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Property Q on G-metric spaces via C-class functions

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Abstract

In this paper, some coincidence and common fixed point theorems are established for two mappings satisfying a nonlinear contractive

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condition in G-metric spaces, generalizing and extending the work of Rashwan and Saleh [J. Nonlinear Convex Analysis, 7 (2006), 289-297]. We also give some examples to support the usability of our results. Moreover, it is proved that these mappings satisfy property Q.

1 Introduction and preliminaries

Fixed point theorems in various generalized metric spaces provide a tool to solve many problems and have applications in the nonlinear analysis and many researchers tried to generalize new contractive mappings to demonstrate the existence of fixed point results (for examples see [2],[3],[4],[5],[6],[7]). Jungck [17] proved a common fixed point theorem for commuting mappings as a generalization of the Banach's fixed point theorem. The concept of the commutativity has generalized in several ways. Sessa [30] introduced the concept of weakly commuting mappings, Jungck [18] extended this concept to compatible maps. In 1998, Jungck and Rhoades [19] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible, but the converse need not to be true, for example see [27].

The notion of a G-metric space was introduced by Mustafa and Sims [24] and [25] as a generalization of the notion of metric spaces. Afterwards, many authors introduced and developed several fixed point theorems for mappings satisfying different contractive conditions in G-metric spaces, see [12, 8, 9, 10, 11, 14, 21, 22, 23, 26, 31]. The study of common fixed points of mappings satisfying strict contractive conditions has been at the center of rigorous research activity. Study of common fixed point theorems in G-metric spaces was initiated by Abbas and Rhoades [1].

We will denote the set all fixed points of a self mapping f from X into itself by $Fix(f)$, i.e., $Fix(f) = \{x \in X : fx = x\}$. It is obvious that if x is a fixed point of f, then it is also a fixed point of f^n for each n, i.e., $Fix(f) \subseteq Fix(f^n)$ if $Fix(f) \neq \emptyset$. However, the converse is false. Indeed, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $fx = \frac{1}{2} - x$ has a unique fixed point $x = \frac{1}{4}$, but every $x \in \mathbb{R}$ is a fixed point for f^n, for each even $n > 1$. Jeong and Rhoades [15] showed that maps satisfying many contractive conditions have property P. They [16] also have shown that a number of contractive conditions involving pairs of maps, have property Q. Several works have been done related to property P and Q (For instance, see [14], [13], [21] and [29]). The following definitions and results will be needed in the sequel. $\mathbb{N}$ and $\mathbb{R}$ will denote the set of natural and real numbers, respectively.

Definition 1.1. *Let X be a nonempty set and let $G : X^3 \to [0, \infty)$ be a function satisfying:*

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$;

(G_2) $0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$;

(G_3) $G(x, x, y) \leq G(x, y, z), \forall x, y, z \in X$, *with* $z \neq y$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, *(symmetry in all three variables)*;

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z), \forall x, y, z, a \in X$, *(rectangle inequality).*

The function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is said to be*

(i) *G-convergent if for every $\varepsilon > 0$, there exist $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \geq k, G(x, x_n, x_m) < \varepsilon$;*

(ii) *G-Cauchy if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n, p \geq k, G(x_m, x_n, x_p) < \varepsilon$, that is, $G(x_m, x_n, x_p) \to 0$ as $m, n, p \to \infty$.*

A space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent.

Lemma 1.3. *Let (X, G) be a G-metric space. The following statements are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x;*

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$;

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$;

(iv) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Lemma 1.4. *Let (X, G) be a G-metric space. The following statements are equivalent:*

(i) *the sequence $\{x_n\}$ is G-Cauchy;*

(ii) *for every* $\varepsilon > 0$, *there exists* $k \in \mathbf{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for* $m, n \geq k$.

Lemma 1.5. *Let* (X, G) *be a* G*-metric space. Then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 1.6. *A* G*-metric space* X *is symmetric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

Proposition 1.7. *Every* G*-metric space* (X, G) *defines a metric space* (X, d_G) *where*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \qquad \forall x, y \in X.$$

Proposition 1.8. *Let* (X, G) *be a* G*-metric space. For any* $x, y, z, a \in X$, *we have*

(i) *if* $G(x, y, z) = 0$, *then* $x = y = z$;

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $G(x, y, y) \leq 2G(x, x, y)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$;

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 1.9. *Let* f *and* g *be self-maps of a nonempty set* X. *If* $w = fx = gx$ *for some* $x \in X$, *then* x *is called a coincidence point of* f *and* g. w *is called a point of coincidence of* f *and* g.

Definition 1.10. *[19] Two self-mappings* f *and* g *are said to be weakly compatible if they commute at their coincidence points, that is,* $fx = gx$ *implies that* $fgx = gfx$.

Definition 1.11. *(Property* P *[15]) Let* f *be a self-mapping of metric space such that fixed point set* $Fix(f) \neq \emptyset$. *Then* f *is said to have the property* P *if* $Fix(f^n) = Fix(f)$, *for each* $n \in \mathbb{N}$. *Equivalently, a mapping has the property* P *if every periodic point is a fixed point.*

Definition 1.12. *(Property* Q *[16]) Let* f *and* g *be self-mappings of a metric space such that* $Fix(f) \cap Fix(g) \neq \emptyset$. f *and* g *are said to have the property* Q *if* $Fix(f^n) \cap Fix(g^n) = Fix(f) \cap Fix(g)$, *for each* $n \in \mathbb{N}$

Recently, Rashwan and Saleh [28] proved the following theorem.

Theorem 1.13. *Let (X,G) be a G-metric space. Let $f,g:(X,G)\to(X,G)$ be satisfying*

$$\psi(G(fx,fy,fz))\leq\psi(N(x,y,z))-\varphi(N(x,y,z)),$$

where

$$\begin{aligned}N(x,y,z)=\max\{&G(gx,gy,gz),G(gx,fx,fx),G(gy,fy,fy),\\&G(gz,fz,fz),\ \alpha G(fx,fx,gy)+(1-\alpha)G(fy,fy,gz),\\&\beta G(gx,fx,fx)+(1-\beta)G(gy,fy,fy)\},\end{aligned}$$

for all $x,y,z\in X$, where, $0<\alpha,\beta<1$ and $\psi,\varphi:[0,\infty)\to[0,\infty)$ are continuous and non-decreasing with $\varphi(t)=0$ if and only if $t=0$. If $f(X)\subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete G-metric subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Lemma 1.14. *[28] Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}G(x_n,x_{n+1},x_{n+1})=0$. If $\{x_n\}$ is not a G-Cauchy sequence in (X,G), then there exists $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k\to\infty$*

$$G(x_{m_k},x_{n_k},x_{n_k}),G(x_{mk},x_{n_k-1},x_{n_k-1}),G(x_{m_k+1},x_{n_k},x_{n_k}),G(x_{n_k-1},x_{m_k+1},x_{m_k+1}).$$

In 2014, A.H. Ansari [2] introduced the concept of C-class functions.

Definition 1.15. *[2] A mapping $F:[0,\infty)^2\to\mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*

(1) $F(s,t)\leq s$ for all $s,t\in[0,\infty)$;

(2) $F(s,t)=s$ implies that either $s=0$ or $t=0$.

We denote C-class functions as $\mathcal{C}$.

Example 1.16. *[2] The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathcal{C}$:*

(1) $F(s,t)=s-t$;

(2) $F(s,t)=ms$ with $0<m<1$;

(3) $F(s,t)=s\beta(s)$ where $\beta:[0,\infty)\to[0,1)$ is continuous;

(4) $F(s,t)=s-\varphi(s)$ where $\varphi:[0,\infty)\to[0,\infty)$ is continuous and $\varphi(t)=0\Leftrightarrow t=0$.

Definition 1.17. *[20] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote Ψ the set of altering distance functions.

Definition 1.18. *A tripled (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ is said to be monotone if for any $x, y \in [0,\infty)$*

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

Example 1.19. *Let $F(s,t) = s - t, \varphi(x) = \sqrt{x}$*

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2 & \text{if } x > 1, \end{cases}$$

then (ψ, φ, F) is monotone.

The rest of this paper is organized as follows: in section 2 we establish some coincidence and common fixed point results for two weakly compatible mappings satisfying a nonlinear contraction condition on G-metric spaces. The results in this section generalize and extend the work of Rashwan and saleh [28] by using C-class functions. Also we give some examples satisfying all requirements of our results. Finally, in section 3 we prove a result of Q property.

2 Main results

First of all, we state the following Lemmas which are fundamental in the sequel.

Lemma 2.1. *[1] Let f and g be weakly compatible self-mappings of nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 2.2. *Let (X, G) be a G-metric space and $f, g : (X, G) \to (X, G)$ be two mappings satisfying*

$$\psi(G(fx, fy, fz)) \leq F\Big(\psi(M(x,y,z)), \varphi(M(x,y,z))\Big), \tag{2.1}$$

for all $x, y, z \in X$, where $\psi, \varphi \in \Psi$ and $F \in \mathcal{C}$ such that (F, ψ, φ) is monotone and

$$M(x,y,z) = \frac{1}{a+b+c+d+e+h}(aG(gx,gy,gz) + bG(gx,fx,fx) + cG(gy,fy,fy)+ dG(gz,fz,fz)+e[\alpha G(fx,fx,gy) + (1-\alpha)G(fy,fy,gz)]+ h[\beta G(gx,fx,fx) + (1-\beta)G(gy,fy,fy)]),$$

where $b, c, d, h \geq 0$, $e > 0$ and $0 < \alpha < 1$. Then f and g have at most a point of coincidence.

Proof. Suppose that $u = fp = gp$ and $v = fq = gq$. Then by (2.1), we have

$$\psi(G(fp,fp,fq)) \leq F\Big(\psi(M(p,p,q)), \varphi(M(p,p,zq))\Big),$$

where

$$M(p,p,q) = \frac{1}{a+b+c+d+e+h}(aG(gp,gp,gq) + bG(gp,fp,fp) + cG(gp,fp,fp)+ + dG(gq,fq,fq) + e[\alpha G(fp,fp,gp) + (1-\alpha)G(fp,fp,gq)]+ h[\beta G(gp,fp,fp) + (1-\beta)G(gp,fp,fp)]).$$

We have

$$\begin{aligned} M(p,p,q) &= \frac{1}{a+b+c+d+e+h}(aG(u,u,v) + e(1-\alpha)G(u,u,v)) \\ &= \frac{a+e(1-\alpha)}{a+b+c+d+e+h}G(u,u,v). \end{aligned}$$

Then

$$\begin{aligned} &\psi(G(u,u,v)) \\ &\leq F\Big(\psi(\frac{a+e(1-\alpha)}{a+b+c+d+e+h}G(u,u,v)), \varphi(\frac{a+e(1-\alpha)}{a+b+c+d+e+h}G(u,u,v))\Big) \\ &\leq F\Big(\psi(G(u,u,v)), \varphi(G(u,u,v))\Big) \\ &\leq \psi(G(u,u,v)), \end{aligned}$$

so, $\psi(G(u,u,v)) = 0$ or $\varphi(G(u,u,v)) = 0$, that is $G(u,u,v) = 0$. Hence, $u = v$. □

Remark 2.3. *In Lemma 2.2, no required hypotheses on the parameters a and β.*

Theorem 2.4. *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ satisfying inequality (2.1) . If $f(X)\subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete G-metric subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. Since $f(X)\subseteq g(X)$, we can choose $x_1\in X$ such that $y_0=fx_0=gx_1$. Continuing in this process, choose $y_n=fx_n=gx_{n+1}$. If $y_n=y_{n+1}$ for some n, then $y_{n+1}=fx_{n+1}=gx_{n+1}$, and so f and g have a coincidence point. From now on, we may assume that $y_n\neq y_{n+1}$ for each n. Then from (2.1), we have

$$\begin{aligned}\psi(G(y_n,y_{n+1},y_{n+1}))&=\psi(G(fx_n,fx_{n+1},fx_{n+1}))\\&\le F\Big(\psi(M(x_n,x_{n+1},x_{n+1})),\varphi(M(x_n,x_{n+1},x_{n+1}))\Big),\end{aligned}\tag{2.2}$$

so,

$$\psi(G(y_n,y_{n+1},y_{n+1}))\le F\Big(\psi(M(x_n,x_{n+1},x_{n+1})),\varphi(M(x_n,x_{n+1},x_{n+1}))\Big),\tag{2.3}$$

where

$$\begin{aligned}&M(x_n,x_{n+1},x_{n+1})\\&=\frac{1}{a+b+c+d+e+h}(aG(gx_n,gx_{n+1},gx_{n+1})+bG(gx_n,fx_n,fx_n)+\\&cG(gx_{n+1},fx_{n+1},fx_{n+1})+dG(gx_{n+1},fx_{n+1},fx_{n+1})+\\&e[\alpha G(fx_n,fx_n,gx_{n+1})+(1-\alpha)G(fx_{n+1},fx_{n+1},gx_{n+1})]+\\&h[\beta G(gx_n,fx_n,fx_n)+(1-\beta)G(gx_{n+1},fx_{n+1},fx_{n+1})]).\end{aligned}$$

Therefore,

$$\begin{aligned}M(x_n,x_{n+1},x_{n+1})\;=\;&\frac{1}{a+b+c+d+e+h}(aG(y_{n-1},y_n,y_n)+bG(y_{n-1},y_n,y_n)+\\&cG(y_n,y_{n+1},y_{n+1})+dG(y_n,y_{n+1},y_{n+1})+\\&e[\alpha G(y_n,y_n,y_n)+(1-\alpha)G(y_{n+1},y_{n+1},y_n)]+\\&h[\beta G(y_{n-1},y_n,y_n)+(1-\beta)G(y_n,y_{n+1},y_{n+1})]).\end{aligned}$$

From (2.3), we obtain

$$\begin{aligned}
&\psi(G(y_n, y_{n+1}, y_{n+1}))\\
&\leq F\Big(\psi(\frac{a+b+h\beta}{a+b+e\alpha+h\beta}G(y_{n-1}, y_n, y_n)), \varphi(\frac{a+b+h\beta}{a+b+e\alpha+h\beta}G(y_{n-1}, y_n, y_n))\Big)\\
&\leq \psi(\frac{a+b+h\beta}{a+b+e\alpha+h\beta}G(y_{n-1}, y_n, y_n)).
\end{aligned}$$

By properties of ψ, we have

$$\begin{aligned}
G(y_n, y_{n+1}, y_{n+1}) &\leq \frac{a+b+h\beta}{a+b+e\alpha+h\beta}G(y_{n-1}, y_n, y_n)\\
&\vdots\\
&\leq \Big(\frac{a+b+h\beta}{a+b+e\alpha+h\beta}\Big)^n G(y_0, y_1, y_1)\\
&\to 0 \ \text{ as } n \to \infty,
\end{aligned}$$

because that

$$k = \frac{a+b+h\beta}{a+b+e\alpha+h\beta} < 1.$$

Using Lemma 1.14, we shall prove that $\{y_n\}$ is a G-Cauchy sequence. Suppose this is not true. Then by Lemma 1.14, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$,

$$G(y_{m_k}, y_{n_k}, y_{n_k}), G(y_{m_k}, y_{n_k-1}, y_{n_k-1}), G(y_{m_k+1}, y_{n_k}, y_{n_k}), G(y_{n_k-1}, y_{m_k+1}, y_{m_k+1}).$$

Putting $x = x_{m_k}, y = x_{m_k}$ and $z = x_{n_k}$ in (2.1), we conclude that

$$\begin{aligned}
&\psi(G(y_{m_k}, y_{m_k}, y_{n_k})) = \psi(G(fx_{m_k}, fx_{m_k}, fx_{n_k}))\\
&\leq F\Big(\psi(M(x_{m_k}, x_{m_k}, x_{n_k})), \varphi(M(x_{m_k}, x_{m_k}, x_{n_k}))\Big),
\end{aligned}$$

where

$$\begin{aligned}
&M(x_{m_k}, x_{m_k}, x_{n_k})\\
&= \frac{1}{a+b+c+d+e+h}(aG(gx_{m_k}, gx_{m_k}, gx_{n_k}) + bG(gx_{m_k}, fx_{m_k}, fx_{m_k}) +\\
&cG(gx_{m_k}, fx_{m_k}, fx_{m_k}) + dG(gx_{n_k}, fx_{n_k}, fx_{n_k}) +\\
&e[\alpha G(fx_{m_k}, fx_{m_k}, gx_{m_k}) + (1-\alpha)G(fx_{m_k}, fx_{m_k}, gx_{n_k})] +\\
&h[\beta G(gx_{m_k}, fx_{m_k}, fx_{m_k}) + (1-\beta)G(gx_{m_k}, fx_{m_k}, fx_{m_k})]).
\end{aligned}$$

We have

$$\begin{aligned}
&M(x_{m_k}, x_{m_k}, x_{n_k})\\
&= \frac{1}{a+b+c+d+e+h}(aG(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}) + bG(y_{m_k-1}, y_{m_k}, y_{m_k})+\\
&cG(y_{m_k-1}, y_{m_k}, y_{m_k}) + dG(y_{n_k-1}, y_{n_k}, y_{n_k})+\\
&e[\alpha G(y_{m_k}, y_{m_k}, y_{m_k-1}) + (1-\alpha)G(y_{m_k}, y_{m_k}, y_{n_k-1})]+\\
&h[\beta G(y_{m_k-1}, y_{m_k}, y_{m_k}) + (1-\beta)G(y_{m_k-1}, y_{m_k}, y_{m_k})])\\
&\to \frac{a+e(1-\alpha)}{a+b+c+d+e+h}\varepsilon.
\end{aligned}$$

Consequently, since (F, ψ, φ) is monotone, we have

$$\begin{aligned}
\psi(\varepsilon) &\leq F\Big(\psi(\frac{a+e(1-\alpha)}{a+b+c+d+e+h}\varepsilon), \varphi(\frac{a+e(1-\alpha)}{a+b+c+d+e+h}\varepsilon)\Big)\\
&\leq F\Big(\psi(\varepsilon), \varphi(\varepsilon)\Big) \leq \psi(\varepsilon).
\end{aligned}$$

Thus, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, that is, $\varepsilon = 0$, which is a contradiction. Therefore, $\{y_n\}$ is a G-Cauchy sequence. Suppose that $g(X)$ is a G-complete subspace of X, so there exists a point $q \in g(X)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = q$. Also, we can find a point $p \in X$ such that $gp = q$. Now, we prove that $fp = q$. By (2.1), we have

$$\psi(G(fx_n, fp, fp)) \leq F\Big(\psi(M(x_n, p, p)), \varphi(M(x_n, p, p))\Big), \tag{2.4}$$

where

$$\begin{aligned}
&M(x_n, p, p)\\
&= \frac{1}{a+b+c+d+e+h}(aG(gx_n, gp, gp) + bG(gx_n, fx_n, fx_n)+\\
&cG(gp, fp, fp) + dG(gp, fp, fp) + e[\alpha G(fx_n, fx_n, gp) + (1-\alpha)G(fp, fp, gp)]+\\
&h[\beta G(gx_n, fx_n, fx_n) + (1-\beta)G(gp, fp, fp)])\\
&= \frac{1}{a+b+c+d+e+h}(aG(gx_n, q, q) + bG(gx_n, fx_n, fx_n) + cG(q, fp, fp)+\\
&dG(q, fp, fp) + e[\alpha G(fx_n, fx_n, q) + (1-\alpha)G(fp, fp, q)]+\\
&h[\beta G(gx_n, fx_n, fx_n) + (1-\beta)G(q, fp, fp)]).
\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, p, p) = \frac{c+d+e(1-\alpha)+h(1-\beta)}{a+b+c+d+e+h}G(q, fp, fp). \tag{2.5}$$

From (2.4), letting $n \to \infty$, one gets using the fact that (F, ψ, φ) is monotone

$$\begin{aligned}\psi(G(q, fp, fp)) &\le F\Big(\psi(\frac{c+d+e(1-\alpha)+h(1-\beta)}{a+b+c+d+e+h}G(q, fp, fp)),\\ &\quad \varphi(\frac{c+d+e(1-\alpha)+h(1-\beta)}{a+b+c+d+e+h}G(q, fp, fp))\Big)\\ &\le F\Big(\psi(G(q, fp, fp)), \varphi(G(q, fp, fp))\Big)\\ &\le \psi(G(q, fp, fp)).\end{aligned}$$

Therefore, $\psi(G(q, fp, fp)) = 0$ or $\varphi(G(q, fp, fp)) = 0$. Hence, $G(q, fp, fp) = 0$, and then $fp = q$. Then q is a point of coincidence of f and g. From Lemma 2.2, q is the unique point of coincidence. Moreover, if f and g are weakly compatible, then by Lemma 2.1, q is the unique common fixed point of f and g. The proof is similar in the case that $f(X)$ is G-complete. □

If we put $a = b = c = d = h = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5. *Let (X, G) be a G-metric space and $f, g : (X, G) \to (X, G)$ be two mappings such that*

$$\psi(G(fx, fy, fz)) \le$$
$$F\Big(\psi(\alpha G(fx, fx, gy) + (1-\alpha)G(fy, fy, gz)), \varphi(\alpha G(fx, fx, gy) + (1-\alpha)G(fy, fy, gz))\Big),$$

for all $x, y, z \in X$, where $\psi, \varphi \in \Psi$ and $F \in \mathcal{C}$ such that (F, ψ, φ) is monotone and $0 < \alpha < 1$. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a G-complete G-metric subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

If we put $g = I$ in Theorem 2.4, where I is the identity mapping, we have the following corollary.

Corollary 2.6. *Let (X, G) be a complete G-metric space. Let f be a self-mapping on X satisfying*

$$\psi(G(fx, fy, fz)) \le F\Big(\psi(N(x, y, z)), \varphi(N(x, y, z))\Big),$$

for all $x, y, z \in X$, where $\psi, \varphi \in \Psi$ and $F \in \mathcal{C}$ such that (F, ψ, φ) is monotone and

$$N(x,y,z) = \frac{1}{a+b+c+d+e+h}(aG(x,y,z) + bG(x,fx,fx) + cG(y,fy,fy) + dG(z,fz,fz) + e[\alpha G(fx,fx,y) + (1-\alpha)G(fy,fy,z)] + h[\beta G(x,fx,fx) + (1-\beta)G(y,fy,fy)]),$$

for all $x, y, z \in X$, where $a, b, c, d, h, \beta \geq 0$, $e > 0$ and $0 < \alpha < 1$. Then f has a unique fixed point.

We also have

Corollary 2.7. *Let (X, G) be a complete G-metric space. Let f be a self-mapping on X satisfying*

$$\psi(G(fx,fy,fz)) \leq F\Big(\psi(\alpha G(fx,fx,y) + (1-\alpha)G(fy,fy,z)), \varphi(\alpha G(fx,fx,y) + (1-\alpha)G(fy,fy,z))\Big),$$

for all $x, y, z \in X$, where $\psi, \varphi \in \Psi$ and $F \in \mathcal{C}$ such that (F, ψ, φ) is monotone and $0 < \alpha < 1$. Then f has a unique fixed point.

The following example illustrates Theorem 2.4.

Example 2.8. *Let $X = \{1, 2, 3\}$ be the set endowed with the G-metric defined by*

(x,y,z)	$G(x,y,z)$
$(1,1,1), (2,2,2), (3,3,3),$	0
$(1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1), (1,1,3), (1,3,1), (3,1,1),$	2
$(1,3,3), (3,1,3), (3,3,1), (2,2,3), (2,3,2), (3,2,2), (2,3,3), (3,2,3), (3,3,2),$	8
$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1),$	8

Let f and g be two self-mappings on X defined by

x	$f(x)$	$g(x)$
1	*1*	*1*
2	*2*	*3*
3	*1*	*2*

It is clear that $f(X) \subseteq g(X)$ and the pair of mappings (f, g) is weakly compatible. Note that 1 is the only coincidence point of f and g and $fg1 =$

$f1 = 1 = g1 = gf1$. Suppose that $a = b = c = h = 0$, $d = e = 1$ and $\alpha = \beta = \frac{1}{2}$. Define $F : [0,\infty)^2 \to \mathbb{R}$ by $F(s,t) = s - t$. Consider $\psi, \varphi : [0,\infty) \to [0,\infty)$ as $\psi(t) = t$ and $\varphi(t) = \frac{1}{2}t$. Then (F,ψ,φ) is monotone. Also, $0 \le M(x,y,z) \le 16$ for all $x,y,z \in X$.
Case 1: If

$$\begin{aligned}(x,y,z) \in &\{(1,1,1),(2,2,2),(3,3,3),(1,1,3),(1,3,1),\\ &(3,1,1),(1,3,3),(3,1,3),(3,3,1)\},\end{aligned}$$

we have $G(fx,fy,fz) = 0$ and $0 \le M(x,y,z) \le 16$. Therefore,

$$\psi(G(fx,fy,fz)) = 0 \le F\Big(\psi(M(x,y,z)),\varphi(M(x,y,z))\Big),$$

so, (2.1) holds.
Case 2: If

$$\begin{aligned}(x,y,z) \in \{&(1,1,2),(1,2,1),(2,1,1),(1,2,2),(2,1,2),(2,2,1),\\ &(2,2,3),(2,3,2),(3,2,2),(2,3,3),(3,2,3),(3,3,2),\\ &(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\},\end{aligned}$$

we have $G(fx,fy,fz) = 2$. So

$$\psi(G(fx,fy,fz)) = 2 \le F\Big(\psi(M(x,y,z)),\varphi(M(x,y,z))\Big).$$

Then condition (2.1) is satisfied for all $x,y,z \in X$. Hence, all the hypotheses of Theorem 2.4 are satisfied and 1 is the unique common fixed point of f and g.

3 Mappings with Property Q

Theorem 3.1. *Under the condition of Theorem 2.4 with $a,b,c,d,h \ge 0$, $e > 0$, $0 < \alpha,\beta < 1$ and if f and g are commuting, then f and g have the property Q.*

Proof. From Theorem 2.4, $Fix(f) \cap Fix(g) \ne \emptyset$. Therefore $Fix(f^n) \cap Fix(g^n) \ne \emptyset$ for each positive integer n. We shall prove the converse. For this, let n be a fixed positive integer greater than 1. Suppose that $p \in Fix(f^n) \cap Fix(g^n)$. We claim that $p \in Fix(f) \cap Fix(g)$. If $n = 1$, nothing to prove. From now on, suppose that $n \ge 2$.

For any positive integers i, j, k, l, r, s satisfying $1 \leq i, j, k, l, r, s \leq n-1$ we have

$$\begin{aligned}\psi(G(f^i g^j p, f^k g^l p, f^r g^s p)) &= \psi(G(f(f^{i-1} g^j p), f(f^{k-1} g^l p), f(f^{r-1} g^s p))) \\ &\leq F\Big(\psi(M(f^{i-1} g^j p, f^{k-1} g^l p, f^{r-1} g^s p)), \varphi(M(f^{i-1} g^j p, f^{k-1} g^l p, f^{r-1} g^s p))\Big),\end{aligned}$$

where

$$\begin{aligned}&M(f^{i-1} g^j p, f^{k-1} g^l p, f^{r-1} g^s p) = \\ &\frac{1}{a+b+c+d+e+h}(aG(f^{i-1} g^{j+1} p, f^{k-1} g^{l+1} p, f^{r-1} g^{s+1} p)+ \\ &bG(f^{i-1} g^{j+1} p, f^i g^j p, f^i g^j p) + cG(f^{k-1} g^{l+1} p, f^k g^l p, f^k g^l p)+ \\ &\quad dG(f^{r-1} g^{s+1} p, f^r g^s p, f^r g^s p)+ \\ &e[\alpha G(f^i g^j p, f^i g^j p, f^{k-1} g^{l+1} p) + (1-\alpha) G(f^k g^l p, f^k g^l p, f^{r-1} g^{s+1} p)]+ \\ &h[\beta G(f^{i-1} g^{j+1} p, f^i g^j p, f^i g^j p) + (1-\beta) G(f^{k-1} g^{l+1} p, f^k g^l p, f^k g^l p)]).\end{aligned}$$

Define

$$\delta = \max_{1 \leq i,j,k,l,r,s \leq n} \{G(f^i g^j p, f^k g^l p, f^r g^s p)\}.$$

Applying the $\max_{1 \leq i,j,k,l,r,s \leq n}$, we get

$$\psi(\delta) \leq F\Big(\psi(\delta), \varphi(\delta)\Big) \leq \psi(\delta). \tag{3.6}$$

Hence, $\delta = 0$. In particular, if we consider the cases

$$(i = 1, j = l = k = s = r = n) \quad \text{and} \quad (j = 1, i = l = k = s = r = n),$$

we conclude that

$$G(fg^n p, f^n g^n p, f^n g^n p) = 0 \quad \text{and} \quad G(f^n gp, f^n g^n p, f^n g^n p) = 0.$$

Since $p \in Fix(f^n) \cap Fix(g^n)$, we get

$$G(fp, p, p) = 0 \quad \text{and} \quad G(gp, p, p) = 0,$$

that is, $fp = gp = p$, i.e., $p \in Fix(f) \cap Fix(g)$. So f and g have the property Q.

□

Example 3.2. *Let $X = \mathbb{R}$ be endowed with the G-metric*

$$G(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Take f and g defined by $f(x) = 2$ and $g(x) = 2x - 2$. Consider $\alpha, \beta \in (0, 1)$, $a, b, c, d, h \geq 0$ and $e > 0$. Define $F : [0, \infty)^2 \to \mathbb{R}$ by $F(s, t) = ws$ with $0 < w < 1$. Consider $\psi, \varphi : [0, \infty) \to [0, \infty)$ as $\psi(t) = \frac{1}{w}t$ and $\varphi(t) = \frac{1}{2w}t$. Then (F, ψ, φ) is monotone. We have $f(X) \subseteq g(X)$. Also (f, g) is commuting, and so it is weakly compatible. Moreover

$$\begin{aligned} 0 = \psi(G(fx, fy, fz)) &\leq M(x, y, z) \\ &= w(\frac{1}{w}M(x, y, z)) \\ &= F\Big(\psi(M(x, y, z)), \varphi(M(x, y, z))\Big), \quad \forall x, y, z \in X. \end{aligned}$$

Therefore, condition (2.1) holds for all $x, y, z \in X$. All hypotheses of Theorem 2.4 are satisfied, and 2 is the unique common fixed point of the mappings f and g.
In addition $p = 2 \in Fix(f^n) \cap Fix(g^n)$ for any integer n, and so f and g have property Q.

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