International Journal of Mathematics and
Computer Science, **14**(2019), no. 3, 547–556

A Subclass of Meromorphic bi-univalent functions

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(Received February 27, 2019, Revised March 30, 2019,
Accepted April 2, 2019)

Abstract

In this article, two subclasses of meromorphic bi-univalent functions are introduced and discussed. The first three coefficient bounds are obtained and proved.

1 Introduction

Let A denote the class of all analytic functions which are defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and which can be written in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}. \tag{1.1}$$

Let S be the class of all normalized analytic univalent functions in A. An analytic function f is subordinate to an analytic function g, written $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f = (g(w))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$. A function

Key words and phrases: Meromorphic functions, Bi-univalent function, k-symmetric star-like functions with respect to points, Differential operator $\mho_\lambda$.
AMS (MOS) Subject Classifications: 30C45, 30C50, 30C80.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

$f \in S_s^*(\alpha)$ is strongly starlike of order $\alpha (0 < \alpha \leq 1)$ if $\left| arg(\frac{zf'(z)}{f(z)}) \right| < \frac{\alpha\pi}{2}$, for $z \in \mathcal{U}$. Alternatively, $f \in S_s^*(\alpha)$ if $\frac{zf'(z)}{f(z)} \prec (\frac{1+z}{1-z})^\alpha$. A subclass $S_s(\varphi)$ of starlike functions with respect to symmetric points satisfies the condition $\frac{zf'(z)}{f(z)-f(-z)} \prec \varphi(z)$, for all $z \in \mathcal{U}$. A subclass $K_s(\varphi)$ of S is a convex function with respect to symmetric points satisfies the condition $\frac{(zf'(z))'}{(f(z)-f(-z))'} \prec \varphi(z)$, for all $z \in \mathcal{U}$. The Koebe one-quarter theorem [1] ensures that the image of $\mathcal{U}$ under every univalent function $f \in A$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying

$$\begin{aligned} &f^{-1}(f(z)) = z, \ \ (z \in \mathcal{U}) \ \ and \\ &f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4) \ \ where \\ &f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - ..., \end{aligned}$$

A function $f \in A$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathcal{U}$. In 1967, Lewin [2] introduced the class σ of bi-univalent functions and he proved the second coefficient for a function f in (1.1). Brannan and Clunie [9] proved Lewin's result by proving $|a_2| \leq \sqrt{2}$. Many authors [3, 4, 5, 6, 7, 8, 9, 10] investigated classes of bi-univalent analytic functions and found estimates on the coefficients for functions in these subclasses. Let Σ denote the class of meromorphic univalent analytic functions f that are defined in the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$ and have the Laurent series expansion

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \quad b_n \in \mathbb{C} \tag{1.2}$$

Estimates of meromorphic univalent functions coefficients were widely investigated. Schiffer [11] obtained that $|b_2| \leq \frac{2}{3}$ for a meromorphic function $g \in \Sigma$. If $b_0 = 0$, Duren [2] derived the inequality $|b_n| \leq \frac{2}{n+1}$ for the coefficient of meromorphic univalent functions with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. Springer [18] showed that $|B_3| \leq 1$ and $|B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$ and conjectured that $|B_{2n-1}| \leq \frac{(2n1)}{n!(n-1)!}$, $(n = 1, 2, ...)$. In this article, ΣS^* and ΣK denote the subclasses of meromorphic starlike and convex functions in S respectively. The concept of meromorphic bi-univalent funtions is given and studied by Suzeini et. al. [16] in the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$, since $h \in \Sigma$ of the form (1.2) is univalent, then it has an inverse h^{-1} satisfying

$$h^{-1}(h(z)) = z \ \ (for \ \ z \in \Delta)$$

and

$$h(h^{-1}(w)) = w \quad (for \;\; 0 < M < |w| < \infty)$$

The inverse function h^{-1} can be written in the form

$$h^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \quad 0 < M < |w| < \infty \tag{1.3}$$

So that a function $g \in \Sigma$ is meoromorphic bi-univalent if $g^{-1} \in \Sigma$, and the class of all meromorphic functions is denoted by $\Sigma_{\mathfrak{B}}$. By a simple calculations we get:

$$h^{-1}(w) = w - b_0 - \frac{b_1}{w^1} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + ... \tag{1.4}$$

For a positive integer k, let $\varepsilon = e^{\frac{2\pi i}{k}}$ then a k-symmetric function f_k with respect to points is given by

$$f_k(z) = \frac{1}{k} \sum_{\upsilon=0}^{k-1} \varepsilon^{\upsilon} f(\varepsilon^{\upsilon} z) = z + \sum_{i=1}^{\infty} \frac{a_{k(i-1)}}{z^{k(i-1)}}$$

It is obvious that $f_1(z) = f(z)$ and $f_2(z) = \frac{f(z)-f(-z)}{2}$. In this paper, the differential operator $\mho_\lambda$ of a function f is defined as

$$\mho_\lambda f_k(z) = (1-\lambda) f_k(z) + \lambda z f_k'(z)$$

$$= z + (1-\lambda) b_0 + \sum_{n=2}^{\infty} [1 - (k(i-1)+1)\lambda] \frac{b_{k(i-1)}}{z^{k(i-1)}}$$

and

$$\mho_\lambda (zf'(z)) = (1-\lambda) zf'(z) + \lambda z (zf'(z))' = z + \sum_{i=2}^{\infty} k(i-1)[\lambda(k(i-1)+1)-1] \frac{b_{k(i-1)}}{z^{k(i-1)}}$$

for $0 \leq \lambda < \frac{1}{n+1}$.

Suppose that $\mathcal{P}$ is the class of all functions with positive real part.

Lemma 1.1. *[13] If $s(z) \in \mathcal{P}$ for $z \in \Delta$ such that $Re(p(z) > 0$, then $|s_i| \leq 2$, where*

$$s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + ...$$

Definition 1.2. *A function $f \in A$ is in the class $\Sigma\mho_\lambda(k, \varphi)$ for $0 \le \lambda \le 1$ if*

$$\left(\frac{z(\mho_\lambda(zf_k'(z))}{\mho_\lambda(f_k(z))}\right) \prec \varphi(z).$$

The class $\Sigma\mho_\lambda(k, \varphi)$ is a generalization of various subclasses of strongly starlike and convex functions with respect to symmetric points. It is easy to note that if $\lambda = 0$, then $\Sigma\mho_0(k, \frac{1+Az}{1+Bz}) \equiv S_s^k(A, B)$ due to [14], if $\lambda = 0$, then $\Sigma\mho_0(2, \varphi) \equiv S_s^*(\varphi)$ due to [15]. Also, if $\lambda = 1$, then $\Sigma\mho_1(2, \varphi) \equiv C_s(\varphi)$ due to [15].

One objective of this paper is to introduce a new subclass of a function f in the class $\Sigma\mho_\lambda(\varphi)$ and determine estimates on the coefficients $|b_1|$ and $|b_2|$.

2 The main results

A subclass of meromorphic bi-univalent functions is introduced.

Definition 2.1. *A function $f \in \Sigma$ given in (1.2) is in the class $\Sigma\mho_\lambda(k, \alpha)$ for $0 \le \lambda \le 1$ and $0 < \alpha \le 1$ if it satisfies*

$$\left|arg\left(\frac{z(\mho_\lambda(zf_k'(z))}{\mho_\lambda(f_k(z))}\right)\right| < \frac{\alpha\pi}{2}, \ for\, z \in \Delta \tag{2.5}$$

$$and \quad \left|arg\left(\frac{w(\mho_\lambda(wh_k'(w))}{\mho_\lambda(h_k(w))}\right)\right| < \frac{\alpha\pi}{2}, \ for\, w \in \Delta \tag{2.6}$$

where

$$h(w) = w - b_0 - \frac{b_{k1}}{w^1} - \frac{b_{k2} + b_0 b_{k1}}{w^2} - \frac{b_{k3} + 2b_0 b_{k1} + b_0^2 b_{k1} + b_{k1}^2}{w^3} + ...$$

After some calculations, the coefficient bounds for a function $f \in \Sigma\mho_\lambda(k, \alpha)$ are estimated.

Theorem 2.2. *Consider a function $f \in \Sigma\mho_\lambda(k, \alpha)$ for $k \in N$, $0 \le \lambda < 1$ and $0 < \alpha \le 1$. Then*

$$|b_0| \le \frac{2\,\alpha}{(1-\lambda)} \tag{2.7}$$

$$|b_{k1}| \leq \frac{2\alpha^2}{|\lambda(k+3)-2|} \tag{2.8}$$

and

$$|b_{k2}| \leq \frac{\frac{4}{3}\alpha^3 - \frac{14}{3}\alpha^2 + 2\alpha}{|(2k+7)\lambda - 3|} \tag{2.9}$$

Proof. From the above definition it follows that

$$\frac{z(\mho_\lambda(zf_k'(z))}{\mho_\lambda(f_k(z))} = (s(z))^\alpha, \ \ for \ \ z \in \Delta, \tag{2.10}$$

$$and \qquad \frac{w(\mho_\lambda(wh_k'(w))}{\mho_\lambda(h_k(w))} = (r(w))^\alpha \ \ for \ \ w \in \Delta, \tag{2.11}$$

where $s(z), \ r(w) \in \mathcal{P}$ and have the forms

$$s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + ... \tag{2.12}$$

$$r(z) = 1 + \frac{r_1}{w} + \frac{r_2}{w^2} + ... \tag{2.13}$$

and

$$[(s(z)]^\alpha = 1 + \frac{\alpha s_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)s_1^2 + \alpha s_2}{z^2} + ...$$

$$[(r(w)]^\alpha = 1 + \frac{\alpha r_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)r_1^2 + \alpha r_2}{w^2} + ...$$

From (2.3) and (2.4), it follows that

$$(\lambda - 1)b_0 = \alpha\, s_1 \tag{2.14}$$

$$(\lambda(k+3)-2)\, b_{k1} = (\alpha s_2 - \frac{1}{2}\alpha(\alpha-1)s_1^2) \tag{2.15}$$

$$((2k+7)\lambda - 3)b_{k2} = \frac{1}{6}\alpha(\alpha-1)(\alpha-2)s_1^3 + \alpha(\alpha-1)s_1 s_2$$

$$+\alpha s_3 + \alpha s_1 b_{k1}(1-(k+1)\lambda) - \frac{1}{2}\alpha(\alpha-1)(\lambda-1)s_2 b_0 \tag{2.16}$$

and

$$-(\lambda-1)b_0 = \alpha\, r_1 \tag{2.17}$$

$$-(\lambda(k+3)-2)\, b_{k1} = (\alpha r_2 - \frac{1}{2}\alpha(\alpha-1)r_1^2) \tag{2.18}$$

$$-((2k+7)\lambda-3)b_{k2} = \frac{1}{6}\alpha(\alpha-1)(\alpha-2)r_1^3 + \alpha(\alpha-1)r_1 r_2$$

$$+\alpha r_3 + \alpha r_1 b_{k1}(1-(k+1)\lambda) - \frac{1}{2}\alpha(\alpha-1)(\lambda-1)r_2 b_0 \tag{2.19}$$

From (2.5) and (2.8), we have

$$s_1 = r_1 \tag{2.20}$$

and

$$b_0^2 = \frac{\alpha^2(s_1^2 + r_1^2)}{2(1-\lambda)^2} \tag{2.21}$$

By Lemma 1.1, it follows that

$$|b_0| \leq \frac{2\,\alpha}{(1-\lambda)} \tag{2.22}$$

From (2.6) and (2.9) it follows that

$$b_{k1}^2 = \frac{[\alpha s_2 - \frac{1}{2}\alpha(\alpha-1)s_1^2]^2 + [\alpha r_2 - \frac{1}{2}\alpha(\alpha-1)r_1^2]^2}{2(\lambda(k+3)-2)^2} \tag{2.23}$$

By Lemma 1.1, we have

$$|b_{k1}| \leq \frac{2\alpha^2}{|\lambda(k+3)-2|} \tag{2.24}$$

In addition, from (2.7) and (2.10), it follows that

$$b_{k2}^2 = \frac{[\frac{1}{6}\alpha(\alpha-1)(\alpha-2)s_1^3+\alpha(\alpha-1)s_1s_2+\alpha s_3+\alpha s_1 b_{k1}(1-(k+1)\lambda)-\frac{1}{2}\alpha(\alpha-1)(\lambda-1)s_2b_0]^2}{2((2k+7)\lambda-3)^2}$$
$$+\frac{[\frac{1}{6}\alpha(\alpha-1)(\alpha-2)r_1^3+\alpha(\alpha-1)r_1r_2+\alpha r_3+\alpha r_1 b_{k1}(1-(k+1)\lambda)-\frac{1}{2}\alpha(\alpha-1)(\lambda-1)r_2b_0]^2}{2((2k+7)\lambda-3)^2}$$

By Lemma 1.1, we have

$$|b_{k2}| \leq \frac{\frac{4}{3}\alpha^3 - \frac{14}{3}\alpha^2 + 2\alpha}{|(2k+7)\lambda-3|} \tag{2.25}$$

This completes the proof of the theorem.□

Definition 2.3. *A function $f \in \Sigma$ given in (1.2) is in the class $\Sigma\mho_\lambda(k, \beta)$ for $0 \leq \lambda < 1$ and $0 \leq \beta < 1$ if*

$$\left(\frac{z(\mho_\lambda(zf_k'(z))}{\mho_\lambda(f_k(z))}\right) > \beta, \ \ for\, z \in \mathcal{U} \tag{2.26}$$

and

$$\left(\frac{w(\mho_\lambda(wf_k'(w))}{\mho_\lambda(f_k(w))}\right) > \beta, \ \ for\, w \in \mathcal{U}.$$

Theorem 2.4. *Let a function $f \in \Sigma\mho_\lambda(k,\beta)$ for $k \in N$, $0 \leq \lambda < 1$ and $0 \leq \beta < 1$. Then*

$$|b_0| \leq \frac{2(1-\beta)}{(1-\lambda)} \tag{2.27}$$

$$|b_{k1}| \leq \frac{2(1-\beta)|1-2\beta|}{|\lambda(k+3)-2|} \tag{2.28}$$

and

$$|b_{k2}| \leq \frac{2(1-\beta)[(1-2\beta)(1-\lambda)+2(1-\beta)+1]}{|(2k+7)\lambda-3|} \tag{2.29}$$

Proof. From Definition 2.3 it follows that

$$\frac{z(\mathcal{D}_\lambda f(z))}{(1-\lambda)\,K_0(f(z))+\lambda\, z\, K_1(f(z))} = \beta + (1-\beta)s(z) \tag{2.30}$$

and

$$\frac{w(\mathcal{D}_\lambda g(w))}{(1-\lambda)\,K_0(g(w))+\lambda\, w\, K_1(g(w))} = \beta + (1-\beta)s(z) \tag{2.31}$$

where $s(z)$ and $r(z)$ are given in (2.11) and (2.12) respectively. From (2.26) and (2.27), it follows that

$$(\lambda-1)b_0=(1-\beta)s_1 \tag{2.32}$$

$$(\lambda(k+3)-2)\,b_{k1}=(1-\beta)(s_2-(1-\beta)s_1^2) \tag{2.33}$$

$$((2k+7)\lambda-3)b_{k2}=(1-\beta)[s_1(1-(k+1)\lambda)b_{k1}+(1-\lambda)s_1b_0+s_3] \tag{2.34}$$

and

$$-(\lambda-1)b_0=(1-\beta)\,r_1 \tag{2.35}$$

$$-(\lambda(k+3)-2)\,b_{k1}=(1-\beta)(r_2-(1-\beta)r_1^2) \tag{2.36}$$

$$-((2k+7)\lambda-3)b_{k2}=(1-\beta)[r_1(1-(k+1)\lambda)b_{k1}+(1-\lambda)r_1b_0+r_3] \tag{2.37}$$

From (2.28) and (2.31), we have

$$s_1=-r_1 \quad \text{and} \quad 2b_0^2=\frac{(1-\beta)^2(s_1^2+r_1^2)}{(1-\lambda)^2} \tag{2.38}$$

By Lemma 1.1, we get

$$|b_0|\leq\frac{2(1-\beta)}{(1-\lambda)} \tag{2.39}$$

From (2.29) and (2.32), we have

$$b_{k1}^2=\frac{(1-\beta)^2[(s_2-(1-\beta)s_1^2)^2+(r_2-(1-\beta)r_1^2)^2]}{2(\lambda(k+3)-2)^2} \tag{2.40}$$

By Lemma 1.1, we deduce that

$$|b_{k1}|\leq\frac{2(1-\beta)|1-2\beta|}{|\lambda(k+3)-2|} \tag{2.41}$$

It follows form (2.30), (2.33) and by using Lemma 1.1 that

$$|b_{k2}|\leq\frac{2(1-\beta)[(1-2\beta)(1-\lambda)+2(1-\beta)+1]}{|(2k+7)\lambda-3|} \tag{2.42}$$

This completes the proof of the theorem.□

3 Acknowledgment

The author thanks the referees for their useful suggestions to improve this paper.

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