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Peakon and Solitary Wave Solutions for The Modified Fornberg-Whitham Equation using Simplest Equation Method

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Abstract

In this paper, the simplest equation method is implemented to construct exact traveling-wave solutions to the modified Fornberg-Whitham equation. With the help of Mathematica symbolic computation software, some new classes of explicit exact solutions are derived. Graphical representations of some obtained solutions are displayed. The method demonstrates the applicability, reliability and efficiency in treating a wide range of nonlinear evolution equations.

1 Introduction

Nonlinear partial differential equations (NPDEs) model a wide range of real-life phenomena arise in physics, biology, chemistry, and several other fields. Constructing exact traveling wave solutions for such equations is an ongoing research. Explicit exact solutions help scientists to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by NPDEs. Over the last four decades, many significant methods for obtaining exact solutions of NPDEs have been proposed.

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Among these Algorithms, the Hirota bilinear method [1], Backlund transformation method [2, 3], Adomian decomposition method and its variants [4, 5, 6, 7, 8, 9], Jacobi expansion method [10], sine-cosine method [11], variational iteration method [12, 13], He's homotopy perturbation and analysis methods [14], $exp\,(-\Phi\,(\xi))$-expansion method [15], tanh-function method [16, 17], F-expansion method [18], differential and reduced differential transform methods [19, 20, 21, 22], G'/G-expansion method [23], residual power series method [24, 25, 26, 27, 28, 29], and the simplest equation method [30, 31].

This paper is basically motivated to apply the simplest equation method (SEM), developed by Kudryashov [30, 31], to formally construct explicit peakon and solitary wave solutions for the modified Fornberg-Whitham (mFW) equation

$$u_t - u_{xxt} + u_x + u^2 u_x = u\,u_{xxx} + 3u_x u_{xx}. \tag{1.1}$$

The mFW equation was investigated by He et al [32]. In that work, many peakons and solitary waves were obtained by applying the bifurcation theory and phase portraits analysis methods. Variety of peakon, periodic, and solitary exact solutions using the unified approach [33], the factorization technique combined with the method of complete discrimination system for polynomial [34], the $\tan(\phi/2)$-expansion and $\tanh(\phi/2)$-expansion methods [35], were also constructed. The reduced differential transform method is applied to tackle the Cauchy version of Eq.(1.1) [36]. The time-fractional mFW equation was processed by wavelet method [37].

Recently, the SEM and its variants have been successfully applied for finding exact solutions of nonlinear Schrodinger equation [38], coupled Konno-Oono equations [39], nonlinear telegraph equation [40], generalized Davey-Stewartson and Zakharov equations [41], dispersive water wave equations [42], generalized (2+1)-dimensional nonlinear evolution equations [43] and the (4+1)-dimensional Fokas equation [44]. See also the included bibliography therein.

The remainder of this paper is organized as follows. Section 2 deals with the general description of the used method. In section 3, the proposed scheme is applied to find some exact traveling-wave solutions of the (1+1)-dimensional mFW equation Eq.(1.1). Discussion and conclusions, with graphical representations of some obtained results, are included in section 4.

2 The Simplest Equation Method

In this part, the major steps of the SEM are described. As a generic example, consider the (1+1)-dimensional nonlinear evolution equation (NLEE) of the form

$$P\left(v, \partial_t v, \partial_x v, \partial_t^2 v, \partial_t \partial_x v, \partial_x^2 v, \ldots\right) = 0, \tag{2.2}$$

where P is a polynomial in $v\,(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The simplest equation algorithm includes the following steps:

Step 1. Define the wave variable $\xi = x \pm \alpha\, t$, where α is the wave speed, to reduce the NLEE Eq.(2.2) to a nonlinear ordinary differential equation (NODE) in $v\,(\xi)$ and its total derivatives $v'\,(\xi)\,, v''\,(\xi)\,, \ldots$ as

$$F\,(v, v', v'', \ldots) = 0, \tag{2.3}$$

Integrate Eq.(2.3) as many times as is applicable and set the constants of integration to be zeros.

Step 2. The SEM assumes the solution of Eq.(2.3) as a polynomial in $\phi\,(\xi)$ as follows:

$$v\,(\xi) = \sum_{i=0}^{m} A_i\, \phi^i\,(\xi),\ A_m \neq 0, \tag{2.4}$$

where $A_i, i = 0, 1, \ldots, m$ are parameters to be determined, $\phi\,(\xi)$ is the function that satisfies some ODE, known as the simplest equation. The simplest equation must be of lesser order than the resulted completely-integrated form of Eq.(2.3), and its solutions can be expressed by elementary functions. In current work, the equation of Bernoulli

$$\phi'\,(\xi) = \lambda\, \phi\,(\xi) + \mu\, \phi^2\,(\xi)\,, \tag{2.5}$$

is considered as a simplest equation. The solution function admits the following rational and rational-exponential forms:

$$\phi\,(\xi) = \frac{1}{\mu\,(\xi_0 - \xi)}, \tag{2.6}$$

when $\lambda = 0$, and

$$\phi\,(\xi) = \frac{\lambda\, e^{\lambda(\xi+\xi_0)}}{1 - \mu\, e^{\lambda(\xi+\xi_0)}}, \tag{2.7}$$

when $\lambda > 0$ and $\mu < 0$, and

$$\phi(\xi) = -\frac{\lambda\, e^{\lambda(\xi+\xi_0)}}{1+\mu\, e^{\lambda(\xi+\xi_0)}}, \tag{2.8}$$

when $\lambda < 0$ and $\mu > 0$, and ξ_0 is an arbitrary constant.

Step 3. Determine the positive integer m by considering the homogeneous balance between the highest order derivative and the linear term of highest order in the resulting equation of Eq.(2.3).

Step 4. Along Eq.(2.5), substitute Eq.(2.4) into Eq.(2.3) to be converted into polynomial in $\phi(\xi)$. Collect all coefficients with the same power of $\phi(\xi)$ and let each coefficient to be vanished. A system of algebraic equations involving the coefficients μ, λ, and A_i, $i = 0, 1, \ldots, m$, would be obtained.

Step 5. Solve the resulting system in Step 4 to get the values of μ, λ, and A_i. Substituting the obtained values into Eq.(2.4), along with general solutions of Eq.(2.5) in Eqs.(2.6)-(2.8), would complete determining the exact travelling-wave solutions of Eq.(2.2).

3 The Modified Fornberg-Whitham Equation

In the current section, an exertion of the simplest equation scheme to construct exact analytic solutions to the (1+1)-dimensional mFW equation Eq.(1.1) is considered.

Utilizing the traveling-wave variable $\xi = x + \alpha\, t$, Eq.(1.1) is carried into the following NODEs:

$$(\alpha+1)\; u' - \alpha\, u^{(3)} + u^2 u' - \left(u\, u^{(3)} + u'u''\right) - 2\, u'u'' = 0. \tag{3.9}$$

Integrating with respect to ξ once and equating the integration constants to zero give

$$(\alpha+1)\; u - \alpha\, u'' + \frac{1}{3}u^3 - u\, u'' - (u')^2 = 0, \tag{3.10}$$

Making balance between u'' and u^3, Eq.(3.10) owns the formal solution

$$u(\xi) = A_0 + A_1\, \phi(\xi) + A_2\, \phi(\xi)^2. \tag{3.11}$$

Substituting Eq.(3.11) into Eq.(3.10), making use of the Bernoulli equation Eq.(2.5), and equating the coefficients of $\phi^i, i = 0, 1, \ldots, 6$ to zero, result the following set of simultaneous algebraic equations in terms of A_0, A_1, A_2, λ, μ and α:

$$A_2^2\left(\frac{1}{3}A_2 - 10\mu^2\right) = 0, \tag{3.12}$$

$$A_2^2\left(A_1 - 18\lambda\mu\right) - 12A_1A_2\mu^2 = 0, \tag{3.13}$$

$$A_1^2\left(A_2 - 3\mu^2\right) + A_2^2\left(A_0 - 8\lambda^2\right) - A_2\left(21A_1\lambda\mu - 6A_0\mu^2 - 6_2\alpha\mu^2\right) = 0, \tag{3.14}$$

$$A_1\left(A_2\left(2A_0 - 9\lambda^2\right) + A_1\left(\frac{1}{3}A_1 - 5\lambda\mu\right) - 2\mu^2\left(A_0 + 2\alpha\right)\right) - 10A_2\lambda\mu\left(A_0 + \alpha\right) = 0, \tag{3.15}$$

$$A_1^2\left(A_0 - 2\lambda^2\right) + A_2\left(A_0^2 - 4A_0\lambda^2 + \alpha\left(1 - 4\lambda^2\right) + 1\right) - 3A_1\lambda\mu\left(A_0 + \alpha\right) = 0, \tag{3.16}$$

$$A_1\left(A_0^2 - A_0\lambda^2 - \alpha\left(\lambda^2 - 1\right) + 1\right) = 0, \tag{3.17}$$

$$A_0\left(\frac{1}{3}A_0^2 + (1 + \alpha)\right) = 0. \tag{3.18}$$

Solving Eq.(3.12) gives that $A_2 = 30\mu^2$. For $A_2 = 0$, a trivial solution would be obtained, so this case will be neglected. Eq.(3.18) implies $A_0 = 0, \pm\sqrt{-3\left(\alpha + 1\right)}$. The algebraic system Eqs.(3.13)-(3.17) is insolvable when $A_0 = \pm\sqrt{-3\left(\alpha + 1\right)}$. For $A_0 = 0$, four classes of solitary wave solutions are obtained with $A_1 = 30\,\lambda\,\mu$, $\alpha = -10\,\lambda^2$, $\lambda = \pm\sqrt{0.5 \pm \sqrt{0.15}}$, and μ, with $\lambda\,\mu < 0$, is an arbitrary.

Consequently, the traveling-wave solutions of Eq.(1.1) have the following forms:

$$u_1\left(x, t\right) = \frac{30\mathrm{e}^{\lambda\left(x - 10\ t\,\lambda^2 + \xi_0\right)}\lambda^2\mu}{\left(1 - \mathrm{e}^{\lambda\left(x - 10\ t\,\lambda^2 + \xi_0\right)}\mu\right)^2}, \tag{3.19}$$

and

$$u_2\left(x, t\right) = -\frac{30\,\mathrm{e}^{\lambda\left(x - 10\ t\,\lambda^2 + \xi_0\right)}\lambda^2\mu}{\left(1 + \mathrm{e}^{\lambda\left(x - 10\ t\,\lambda^2 + \xi_0\right)}\mu\right)^2}. \tag{3.20}$$

4 Discussion and Conclusion

The SEM is effectively employed to construct analytic traveling wave solutions for the (1+1)-dimensional modified Fornberg-Whitham equation. Four different classes of soliton solutions, namely the peakon and solitary, are formally derived. Figure 1 shows the behavior of $u\left(x, t\right)$ in Eq.(3.19) in 3D and

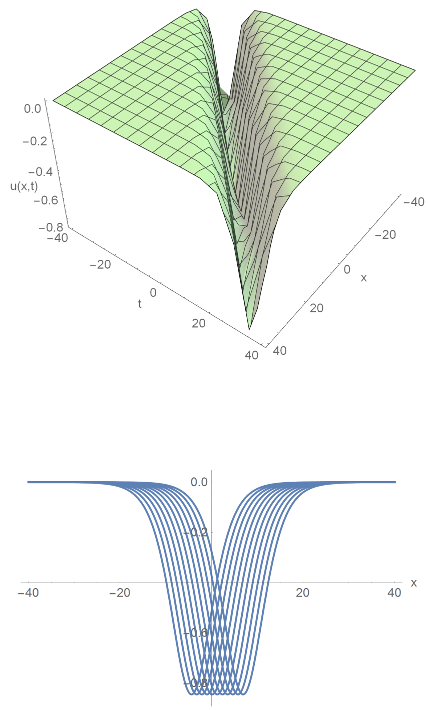

Figure 1: 3D and 2D peakon soliton profiles of $u\left(x,t\right)$ in Eq.(3.19) with $\lambda=-\mu=\sqrt{0.5-\sqrt{0.15}}$, and $\xi_0=2$ for $-40\leqslant x,t\leqslant 40$.

the corresponding x-curves in 2D. In the same manner, Figure 2 represents the profile of solitary wave solution obtained in Eq.(3.20). In the other two cases, i.e. for $\lambda=-\mu=-\sqrt{0.5\pm\sqrt{0.15}}$, same profile solutions can be obtained as in the others with translating.

These solutions have different physical structures depend on the real parameters. The presented method is effective, simple, and applicable to handle many other NPDEs. With aid of the Mathematica Software Package 11, we have checked our solutions by putting them back into the original equations.

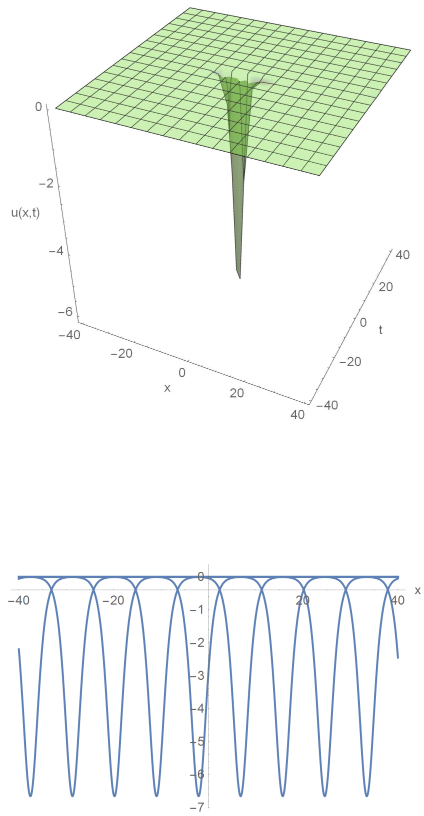

Figure 2: 3D and 2D solitary wave profiles of $u(x,t)$ in Eq.(3.20) with $\lambda = -\mu = -\sqrt{0.5+\sqrt{0.15}}$, and $\xi_0 = 2$ for $-40 \leqslant x, t \leqslant 40$.

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