{k_m-1}},$$

constructed such that $f_1(z), f_2(z)$ and $f_1^*(z), f_2^*(z)$ have the same principal indices respectively; i.e., they have the same maximum modulus terms. Taking into account Lemma C, we obtain

$$\lambda_{\alpha,\beta}(f_1) = \lambda_{\alpha,\beta}(f_2) \geq \liminf_{m\to\infty} \frac{\alpha[\lambda_{k_{m-1}}]}{\beta[\frac{1}{\lambda_{k_m}}\log[R^{-\lambda_{k_m}}E_R^{k_m}(u)]^{-1}]} \tag{3.16}$$

and

$$\lambda_{\alpha,\beta}(f_1) = \lambda_{\alpha,\beta}(f_2) \geq \liminf_{m\to\infty} \frac{\alpha[\lambda_{k_{m-1}}]}{\beta[\frac{1}{(\lambda_{k_m}-\lambda_{k_{m-1}})}\log|\frac{RE_R^{k_m-1}(u)}{E_R^{k_m}(u)}|]}, \tag{3.17}$$

for every subsequence $\{\lambda_{k_m}\}; \{\lambda_k\}$ and $(E_R^{k_m}(u))$ are corresponding coefficients. But as $f_1^*(z)$ and $f_2^*(z)$ satisfies all the conditions of Lemma D, hence from (3.4) and (3.5) we obtain

$$\lambda_{\alpha,\beta}(f_1) = \lambda_{\alpha,\beta}(f_2) = \lambda_{\alpha,\beta}(u) = \max_{\{k_m\}}\{\liminf_{m\to\infty} \frac{\alpha[\lambda_{k_m-1}]}{\beta[\frac{1}{\lambda_{k_m}}\log[R^{-\lambda_{k_m}}E_R^{k_m}(u)]^{-1}]}\}$$
$$= \max_{\{k_m\}}\{\liminf_{m\to\infty} \frac{\alpha[\lambda_{k_m-1}]}{\beta[\frac{1}{(\lambda_{k_m}-\lambda_{k_m-1})}\log|\frac{RE_R^{k_m-1}(u)}{E_R^{k_m}(u)}|]}\}. \tag{3.18}$$

Now using (3.2) we have

$$\lambda_{\alpha,\beta}(f_1) \leq \lambda_{\alpha,\beta}(u) \leq \lambda_{\alpha,\beta}(f_2)$$

it follows from (3.8) the completeness of the proof of Theorem 3.2.
Remark 3.1. For $\alpha(t) = \beta(t) = \log t$ the Theorem 3.1 gives the following formula for the order $\rho_{\alpha,\beta}(u)$ of $u \in \mathbb{R}^n$:

$$\rho_{\alpha,\beta}(u) = \limsup_{k\to\infty} \frac{\log \lambda_k}{\log\log[R\frac{E_R^{\lambda_{k-1}}(u)}{E_R^{\lambda_k}(u)}]^{\frac{1}{(\lambda_k-\lambda_{k-1})}}}.$$

4 Growth of Entire Harmonic Functions in Terms of L^2-Approximation Errors

Let $L^2(S^n)$ denote the class of real valued functions $u(rx)$ which are entire harmonic in the unit sphere S^n and for which $\int\int_{S^n} |u(ry)|^2 ds(y) < \infty$. Let

$$e_R^{2,k}(u) = \{\min_{P\in\pi_k} \int\int_{S^n} |u(y) - P(y)|^2 ds(y)\}^{\frac{1}{2}}.$$

It is known that if $u \in L^2(S^n)$ and $\{e_R^{2,k}(u)\}^{\frac{1}{k}} \to 0$, then $u(rx)$ is an entire function.
Theorem 4.1. Let $u \in L^2(S^n)$ and $\{e_R^{2,k}(u)\}^{\frac{1}{k}} \to 0$ as $k \to \infty$. Then $u(rx)$ is an entire harmonic function. Furthermore,

$$G(rx) = \sum_{k=0}^{\infty} e_R^{2,k}(u) r^k$$

is entire and we have

$$\rho_{\alpha,\beta}(G) = \rho_{\alpha,\beta}(u), \lambda_{\alpha,\beta}(G) = \lambda_{\alpha,\beta}(u) \tag{4.19}$$

and formulae similar to (3.1),(3.4) and (3.5) $E_R^k(u)$ replaced by $e_R^{2,k}(u)$ hold.
Proof. From [3] we have

$$Y^{(k)}(x;u) = \frac{k+\nu}{\nu w_n}\int_{S^n} c_k^{\nu}[(x,y)]u(y)ds(y) \tag{4.20}$$

where $k \in Z_+, x \in S^n$, (.) is the scalar product in $\mathbb{R}^n$ and c_k^{ν} are the Gegenbauer polynomials of degree k and $\nu = \frac{n-2}{2}$. It is known that a harmonic polynomial is the sum of homogeneous harmonic polynomials, using the summation theorem [2, p.235], for the Gegenbauer polynomials c_k^{ν}, we obtain

$$\int_{S^n} c_k^{\nu}[(\xi,\eta)]P(\tau\eta)ds(\eta) = 0, \tag{4.21}$$

where $P \in \pi_{k-1}, 0 < \tau < R$ and $\xi \in S^n$. Using (4.3) with Schwartz inequality in (4.2), we get

$$Y^{(k)}(\xi;u)\tau^k = \frac{k+\nu}{\nu w_n}(\int_{S^n} |c_k^{\nu}[(\xi,\eta)]u(\tau\eta) - P(\tau\eta)|^2 ds(\eta))^{\frac{1}{2}}(\int_{S^n} ds(\eta))^{\frac{1}{2}}.$$

From [2, p.176] we have

$$\max_{1\le t\le 1} |c_k^{\nu}(t)| \le c_k^{\nu}(1) = \frac{(k+2\nu-1)!}{(2\nu-1)!k!},$$

it gives

$$|Y^{(k)}(\xi;u)|\tau^k \le \frac{k+\nu}{\nu w_n} c_k^{\nu}(1) w_n(\int_{S^n} |u(\tau\eta) - P(\tau\eta)|^2 ds(\eta))^{\frac{1}{2}}$$

$$\le \frac{2(k+2\nu)!}{(2\nu)!k!} e_R^{2,k}(u)$$
$$\le \frac{2(k+2\nu)^{2\nu}}{(2\nu)!} e_R^{2,k}(u).$$

Using the definition of $e_R^{2,k}(u)$ we obtain

$$e_R^{2,k}(u) \le \| u(y) - P(u) \|_2 \le A^{\frac{1}{2}} \max_{y\in \overline{B^n}_R} |u(y) - P(y)| = A^{\frac{1}{2}} E_R^k(u)$$

where A is the area of B^n_R. Now using Lemma 2.2, we have

$$e^{2,k}_R(u) \leq \sqrt{\frac{2A}{(2\nu)!}}(2\nu+1)!(k+2\nu)^{2\nu}M(r;u)(\frac{R}{r})^k.$$

Hence

$$M(r;G) \leq \sqrt{\frac{2A}{(2\nu)!}}(2\nu+1)!\sum_{k=0}^{\infty}(k+2\nu)^{2\nu}M(r;u)(\frac{R}{r})^k.$$

so for all sufficiently large r

$$M(\frac{r}{e},G) \leq M(r;u). \tag{4.22}$$

Further

$$\begin{aligned} M(r,G') &\leq \sum_{k=1}^{\infty} k e^{2,k}_R(u) r^{k-1} \\ &\geq \frac{1}{r}\sum_{k=1}^{\infty} k\frac{(2\nu)!}{2(k+2\nu)^{2\nu}}|Y^{(k)}(x;u)|\tau^k r^k \\ &\geq \frac{1}{r}\sum_{k=1}^{\infty}|Y^{(k)}(x;u)|r^k \\ &\geq (1+o(1))\frac{M(r;u)}{r} \end{aligned}$$

it gives

$$M(r+1,G) > (1+o(1))\frac{M(r;u)}{r}. \tag{4.23}$$

Now the result (4.1) follows from (4.4) and (4.5). The remaining proof follows as in Theorem 3.1 and 3.2.

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