International Journal of Mathematics and
Computer Science, **14**(2019), no. 3, 591–599

LWPC Quasigroups

George Olufemi[1], Adeniran Olusola[2], Olaleru Johnson[1]

[1]Department of Mathematics
University of Lagos
Akoka, Yaba Lagos, Nigeria

[2]Department of Mathematical Sciences
Federal University of Agriculture
Abeokuta, Nigeria

email: oogeorge@unilag.edu.ng, adeniranoj@funaab.edu.ng,
jolaleru@unilag.edu.ng

(Received September 3, 2018, Revised October 18, 2018,
Revised April 11, 2019, Accepted April 17, 2019)

Abstract

Left weak inverse property, power associative conjugacy closed (LWPC) quasigroups are quasigroups that satisfy the identity $(xy \cdot x) \cdot xz = x((yx \cdot x)z)$. Right weak inverse property, power associative conjugacy closed (RWPC) quasigroups satisfy the mirror identity $zx \cdot (x \cdot yx) = (z(x \cdot xy))x$. It is shown that LWPC quasigroups are flexible, satisfy the inverse property and are alternative. It is shown that an LWPC quasigroup is a loop.

1 Introduction

In [1], Belousov asked the question of determining which identity forces a quasigroup to be a loop.
It is known that an associative quasigroup is a loop, and is therefore a group.
Kunen [3] showed that any of the four Moufang identities imply that a quasigroup is a loop.

Key words and phrases: LWPC, RWPC quasigroups, WIP PACC loops.
AMS (MOS) Subject Classifications: 20N05
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

Phillips in [6], axiomatized the variety of weak inverse property, power associative conjugacy closed (WIP PACC) loop by the LWPC and the RWPC identities. He showed that LWPC implies LCC and RWPC implies RCC. He showed that in LWPC, $x^\rho = x^\lambda$, which by [4] guarantees power associativity. He offered an otter output showing that LWPC and RWPC together imply weak inverse property. He asked the question " Is a quasigroup that satisfies the LWPC and RWPC identities a loop?" In this paper, we investigate LWPC quasigroup and provide an answer to this question.

2 Definitions and Results

A groupoid $(Q, \cdot)$ is a non-empty set Q with a binary operation $(\cdot)$.
A quasigroup $(Q, \cdot)$ is a groupoid such that for all $a,\ b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x,\ y \in Q$.
Let a be a fixed element in Q. A right translation by a in Q is the mapping $R(a) : Q \to Q$ defined by

$$xR(a) = xa.$$

Symmetrically, the left translation by a in Q is the mapping $L(a) : Q \to Q$ defined by

$$xL(a) = ax.$$

A loop is a quasugroup with a unique identity element. For a general overview on the theory of quasigroups and loops, the reader can consult [5].
A quasigroup $(Q, \cdot)$ has the left inverse property if there exists a bijection $J_\lambda : x \to x^\lambda$ such that

$$x^\lambda \cdot xy = y.$$

A quasigroup $(Q, \cdot)$ has the right inverse property if there exists a bijection $J_\rho : x \to x^\rho$ such that

$$xy \cdot y^\rho = x.$$

A quasigroup Q is an inverse property quasigroup if it satisfies the left inverse and the right inverse properties.
A quasigroup Q has the weak inverse property (WIP) if it satisfies

$$x(yx)^\rho = y^\rho \ or \ (xy)^\lambda x = y^\lambda \tag{1}$$

The right (resp. middle, left) nucleus of a quasigroup Q consists of all elements $a \in Q$ such that $xy \cdot a = x \cdot ya$ $(resp.\ xa \cdot y = x \cdot ay,\ a \cdot xy = ax \cdot y)$. It is denoted by $N_\rho(Q)$ $(resp.\ N_\mu(Q),\ N_\lambda(Q)\ )$. The nucleus is the intersection of the three nuclei, i.e., $N(Q) = N_\rho(Q) \cap N_\mu(Q) \cap N_\lambda$. In a loop $(Q, \cdot)$ with identity element e, the left inverse of element $x \in Q$ is the element $x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e,$$

while the right inverse element of $x \in Q$ is the element $x^\rho \in Q$ such that

$$x \cdot x^\rho = e.$$

If $\rho = \lambda$, we write $x^\rho = x^\lambda = x^{-1}$.

A quasigroup Q is said to be power associative if every singleton is associative.

A quasigroup Q is said to satisfy the flexible law if $x \cdot yx = xy \cdot x$ and Q satisfies the right alternative property (RAP) if $y \cdot xx = yx \cdot x$. The left alternative property is the mirror identity of RAP.

A quasigroup Q is conjugacy closed if it satisfies the following equations:

$$(xy)/x \ \cdot xz = x(yz) \qquad (LCC)$$

$$zx \ \cdot x \setminus (yx) = (zy)x \qquad (RCC)$$

This definition is due to Goodaire and Robinson [2].

An autotopism of a quasigroup Q is a triple $(U,\ V,\ W)$ of bijections with the property that

$$xU \cdot yV = (xy)W \qquad (2)$$

for all $x,\ y \in Q$.

Proposition 2.1. *Let Q be a LWPC quasigroup. Then*
(i) $x \cdot xx = xx \cdot x$, (ii) $x^\rho = x^\lambda$.

Proof. (i) Put $y = z = 1$ in the LWPC identity.
(ii) Put $z = 1$ and $y = x^\lambda$ in the LWPC identity, we have

$$(xx^\lambda \cdot x)x = xx,$$
$$xx^\lambda = I,$$
$$x^\lambda = x^\rho.$$

□

Remark 2.2. *The two identities in Proposition 2.1 are equivalent. [4].*

The following lemma restates the definition of LWPC quasigroup in terms of translation maps.

Lemma 2.3. *Let Q be an LWPC quasigroup, then the following identities hold:*
(i) $L(x)R(x)R(x) = R(x)R(x)L(x)$,
(ii) $R^{-1}(x)L(x)R(x) = R(x)L(x)R^{-1}(x)$,
(iii) $L(x) = R^{-1}(x)R^{-1}(x)L(x)R(x)R(x) = R(x)R(x)L(x)R^{-1}(x)R^{-1}(x)$.

Theorem 2.4. *Let Q be an LWPC quasigroup, Then: (ii) $(Q, \cdot)$ is flexible, (ii) Q is an inverse property quasigroup, (iii) Q is left and right alternative.*

Proof. (i) Set $y = (y/x)/x$ in the LWPC identity.

$$(x(y/x)/x)x \cdot xz = x \cdot yz.$$

Put $z = x$,

$$(x(y/x)/x)x \cdot xx = x \cdot yx,$$
$$yR^{-1}(x)R^{-1}(x)L(x)R(x)R(x^2) = yR(x)L(x),$$

since Q is power associative, we have

$$yR^{-1}(x)R^{-1}(x)L(x)R(x)R(x)R(x) = yR(x)L(x),$$

By Lemma 2.3 (iii),

$$yL(x)R(x) = yR(x)L(x)$$

or

$$xy \cdot x = x \cdot yx$$

(ii) The LWPC identity is equivalent to

$$x^\lambda((xy \cdot x) \cdot xz) = (yx \cdot x)z,$$

setting $z = 1$ and using Lemma 2.3 (i) on the L. H. S,

$$x^\lambda(x(yx \cdot x) = yx \cdot x,$$

setting $yx \cdot x = z$, we have $x^\lambda \cdot xz = z$.
Thus, Q satisfies the left inverse property.
Now set $z = x\backslash z$ and then $z = 1$ in the LWPC identity,

$$xy \cdot x = x((yx \cdot x)x\backslash 1),$$
$$x \cdot yx = x(yx \cdot x)x^\rho)$$

setting $yx = z$ and using the left cancellation law, we have

$$zx \cdot x^\rho = z.$$

Thus, Q has the right inverse property and is therefore an inverse property quasigroup.
(iii) Setting $y = 1$ and $z = x \setminus z$ in the LWPC identity,

$$xx \cdot z = x(xx \cdot x\backslash z), \tag{3}$$

and use power associative property on the right hand side of equation (3) to obtain

$$x \cdot xz = xx \cdot z. \tag{4}$$

The right alternative law is obtained by taking the inverses in (4) and replacing x^{-1} and y^{-1} by x and y respectively. □

Corollary 2.5. *The following is true in an LWPC quasigroup Q:*
(i) $R(x) = L^{-1}(x)R(x)L(x)$ from Theorem 2.4 (i),
(ii) $R^{-1}(x) = R(x^{-1})$ and $L^{-1}(x) = L(x^{-1})$, from Theorem 2.4 (ii),
(iii) $JL(x)J = R^{-1}(x)$ and $JR(x)J = L^{-1}(x)$, where $xJ = x^{-1}$, from Theorem 2.4 (ii).

Theorem 2.6. *Let $(Q, \cdot)$ be a quasigroup, then Q is an LWPC quasigroup if and only if $A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$ is an autotopisms of Q.*

Proof. Let Q be an LWPC quasigroup. Set $y = (y/x)/x$ in the LWPC identity,

$$(x((y/x)/x))x \cdot xz = x(yz),$$
$$yR^{-2}(x)L(x)R(x) \cdot zL(x) = (yz)L(x).$$

Thus,

$$A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$$

is an autotopism of Q.
Conversely, if $A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$ is an autotopism of Q. Applying this autotopism to the product yz, to obtain

$$yR^{-2}(x)L(x)R(x) \cdot zL(x) = (yz)L(x)$$

put $y = yx \cdot x$, to obtain the LWPC identity. □

Definition 2.7. *A quasigroup is a Moufang quasigroup if and only if it satisfies the identity*

$$(xy \cdot x)z = x(y \cdot xz) \tag{5}$$

The following lemma is well known.

Lemma 2.8. *Let* $(Q, \cdot)$ *be a quasigroup for* $a \in Q$,

$$a \in N_\lambda \Leftrightarrow (L(a),\ I,\ L(a)) \in\ AtpQ,$$

$$a \in N_\rho \Leftrightarrow (I,\ R(a),\ R(a)) \in\ AtpQ,$$

$$a \in N_\mu \Leftrightarrow (R^{-1}(a),\ L(a),\ I) \in\ AtpQ.$$

Lemma 2.9. *[7] In a weak inverse property quasigroup* $N_\lambda = N_\rho = N_\mu$.

Theorem 2.10. *Let* Q *be an LWPC quasigroup. Then* Q *is Moufang if and only if it is nuclear square.*

Proof. Let Q be an LWPC quasigroup and assume Q is Moufang. Now set $z = x\backslash z$ in the LWPC identity, we have

$$(xy \cdot x)z = x((yx \cdot x)x\backslash z) \tag{6}$$

Since Q is Moufang, we have from equations (5) and (6) $x(y \cdot xz) = x((yx \cdot x)x\backslash z)$ and with $z = x\backslash z$ again, we obtain

$$\begin{aligned} x(yz) &= x((yx \cdot x)x\backslash(x\backslash z)), \\ yz &= (yx \cdot x)x\backslash(x\backslash z). \end{aligned} \tag{7}$$

From equation (7), we have the autotopism $(R(x^2),\ L^{-2}(x),\ I)$ and by power associativity in Q, this autotopism becomes $(R^2(x),\ L^{-2}(x),\ I)$, since the set of autotopisms of a quasigroup forms a group, we have the inverse autotopism $(R^{-2}(x),\ L^2(x),\ I)$ and by Lemmas 2.8 and 2.9, we have $x^2 \in N$.
Conversely, let Q be an LWPC quasigroup and assume squares are nuclear, then $(R^{-2}(x)L(x)R(x),\ L(x),\ L(x))$ and $(R(x^2),\ L^{-2}(x),\ I)$ are autotopisms of$Q$. Since the set of autotopisms of a quasigroup forms a group, we have

$$(R(x^2),\ L^{-2}(x),\ I)(R^{-2}(x)L(x)R(x),\ L(x),\ L(x)) = (L(x)R(x),\ L^{-1}(x),\ L(x)),$$

by applying this autotopism to the product yz gives $(xy \cdot x)z = x(y \cdot xz)$. $\square$

Lemma 2.11. *[7]. If* (U, V, W) *is an autotopism of a weak inverse property quasigroup, then* $(\rho W \lambda, U, \rho V \lambda)$ *is also an autopism of* Q.

Theorem 2.12. *Let* Q *be a quasigroup satisfying the LWPC identity, then* Q *is an RWPC quasigroup.*

Proof. Since an LWPC quasigroup is power associative, has the weak inverse property, and is left conjugacy closed quasigroup, we only need to show that the Q satisfies the RCC identity.
Now apply Lemma 2.11 to the autotopism $A(x)$ to obtain

$$(R^{-1}(x),\ R^{-2}(x)L(x)R(x),\ R^{-1}(x)),$$

taking the inverse of this autotopism and use Theorem 2.3 (i), we obtain

$$(R(x),\ R(x)L^{-1}(x),\ R(x)).$$

Applying this autotopism to the product yz gives the RCC identity. □

Phillips (2006) asked the question " Is a quasigroup that satisfies the LWPC and RWPC a loop?"
We now provide an answer to this question.

Theorem 2.13. *(i) Every quasigroup satisfying the LWPC identity has a right identity element;*
(ii) Every quasigroup satisfying the RWPC has a left identity element.

Proof. (i) Let Q be a quasigroup satisfying the LWPC identity and let x be a fixed element in Q. Define $e \in Q$ as $xe = e$. Now for $w \in Q$, there are $y,\ t \in Q$ such that $w = tx$ and $t = yx$.
Then

$$\begin{aligned} x \cdot we &= x(tx \cdot e) \\ &= x((yx \cdot x)e) \\ &= (xy \cdot x) \cdot xe \\ &= (xy \cdot x)x \\ &= (x \cdot yx)x \\ &= xt \cdot x \\ &= x \cdot tx \\ &= xw, \end{aligned}$$

and by the left cancellation law, we have $we = w$.
Thus, e is a right identity element.
(ii) Similarly, let Q be an RWPC quasigroup and let e be an element in Q

such that $ex = e$. Let $xy = t \quad$ and $\quad xt = w$.
Then

$$\begin{aligned} ew \cdot x &= (e \cdot xt)x \\ &= (e(x \cdot xy))x \\ &= ex(x \cdot yx) \\ &= x(x \cdot yx) \\ &= x(xy \cdot x) \\ &= x \cdot tx \\ &= xt \cdot x \\ &= wx, \end{aligned}$$

and by right cancellation, we see that $ew = w$ and therefore e is a left identity element. □

Corollary 2.14. *Let Q be a quasigroup satisfying the LWPC and the RWPC identities, then Q is a loop.*

An LWPC quasigroup Q of order 8

.	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

This quasigroup is a loop and also satisfies the RWPC identity.

References

[1] V. D. Belousov, *Foundations of the theory of quasigroups and loops.* Izdat, Nauka, Moscow, 1967.

[2] E. G. Goodaire, D. A. Robinson, *A class of loops which are isomorphic to all loop isotopes,* Canad. J. Math., **34,** (1982), 662–672.

[3] K. Kunen, *Moufang quasigroups,* J. Algebra, **183,** (1996), 231–234.

[4] K. Kunen, *The structure of conjugacy closed loops,* Trans. Amer. Math. Soc., **352,** (2000), 2889–2911.

[5] H. O. Pflugfelder, *Quasigroups and loops: Introduction,* Sigma Series in Pure Math., Vol. **7,** Heldermann Verlag, Berlin, 1990.

[6] J. D. Phillips, *A short basis for the variety of WIP PACC- loops,* Quasigroups and Related Systems, **14,** (2006), 259–271.

[7] J. M. Osborn, *Loops with the weak inverse property,* Pacific J. Math., **10,** (1960), 295–304.