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Fuzzy UP-ideals and fuzzy UP-subalgebras of UP-algebras in term of level subsets

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Abstract

In this paper, we study the notions of fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras in term of upper t-(strong) level subsets and lower t-(strong) level subsets of a fuzzy set, and some properties and results are discussed.

1 Introduction and Preliminaries

The concept of a fuzzy subset of a set was first considered by Zadeh [34] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [34], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2000, Jun et al. [15] studied fuzzy $\mathcal{I}$-ideals in IS-algebras. Roh et al. [24] gave a relation between a fuzzy $\mathscr{I}$-ideal and a fuzzy associative $\mathscr{I}$-ideal, and investigated some related properties. In 2001, Lele et al. [19] studied fuzzy ideals and weak ideals in BCK-algebras. Jun

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[11] introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provided some appropriate examples. In 2002, Jun [12] studied the fuzzification of sub-implicative ideals in BCI-algebras, and investigated some related properties. Jun et al. [16] studied fuzzy B-algebras in B-algebras. Yonglin and Xiaohong [33] introduced the notion of fuzzy a-ideals in BCI-algebras, and investigated its properties. In 2004, Jun [13] introduced the concept of (α, β)-fuzzy ideals of BCK/BCI-algebras. In 2005, Akram and Dar [1] introduced the notions of T-fuzzy subalgebras and T-fuzzy H-ideals in BCI-algebras and investigated some of their properties. Jun [17] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Jun [14] introduced the notion of fuzzy subalgebras with thresholds of BCK/BCI-algebras, and obtained its characterizations. Akram and Dar [2] introduced the notion of fuzzy ideals in K-algebras. In 2008, Saeid and Jun [26] introduced the concept of anti fuzzy subalgebras of BCK/BCI-algebras by using the notion of anti fuzzy points. In 2009, Saeid and Rezvani [27] introduced the notion of fuzzy BF-algebra and fuzzy topological BF-algebras. Hadipour [7] introduced the notion of (α, β)-fuzzy BF-algebras. In 2010, Song et al. [31] introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. In 2011, Mostafa et al. [22] introduced the notion of fuzzy KU-ideals in KU-algebras. In 2012, Mostafa et al. [21] introduced the notion of Q-ideals and fuzzy Q-ideals in Q-algebras, and studied their properties. In 2014, Yamini and Kailasavalli [32] introduced the notion of B-ideals and fuzzy B-ideals in B-algebras. Rajam and Chandramouleeswaran [23] introduced the notion of L-fuzzy β-subalgebras on β-algebras, and investigated some of their properties.

Iampan [8] now introduced a new algebraic structure, called a UP-algebra. The notions of fuzzy subalgebras and fuzzy ideals play an important role in studying the many logical algebras. Somjanta et al.[30] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras, and investigated some of its properties. In this paper, we study the notions of fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras in term of upper t-(strong) level subsets and lower t-(strong) level subsets of a fuzzy set, and some properties and results are discussed.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. *[8] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following* *axioms: for any* $x, y, z \in A$*,*

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$*,*

(UP-2) $0 \cdot x = x$*,*

(UP-3) $x \cdot 0 = 0$*, and*

(UP-4) $x \cdot y = y \cdot x = 0$ *implies* $x = y$*.*

Example 1.2. [29] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*, and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 1.3. [6] Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by for all $x, y \in \mathbb{N}$,

$$x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

and

$$x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

Example 1.4. [18] Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	2	5
2	0	1	0	3	1	5
3	0	1	2	0	4	5
4	0	0	0	3	0	5
5	0	0	2	0	2	0

Then $(A, \cdot, 0)$ is a UP-algebra.

For more examples of UP-algebras, see [3, 9, 28, 29].

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.5. *[8, 9] In a UP-algebra A, the following properties hold: for any $a, x, y, z \in A$,*

1. $x \cdot x = 0$,
2. $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
3. $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
4. $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
5. $x \cdot (y \cdot x) = 0$,
6. $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$,
7. $x \cdot (y \cdot y) = 0$,
8. $(x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0$,
9. $(((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0$,
10. $((x \cdot y) \cdot z) \cdot (y \cdot z) = 0$,
11. $x \cdot y = 0$ *implies* $x \cdot (z \cdot y) = 0$,
12. $((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0$, *and*
13. $((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0$.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A [8] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

Definition 1.6. *[8] A subset B of A is called a UP-ideal of A if it satisfies the following properties:*

1. *the constant 0 of A is in B, and*
2. *for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.*

Definition 1.7. *[8] A subset S of A is called a UP-subalgebra of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.*

Proposition 1.8. *[8] A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.9. *[34] A fuzzy set in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f\colon X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line.*

Definition 1.10. *[30] A fuzzy set f in A is called a fuzzy UP-ideal of A if it satisfies the following properties: for any $x, y, z \in A$,*

1. *$f(0) \geq f(x)$, and*
2. *$f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.*

Definition 1.11. *[30] A fuzzy set f in A is called a fuzzy UP-subalgebra of A if for any $x, y \in A$,*

$$f(x \cdot y) \geq \min\{f(x), f(y)\}.$$

Definition 1.12. *[30] Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the complement of f in A.*

Remark 1.13. *For all fuzzy set f in A, we have $f = \overline{\overline{f}}$.*

Definition 1.14. *[30] Let f be a fuzzy set in A. For any $t \in [0, 1]$, the sets*

$$U(f; t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f; t) = \{x \in A \mid f(x) > t\}$$

are called an upper t-level subset and an upper t-strong level subset of f, respectively. The sets

$$L(f; t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f; t) = \{x \in A \mid f(x) < t\}$$

are called a lower t-level subset and a lower t-strong level subset of f, respectively.

Definition 1.15. *[10] Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then fuzzy set β in Y defined by*

$$\beta(y) = \begin{cases} \sup\{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the preimage of β under f.

Definition 1.16. *[25] A fuzzy set f in A has* sup *property if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \sup\{f(t)\}_{t \in T}$.*

Definition 1.17. *[5] A fuzzy relation on a nonempty set X is an arbitrary function $f \colon X \times X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.*

Definition 1.18. *[21] Let f and g be fuzzy sets in nonempty sets A and B, respectively. The Cartesian product of f and g is $f \times g \colon A \times B \to [0,1]$ defined by*

$$(f \times g)(x,y) = \max\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

The dot product of f and g is $f \cdot g \colon A \times B \to [0,1]$ defined by

$$(f \cdot g)(x,y) = \min\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

Definition 1.19. *[21] If f is a fuzzy set in a nonempty set X, the strongest fuzzy relation on X is $\mu_f \colon X \times X \to [0,1]$ defined by $\mu_f(x,y) = \max\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\mu_f(x,y) = \max\{f(x), f(y)\} \in [0,1]$. Hence, μ_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \times f = \mu_f$.*

Definition 1.20. *If f is a fuzzy set in a nonempty set X, the weakness fuzzy relation on X is $\beta_f \colon X \times X \to [0,1]$ defined by $\beta_f(x,y) = \min\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\beta_f(x,y) = \min\{f(x), f(y)\} \in [0,1]$. Hence, β_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \cdot f = \beta_f$.*

Definition 1.21. *[20] Let X and Y be any two nonempty sets and let $f \colon X \to Y$ be any function. A fuzzy set μ in X is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.*

Definition 1.22. *[8] Let $(A, \cdot, 0)$ and $(A', \cdot', 0')$ be UP-algebras. A mapping f from A to A' is called a UP-homomorphism if*

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f \colon A \to A'$ is called a

1. *UP-endomorphism of A if $A' = A$,*
2. *UP-epimorphism if f is surjective,*
3. *UP-monomorphism if f is injective, and*

4. *UP-isomorphism if f is bijective. Moreover, we say A is UP-isomorphic to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.*

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the image of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the image of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the inverse image of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the kernel of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\mathrm{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.23. *[8] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

1. *$f(0_A) = 0_B$, and*
2. *for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$.*

2 Main Results

In this section, we study fuzzy UP-ideals and fuzzy UP-subalgebras of UP-algebras.

Theorem 2.1. *Every fuzzy UP-ideal of A is a fuzzy UP-subalgebra of A.*

Proof. Let f be a fuzzy UP-ideal of A. Let $x, y \in A$. Then

$$\begin{aligned}
f(x \cdot y) &\geq \min\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.10 2)}\\
&= \min\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.5 1)}\\
&= \min\{f(0), f(y)\} && \text{((UP-3))}\\
&= f(y) && \text{(Definition 1.10 1)}\\
&\geq \min\{f(x), f(y)\}.
\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra of A. $\square$

Lemma 2.2. *Let f be a fuzzy UP-ideal of A. If the inequality $x \leq y \cdot z$ holds in A for all $x, y, z \in A$, then $f(z) \geq \min\{f(x), f(y)\}$ for all $x, y, z \in A$.*

Proof. Assume $x \leq y \cdot z$ for all $x, y, z \in A$. Then $x \cdot (y \cdot z) = 0$. By Definition 1.102, we have

$$f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}. \tag{2.1}$$

By (2.1) and (UP-2), let $x = 0$, so

$$f(z) = f(0 \cdot z) \geq \min\{f(0 \cdot (x \cdot z)), f(x)\} = \min\{f(x \cdot z), f(x)\}. \tag{2.2}$$

By (2.1) and Definition 1.10 1, we have

$$f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} = \min\{f(0), f(y)\} = f(y). \tag{2.3}$$

By (2.2) and (2.3), we have

$$f(z) \geq \min\{f(x \cdot z), f(x)\} \geq \min\{f(y), f(x)\} = \min\{f(x), f(y)\}.$$

□

Lemma 2.3. *If f is a fuzzy UP-ideal of A and if $x, y \in A$ is such that $x \leq y$ in A, then $f(x) \leq f(y)$.*

Proof. Let $x, y \in A$ be such that $x \leq y$ in A. Then $x \cdot y = 0$. Thus

$$\begin{aligned}
f(y) &= f(0 \cdot y) && \text{((UP-2))}\\
&\geq \min\{f(0 \cdot (x \cdot y)), f(x)\} && \text{(Definition 1.10 2)}\\
&= \min\{f(0 \cdot 0), f(x)\}\\
&= \min\{f(0), f(x)\} && \text{((UP-2))}\\
&= f(x). && \text{(Definition 1.10 1)}
\end{aligned}$$

□

We can easily prove the lemma.

Lemma 2.4. *Let f be a fuzzy set in A. For any $t \in [0, 1]$, the following properties hold:*

1. $L(f; t) = U(\overline{f}; 1 - t)$,
2. $L^-(f; t) = U^+(\overline{f}; 1 - t)$,
3. $U(f; t) = L(\overline{f}; 1 - t)$, *and*
4. $U^+(f; t) = L^-(\overline{f}; 1 - t)$.

Theorem 2.5. *[30] Let f be a fuzzy set in A. Then the following statements hold:*

1. *f is a fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U(f;t) \neq \emptyset$ implies $U(f;t)$ is a UP-ideal of A,*

2. *f is a fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U^+(f;t) \neq \emptyset$ implies $U^+(f;t)$ is a UP-ideal of A,*

3. *$\overline{f}$ is a fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L(f;t) \neq \emptyset$ implies $L(f;t)$ is a UP-ideal of A, and*

4. *$\overline{f}$ is a fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L^-(f;t) \neq \emptyset$ implies $L^-(f;t)$ is a UP-ideal of A.*

Theorem 2.6. *[30] Let f be a fuzzy set in A. Then the following statements hold:*

1. *f is a fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U(f;t) \neq \emptyset$ implies $U(f;t)$ is a UP-subalgebra of A,*

2. *f is a fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U^+(f;t) \neq \emptyset$ implies $U^+(f;t)$ is a UP-subalgebra of A,*

3. *$\overline{f}$ is a fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L(f;t) \neq \emptyset$ implies $L(f;t)$ is a UP-subalgebra of A, and*

4. *$\overline{f}$ is a fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L^-(f;t) \neq \emptyset$ implies $L^-(f;t)$ is a UP-subalgebra of A.*

Proposition 2.7. *If f is a fuzzy UP-ideal of A, then $f(x \cdot (x \cdot y)) \geq f(y)$ for all $x, y \in A$.*

Proof. Let $x, y \in A$. Then

$$\begin{aligned}
f(x \cdot (x \cdot y)) &\geq \min\{f(x \cdot ((x \cdot y) \cdot (x \cdot y))), f(x \cdot y)\} && \text{(Definition 1.10 2)}\\
&= \min\{f(x \cdot 0), f(x \cdot y)\} && \text{(Proposition 1.5 1)}\\
&= \min\{f(0), f(x \cdot y)\} && \text{((UP-3))}\\
&= f(x \cdot y) && \text{(Definition 1.10 1)}\\
&\geq \min\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.10 2)}\\
&= \min\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.5 1)}\\
&= \min\{f(0), f(y)\} && \text{((UP-3))}\\
&= f(y). && \text{(Definition 1.10 1)}
\end{aligned}$$

□

Corollary 2.8. *Let f be a fuzzy set in A. Then the following statements hold:*

1. *if f is a fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-ideal of A, and*

2. *if $\overline{f}$ is a fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-ideal of A.*

Proof. 1 Assume that f is a fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.5 1, we have $U(f;t)$ is a UP-ideal of A.

2 Assume that $\overline{f}$ is a fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.5 3, we have $L(f;t)$ is a UP-ideal of A. □

Corollary 2.9. *Let f be a fuzzy set in A. Then the following statements hold:*

1. *if f is a fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-subalgebra of A, and*

2. *if $\overline{f}$ is a fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-subalgebra of A.*

Proof. 1 Assume that f is a fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.61, we have $U(f;t)$ is a UP-subalgebra of A.

2 Assume that $\overline{f}$ is a fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.63, we have $L(f;t)$ is a UP-subalgebra of A. □

Corollary 2.10. *Let I be a UP-ideal of A. Then the following statements hold:*

1. *for any $k \in (0,1]$, there exists a fuzzy UP-ideal g of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$, and*
2. *for any $k \in [0,1)$, there exists a fuzzy UP-ideal f of A such that $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.*

Proof. 1 Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $L(f;t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f;t)$. Then $f(x) \leq t < k$, so $f(x) \neq k$. Thus $f(x) = 0$, so $x \in I$. That is $L(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 0 \leq t$, so $x \in L(f;t)$. That is $I \subseteq L(f;t)$. Hence, $L(f;t) = I$ for all $t < k$.

Case 2: To show that $L(f;t) = A$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 0 < t & \text{if } x \in I, \\ k \leq t & \text{if } x \notin I, \end{cases}$$

so $x \in L(f;t)$. That is $A \subseteq L(f;t)$. Hence, $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.53 that $\overline{f}$ is a fuzzy UP-ideal of A. By Remark 1.13, we have $L(\overline{\overline{f}};t) = L(f;t) = I$ for all $t < k$ and $L(\overline{\overline{f}};t) = L(f;t) = A$ for all $t \geq k$. Putting $\overline{f} = g$. Then g is a fuzzy UP-ideal of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$.

2 Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $U(f;t) = I$ for all $t > k$, let $t \in [0,1]$ be such that $t > k$. Let $x \in U(f;t)$. Then $f(x) \geq t > k$, so $f(x) \neq k$. Thus $f(x) = 1$, so $x \in I$. That is $U(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 1 \geq t$, so $x \in U(f;t)$. That is $I \subseteq U(f;t)$. Hence, $U(f;t) = I$ for all $t > k$.

Case 2: To show that $U(f;t) = A$ for all $t \leq k$, let $t \in [0,1]$ be such that $t \leq k$. Clearly, $U(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 1 > t & \text{if } x \in I, \\ k \geq t & \text{if } x \notin I, \end{cases}$$

so $x \in U(f;t)$. That is $A \subseteq U(f;t)$. Hence, $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.51 that f is a fuzzy UP-ideal of A. □

Corollary 2.11. *Let I be a UP-subalgebra of A. Then the following statements hold:*

1. *for any $k \in (0,1]$, there exists a fuzzy UP-subalgebra g of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$, and*

2. *for any $k \in [0,1)$, there exists a fuzzy UP-subalgebra f of A such that $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.*

Proof. 1 Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.101, we have $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.63 that $\overline{f}$ is a fuzzy UP-subalgebra of A. By Remark 1.13, we have $L(\overline{\overline{f}};t) = L(f;t) = I$ for all $t < k$ and $L(\overline{\overline{f}};t) = L(f;t) = A$ for all $t \geq k$. Putting $\overline{f} = g$. Then g is a fuzzy UP-subalgebra of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$.

2 Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.102, we have $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.61 that f is a fuzzy UP-subalgebra of A. □

Theorem 2.12. *Let f be a fuzzy set in A and $s < t$ for $s,t \in [0,1]$. Then the following statements hold:*

1. *$L(f;s) = L(f;t)$ if and only if there is no $x \in A$ such that $s < f(x) \leq t$,*

2. *$L^-(f;s) = L^-(f;t)$ if and only if there is no $x \in A$ such that $s \leq f(x) < t$,*

3. *$U(f;s) = U(f;t)$ if and only if there is no $x \in A$ such that $s \leq f(x) < t$, and*

4. *$U^+(f;s) = U^+(f;t)$ if and only if there is no $x \in A$ such that $s < f(x) \leq t$.*

Proof. 1 Assume that $L(f;s) = L(f;t)$. Suppose that there is $x \in A$ such that $s < f(x) \leq t$. Thus $x \in L(f;t)$ but $x \notin L(f;s)$, so $L(f;s) \neq L(f;t)$ which is a contradiction. Hence, there is no $x \in A$ such that $s < f(x) \leq t$.

Conversely, assume that there is no $x \in A$ such that $s < f(x) \leq t$. Let $x \in L(f;s)$. Then $f(x) \leq s < t$, so $x \in L(f;t)$. Thus $L(f;s) \subseteq L(f;t)$. Suppose that $L(f;t) \nsubseteq L(f;s)$. Then there is $x \in L(f;t)$ but $x \notin L(f;s)$. Thus $f(x) \leq t$ and $f(x) > s$, so $s < f(x) \leq t$ which is a contradiction. Thus $L(f;t) \subseteq L(f;s)$. Hence, $L(f;t) = L(f;s)$.

2 Assume that $L^-(f;s) = L^-(f;t)$. Suppose that there is $x \in A$ such that $s \leq f(x) < t$. Thus $x \in L^-(f;t)$ but $x \notin L^-(f;s)$, so $L^-(f;s) \neq L^-(f;t)$ which is a contradiction. Hence, there is no $x \in A$ such that $s \leq f(x) < t$.

Conversely, assume that there is no $x \in A$ such that $s \leq f(x) < t$. Let $x \in L^-(f;s)$. Then $f(x) < s < t$, so $x \in L^-(f;t)$. Thus $L^-(f;s) \subseteq L^-(f;t)$. Suppose that $L^-(f;t) \nsubseteq L^-(f;s)$. Then there is $x \in L^-(f;t)$ but $x \notin L^-(f;s)$. Thus $f(x) < t$ and $f(x) \geq s$, so $s \leq f(x) < t$ which is a contradiction. Thus $L^-(f;t) \subseteq L^-(f;s)$. Hence, $L^-(f;t) = L^-(f;s)$.

3 Similarly to as in the proof of 1.

4 Similarly to as in the proof of 2. □

Corollary 2.13. *Let f be a fuzzy set in A and $s, t \in [0, 1]$. Then the following statements hold:*

1. *$L(f;s) = L(f;t)$ if and only if $U^+(f;s) = U^+(f;t)$, and*

2. *$U(f;s) = U(f;t)$ if and only if $L^-(f;s) = L^-(f;t)$.*

Proof. 1 It follows from Theorem 2.12 1 and Theorem 2.12 4.

2 It follows from Theorem 2.12 2 and Theorem 2.12 3. □

Theorem 2.14. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

1. *for every fuzzy UP-ideal β of B, μ is a fuzzy UP-ideal of A, and*

2. for every fuzzy UP-subalgebra β of B, μ is a fuzzy UP-subalgebra of A.

Proof. 1 Let β be a fuzzy UP-ideal of B. Let $x \in A$. Then

$$\begin{aligned}
\mu(0_A) &= (\beta \circ f)(0_A) \\
&= \beta(f(0_A)) \\
&\geq \beta(f(x)) && \text{((UP-3), Theorem 1.23 2, Lemma 2.3)} \\
&= (\beta \circ f)(x) \\
&= \mu(x).
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\mu(x \cdot z) &= (\beta \circ f)(x \cdot z) \\
&= \beta(f(x \cdot z)) \\
&= \beta(f(x) * f(z)) \\
&\geq \min\{\beta(f(x) * (f(y) * f(z))), \beta(f(y))\} && \text{(Definition 1.10 2)} \\
&= \min\{\beta(f(x \cdot (y \cdot z))), \beta(f(y))\} \\
&= \min\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y)\} \\
&= \min\{\mu(x \cdot (y \cdot z)), \mu(y)\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-ideal of A.

2 Let β be a fuzzy UP-subalgebra of B. Let $x, y \in A$. Then

$$\begin{aligned}
\mu(x \cdot y) &= (\beta \circ f)(x \cdot y) \\
&= \beta(f(x \cdot y)) \\
&= \beta(f(x) * f(y)) \\
&\geq \min\{\beta(f(x)), \beta(f(y))\} && \text{(Definition 1.11)} \\
&= \min\{(\beta \circ f)(x), (\beta \circ f)(y)\} \\
&= \min\{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-subalgebra of A. □

Lemma 2.15. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* sup *property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$ and $\beta(a \cdot b) = \mu(a_0 \cdot b_0)$.*

Proof. Let $a, b \in B$. Since f is surjective, we have $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(a * b)$ are nonempty subsets of A. By Definition 1.15, we obtain

$$\begin{aligned}\beta(a) &= \sup\{\mu(t)\}_{t\in f^{-1}(a)} \\ &= \mu(a_0) \text{ for some } a_0 \in f^{-1}(a), && \text{(Definition 1.16)} \\ \beta(b) &= \sup\{\mu(t)\}_{t\in f^{-1}(b)} \\ &= \mu(b_0) \text{ for some } b_0 \in f^{-1}(b) && \text{(Definition 1.16)}\end{aligned}$$

and

$$\begin{aligned}\beta(a\cdot b) &= \sup\{\mu(t)\}_{t\in f^{-1}(a\cdot b)} \\ &= \mu(c) \text{ for some } c \in f^{-1}(a * b). && \text{(Definition 1.16)}\end{aligned}$$

Since $f(c) = a * b = f(a_0) * f(b_0) = f(a_0 \cdot b_0)$ and μ is f-invariant, we have $\mu(c) = \mu(a_0 \cdot b_0)$. Hence, $\beta(a \cdot b) = \mu(a_0 \cdot b_0)$. ☐

Theorem 2.16. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

1. *for every f-invariant fuzzy UP-ideal μ of A with* sup *property, β is a fuzzy UP-ideal of B, and*

2. *for every f-invariant fuzzy UP-subalgebra μ of A with* sup *property, β is a fuzzy UP-subalgebra of B.*

Proof. 1 Let μ be an f-invariant fuzzy UP-ideal of A with sup property. By Definition 1.10 1, we have $\mu(0_A) \geq \mu(x)$ for all $x \in A$. By Theorem 1.23 1, we have $0_A \in f^{-1}(0_B)$ and so $f^{-1}(0_B) \neq \emptyset$. Thus $\beta(0_B) = \sup\{\mu(t)\}_{t\in f^{-1}(0_B)} \geq \mu(0_A)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. By Definition 1.10 1, we have $\mu(0_A) \geq \mu(t)$ for all $t \in f^{-1}(y)$. Thus $\mu(0_A)$ is an upper bound of $\{\mu(t)\}_{t\in f^{-1}(y)}$, so $\mu(0_A) \geq \sup\{\mu(t)\}_{t\in f^{-1}(y)} = \beta(y)$. By Proposition 1.5 2, we have $\beta(0_B) \geq \beta(y)$. Let $a, b, c \in B$. By Lemma 2.15, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$ and $c_0 \in f^{-1}(c)$ such that $\beta(b) = \mu(b_0), \beta(a * c) = \mu(a_0 \cdot c_0)$ and $\beta(a * (b * c)) = \mu(a_0 \cdot (b_0 \cdot c_0))$. Thus

$$\begin{aligned}\beta(a * c) &= \mu(a_0 \cdot c_0) \\ &\geq \min\{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0)\} && \text{(Definition 1.10 2)} \\ &= \min\{\beta(a * (b * c)), \beta(b)\}.\end{aligned}$$

Hence, β is a fuzzy UP-ideal of B.

2 Let μ be an f-invariant fuzzy UP-subalgebra of A with sup property. Let $a, b \in B$. Since f is surjective, we have $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(a * b)$ are nonempty subsets of A. By Lemma 2.15, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$ and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned} \beta(a * b) &= \mu(a_0 \cdot b_0) \\ &\geq \min\{\mu(a_0), \mu(b_0)\} && \text{(Definition 1.11)} \\ &= \min\{\beta(a), \beta(b)\}. \end{aligned}$$

Hence, β is a fuzzy UP-subalgebra of B. $\square$

Remark 2.17. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras. We can easily prove that $A \times B$ is a UP-algebra defined by*

$$(x_1, x_2) \diamond (y_1, y_2) = (x_1 \cdot y_1, x_2 * y_2)$$

for all $x_1, y_1 \in A$ and $x_2, y_2 \in B$.

Lemma 2.18. *Let f be a fuzzy set in A and g a fuzzy set in B. For any $t \in [0, 1]$, the following properties hold:*

1. *$f \times f$ is a fuzzy relation on A,*
2. *$L(f \times g; t) = L(f; t) \times L(g; t)$, and*
3. *$L^-(f \times g; t) = L^-(f; t) \times L^-(g; t)$.*

Proof. 1 Let $(x, y) \in A \times A$. Since f is a fuzzy set in A, we have $f(x), f(y) \in [0, 1]$. Thus there exists a unique $\max\{f(x), f(y)\} \in [0, 1]$ such that $(f \times f)(x, y) = \max\{f(x), f(y)\}$. Hence, $f \times f$ is a fuzzy relation on A.

2 For any $(x, y) \in A \times B$,

$$\begin{aligned} (x, y) \in L(f \times g; t) &\Leftrightarrow (f \times g)(x, y) \leq t \\ &\Leftrightarrow \max\{f(x), g(y)\} \leq t \\ &\Leftrightarrow f(x) \leq t \text{ and } g(y) \leq t \\ &\Leftrightarrow x \in L(f; t) \text{ and } y \in L(g; t) \\ &\Leftrightarrow (x, y) \in L(f; t) \times L(g; t). \end{aligned}$$

Hence, $L(f \times g; t) = L(f; t) \times L(g; t)$.

3 For any $(x, y) \in A \times B$,

$$\begin{aligned}(x, y) \in L^-(f \times g; t) &\Leftrightarrow (f \times g)(x, y) < t \\ &\Leftrightarrow \max\{f(x), g(y)\} < t \\ &\Leftrightarrow f(x) < t \text{ and } g(y) < t \\ &\Leftrightarrow x \in L^-(f; t) \text{ and } y \in L^-(g; t) \\ &\Leftrightarrow (x, y) \in L^-(f; t) \times L^-(g; t).\end{aligned}$$

Hence, $L^-(f \times g; t) = L^-(f; t) \times L^-(g; t)$. □

Lemma 2.19. *Let f be a fuzzy set in A and g a fuzzy set in B. For any $t \in [0, 1]$, the following properties hold:*

1. *$f \cdot f$ is a fuzzy relation on A,*
2. *$U(f \cdot g; t) = U(f; t) \times U(g; t)$, and*
3. *$U^+(f \cdot g; t) = U^+(f; t) \times U^+(g; t)$.*

Proof. Similarly to as in the proof of Lemma 2.18. □

Lemma 2.20. *Let f be a fuzzy set in A. For any $t \in [0, 1]$, the following properties hold:*

1. *$L(\mu_f; t) = L(f; t) \times L(f; t)$,*
2. *$L^-(\mu_f; t) = L^-(f; t) \times L^-(f; t)$,*
3. *$U(\beta_f; t) = U(f; t) \times U(f; t)$, and*
4. *$U^+(\beta_f; t) = U^+(f; t) \times U^+(f; t)$.*

Proof. 1 For any $(x, y) \in A \times A$,

$$\begin{aligned}(x, y) \in L(\mu_f; t) &\Leftrightarrow \mu_f(x, y) \leq t \\ &\Leftrightarrow \max\{f(x), f(y)\} \leq t \\ &\Leftrightarrow f(x) \leq t \text{ and } f(y) \leq t \\ &\Leftrightarrow x \in L(f; t) \text{ and } y \in L(f; t) \\ &\Leftrightarrow (x, y) \in L(f; t) \times L(f; t).\end{aligned}$$

Hence, $L(\mu_f; t) = L(f; t) \times L(f; t)$.

2 For any $(x, y) \in A \times A$,

$$\begin{aligned}(x, y) \in L^{-}(\mu_f; t) &\Leftrightarrow \mu_f(x, y) < t \\ &\Leftrightarrow \max\{f(x), f(y)\} < t \\ &\Leftrightarrow f(x) < t \text{ and } f(y) < t \\ &\Leftrightarrow x \in L^{-}(f; t) \text{ and } y \in L^{-}(f; t) \\ &\Leftrightarrow (x, y) \in L^{-}(f; t) \times L^{-}(f; t).\end{aligned}$$

Hence, $L^{-}(\mu_f; t) = L^{-}(f; t) \times L^{-}(f; t)$.

3 Similar to the proof of 1.

4 Similar to the proof of 2. □

Theorem 2.21. *Let f be a fuzzy set in A. Then the following statements hold:*

1. *if μ_f is a fuzzy UP-ideal of $A \times A$, then f is a fuzzy UP-ideal of A, and*
2. *if β_f is a fuzzy UP-ideal of $A \times A$, then f is a fuzzy UP-ideal of A.*

Proof. 1 Assume that μ_f is a fuzzy UP-ideal of $A \times A$. Let $x \in A$. Then

$$\begin{aligned}f(0) &= \max\{f(0), f(0)\} \\ &= \mu_f(0, 0) \\ &\geq \mu_f(x, x) && \text{(Definition 1.10 1)} \\ &= \max\{f(x), f(x)\} \\ &= f(x).\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x, x) \diamond (z, z)) \\ &\geq \min\{\mu_f((x, x) \diamond ((y, y) \diamond (z, z))), \mu_f(y, y)\} && \text{(Definition 1.10 2)} \\ &= \min\{\mu_f(x \cdot (y \cdot z), x \cdot (y \cdot z)), \mu_f(y, y)\} \\ &= \min\{\max\{f(x \cdot (y \cdot z)), f(x \cdot (y \cdot z))\}, \max\{f(y), f(y)\} \\ &= \min\{f(x \cdot (y \cdot z)), f(y)\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal of A.

2 Similar to the proof of 1. □

Theorem 2.22. *Let f be a fuzzy set in A. Then the following statements hold:*

1. *if μ_f is a fuzzy UP-subalgebra of $A \times A$, then f is a fuzzy UP-subalgebra of A, and*

2. *if β_f is a fuzzy UP-subalgebra of $A \times A$, then f is a fuzzy UP-subalgebra of A.*

Proof. 1 Assume that μ_f is a fuzzy UP-subalgebra of $A \times A$. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x, x) \diamond (z, z)) \\ &\geq \min\{\mu_f((x, x), \mu_f(y, y)\} && \text{(Definition 1.11)} \\ &\geq \min\{\max\{f(x), f(x)\}, \max\{f(y), f(y)\}\} \\ &= \min\{f(x), f(y)\}. \end{aligned}$$

Hence, f is a fuzzy UP-subalgebra of A.

2 Similarly to as in the proof of 1. □

Lemma 2.23. *[4] For any $a, b, c, d \in \mathbb{R}$, the following properties hold:*

1. $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$*, and*

2. $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Theorem 2.24. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras. Then the following statements hold:*

1. *if f is a fuzzy UP-ideal of A and g is a fuzzy UP-ideal of B, then $f \cdot g$ is a fuzzy UP-ideal of $A \times B$, and*

2. *if f is a fuzzy UP-subalgebra of A and g is a fuzzy UP-subalgebra of B, then $f \cdot g$ is a fuzzy UP-subalgebra of $A \times B$.*

Proof. 1 Assume that f is a fuzzy UP-ideal of A and g is a fuzzy UP-ideal of B. Let $(x, y) \in A \times B$. Then

$$\begin{aligned} (f \cdot g)(0, 0) &= \min\{f(0), g(0)\} \\ &\geq \min\{f(x), g(y)\} && \text{(Definition 1.10 1)} \\ &= (f \cdot g)(x, y). \end{aligned}$$

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B$. Then

$$\begin{aligned}
&(f \cdot g)((x_1, x_2) \diamond (z_1, z_2)) \\
&\quad = (f \cdot g)(x_1 \cdot z_1, x_2 * z_2) \\
&\quad = \min\{f(x_1 \cdot z_1), g(x_2 * z_2)\} \\
&\quad \geq \min\{\min\{f(x_1 \cdot (y_1 \cdot z_1)), f(y_1)\}, \\
&\qquad \min\{g(x_2 * (y_2 * z_2)), g(y_2)\}\} && \text{(Definition 1.10 2)} \\
&\quad = \min\{\min\{f(x_1 \cdot (y_1 \cdot z_1)), \{g(x_2 * (y_2 * z_2))\}\}, \\
&\qquad \min\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.23 2)} \\
&\quad = \min\{(f \cdot g)(x_1 \cdot (y_1 \cdot z_1), x_2 * (y_2 * z_2)), (f \cdot g)(y_1, y_2)\} \\
&\quad = \min\{(f \cdot g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \cdot g)(y_1, y_2)\}.
\end{aligned}$$

Hence, $f \cdot g$ is a fuzzy UP-ideal of $A \times B$.

2 Let $(x_1, x_2), (y_1, y_2) \in A \times B$. Then

$$\begin{aligned}
&(f \cdot g)((x_1, x_2) \diamond (y_1, y_2)) \\
&\quad = (f \cdot g)(x_1 \cdot y_1, x_2 * y_2) \\
&\quad = \min\{f(x_1 \cdot y_1), g(x_2 * y_2)\} \\
&\quad \geq \min\{\min\{f(x_1), f(y_1)\}, \min\{g(x_2), g(y_2)\}\} && \text{(Definition 1.11)} \\
&\quad = \min\{\min\{f(x_1), g(x_2)\}, \min\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.23 2)} \\
&\quad = \min\{(f \cdot g)(x_1, x_2), (f \cdot g)(y_1, y_2)\}.
\end{aligned}$$

Hence, $f \cdot g$ is a fuzzy UP-subalgebra of $A \times B$. $\square$

Give examples of conflict that f and g are fuzzy UP-ideals (resp., fuzzy UP-subalgebras) of A but $f \times g$ is not a fuzzy UP-ideal (resp., fuzzy UP-subalgebra) of $A \times A$.

Example 2.25. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.5, f(1) = 0.1, g(0) = 0.6 \text{ and } g(1) = 0.2.$$

Using this data, we can show that f and g are fuzzy UP-ideals of A. Let $x_1 = 0, x_2 = 0, y_1 = 1, y_2 = 0, z_1 = 1, z_1 = 1$. Then

$$(f \times g)((x_1, x_2) \diamond (z_1, z_2)) = 0.2 \geq 0.5 = \\ \min\{(f \times g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \times g)(y_1, y_2)\}.$$

Thus Definition 1.10 2 is false. Hence, $f \times g$ is not a fuzzy UP-ideal of $A \times A$.

Example 2.26. Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.5, f(a) = 0.1, f(b) = 0.2, g(0) = 0.5, g(a) = 0.1 \text{ and } g(b) = 0.2.$$

Using this data, we can show that f and g are fuzzy UP-subalgebras of A. Let $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 2$. Then

$$(f \times g)((x_1, x_2) \diamond (y_1, y_2)) = 0.1 \geq 0.2 = \min\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.$$

Thus Definition 1.11 is false. Hence, $f \times g$ is not a fuzzy UP-subalgebra of A.

Theorem 2.27. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f \cdot g$ is a fuzzy UP-ideal of $A \times B$, then the following statements hold:*

1. *either $f(0_A) \geq f(x)$ for all $x \in A$ or $g(0_B) \geq g(y)$ for all $y \in B$,*

2. *if $f(0_A) \geq f(x)$ for all $x \in A$, then either $g(0_B) \geq g(y)$ for all $y \in B$ or $g(0_B) \geq f(x)$ for all $x \in A$,*

3. *if $g(0_B) \geq g(y)$ for all $y \in B$, then either $f(0_A) \geq f(x)$ for all $x \in A$ or $f(0_A) \geq g(y)$ for all $y \in B$, and*

4. *either f is a fuzzy UP-ideal of A or g is a fuzzy UP-ideal of B.*

Proof. 1 Suppose that $f(0_A) < f(x)$ for some $x \in A$ and $g(0_B) < g(y)$ for some $y \in B$. Then $(f \cdot g)(x, y) = \min\{f(x), g(y)\} > \min\{f(0_A), g(0_B)\} = (f \cdot g)(0_A, 0_B)$ which is a contradiction. Hence, either $f(0_A) \geq f(x)$ for all $x \in A$ or $g(0_B) \geq g(y)$ for all $y \in B$

2 Assume that $f(0_A) \geq f(x)$ for all $x \in A$. Suppose that $g(0_B) < g(y)$ for some $y \in B$ and $g(0_B) < f(x)$ for some $x \in A$. Then $g(0_B) < f(x) \leq f(0_A)$. Thus

$$\begin{aligned}
(f \cdot g)(x, y) &= \min\{f(x), g(y)\} \\
&> \min\{g(0_B), g(0_B)\} \\
&= g(0_B) \\
&= \min\{f(0_A), g(0_B)\} \\
&= (f \cdot g)(0_A, 0_B)
\end{aligned}$$

which is a contradiction. Hence, either $g(0_B) \geq g(y)$ for all $y \in B$ or $g(0_B) \geq f(x)$ for all $x \in A$.

3 Assume that $g(0_B) \geq g(y)$ for all $y \in B$. Suppose that $f(0_A) < f(x)$ for some $x \in A$ and $f(0_A) < g(y)$ for some $y \in B$. Then $f(0_A) < g(y) \leq g(0_B)$. Thus

$$\begin{aligned}
(f \cdot g)(x, y) &= \min\{f(x), g(y)\} \\
&> \min\{f(0_A), f(0_A)\} \\
&= f(0_A) \\
&= \min\{f(0_A), g(0_B)\} \\
&= (f \cdot g)(0_A, 0_B)
\end{aligned}$$

which is a contradiction. Hence, either $f(0_A) \geq f(x)$ for all $x \in A$ or $f(0_A) \geq g(y)$ for all $y \in B$.

4 Suppose that f is not a fuzzy UP-ideal of A and g is not a fuzzy UP-ideal of B. By 1, assume that $f(0_A) \geq f(x)$ for all $x \in A$. Then from 2, either $g(0_B) \geq g(y)$ for all $y \in B$ or $g(0_B) \geq f(x)$ for all $x \in A$. If $g(0_B) \geq f(x)$ for all $x \in A$, then for all $x \in A$,

$$(f \cdot g)(x, 0_B) = \min\{f(x), g(0_B)\} = f(x). \tag{2.4}$$

Since $f \cdot g$ is a fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in A$,

$$\begin{aligned} f(x \cdot z) &= (f \cdot g)(x \cdot z, 0_B) && ((2.4)) \\ &= (f \cdot g)(x \cdot z, 0_B * 0_B) && ((\text{UP-3})) \\ &= (f \cdot g)((x, 0_B) \diamond (z, 0_B)) \\ &\geq \min\{(f \cdot g)((x, 0_B) \diamond [(y, 0_B) \diamond (z, 0_B)]), \\ &\quad (f \cdot g)(y, 0_B)\} && (\text{Definition 1.10 2}) \\ &= \min\{(f \cdot g)(x \cdot (y \cdot z), 0_B * (0_B * 0_B)), (f \cdot g)(y, 0_B)\} \\ &= \min\{(f \cdot g)(x \cdot (y \cdot z), 0_B), (f \cdot g)(y, 0_B)\} && ((\text{UP-3})) \\ &= \min\{\min\{f(x \cdot (y \cdot z)), g(0_B)\}, \min\{f(y), g(0_B)\}\} \\ &= \min\{f(x \cdot (y \cdot z)), f(y)\}. \end{aligned}$$

Hence, f is a fuzzy UP-ideal of A which is a contradiction. Assume that $g(0_B) \geq g(y)$ for all $y \in B$. Then from 3, either $f(0_A) \geq f(x)$ for all $x \in A$ or $f(0_A) \geq g(y)$ for all $y \in B$. If $f(0_A) \geq g(y)$ for all $y \in B$, then for all $y \in B$,

$$(f \cdot g)(0_A, y) = \min\{f(0_A), g(y)\} = g(y). \tag{2.5}$$

Since $f \cdot g$ is a fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in B$,

$$\begin{aligned} g(x * z) &= (f \cdot g)(0_A, x * z) && ((2.5)) \\ &= (f \cdot g)(0_A \cdot 0_A, x * z) && ((\text{UP-3})) \\ &= (f \cdot g)((0_A, x) \diamond (0_A, z)) \\ &\geq \min\{(f \cdot g)((0_A, x) \diamond [(0_A, y) \diamond (0_A, z)]), \\ &\quad (f \cdot g)(0_A, y)\} && (\text{Definition 1.10 2}) \\ &= \min\{(f \cdot g)(0_A \cdot (0_A \cdot 0_A)), (x * (y * z)), (f \cdot g)(0_A, y)\} \\ &= \min\{(f \cdot g)(0_A, x * (y * z)), (f \cdot g)(0_A, y)\} && ((\text{UP-3})) \\ &= \min\{\min\{f(0_A), g(x * (y * z))\}, \min\{f(0_A), g(y)\}\} \\ &= \min\{g(x * (y * z)), g(y)\}. \end{aligned}$$

Hence, g is a fuzzy UP-ideal of B which is a contradiction. Since f is not a fuzzy UP-ideal of A and g is not a fuzzy UP-ideal of B, and $f(0_A) \geq f(x)$ for all $x \in A$ and $g(0_B) \geq g(y)$ for all $y \in B$, there exist $x, y, z \in A$ and $x', y', z' \in B$ such that

$$f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$$

and

$$g(x' * z') < \min\{g(x' * (y' * z')), g(y')\}.$$

Thus

$$\min\{f(x \cdot z), g(x' * z')\} < \min\{\min\{f(x \cdot (y \cdot z)), f(y)\}, \min\{g(x' * (y' * z')), g(y')\}\}.$$

Since $f \cdot g$ is a fuzzy UP-ideal of $A \times B$, we have

$$\begin{aligned} \min\{f(x \cdot z), g(x' * z')\} &= (f \cdot g)(x \cdot z, x' * z') \\ &= (f \cdot g)((x, x') \diamond (z, z')) \\ &\geq \min\{(f \cdot g)((x, x') \diamond [(y, y') \diamond (z, z')]), \\ &\quad (f \cdot g)(y, y')\} \qquad \text{(Definition 1.10 2)} \\ &= \min\{(f \cdot g)(x \cdot (y \cdot z), x' * (y' * z')), (f \cdot g)(y, y')\} \\ &= \min\{\min\{f(x \cdot (y \cdot z)), g(x' * (y' * z'))\}, \\ &\quad \min\{f(y), g(y')\}\}. \end{aligned}$$

Thus $\min\{f(x \cdot z), g(x' * z')\} \not< \min\{\min\{f(x \cdot (y \cdot z)), f(y)\}, \min\{g(x' * (y' * z')), g(y')\}\}$ which is a contradiction. Similarly, by 1, if $g(0_B) \geq g(y)$ for all $y \in B$, we have a contradiction. Hence, either f is a fuzzy UP-ideal of A or g is a fuzzy UP-ideal of B. $\square$

Theorem 2.28. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f \cdot g$ is a fuzzy UP-subalgebra of $A \times B$, then either f is a fuzzy UP-subalgebra of A or g is a fuzzy UP-subalgebra of B.*

Proof. Suppose that f is not a fuzzy UP-subalgebra of A and g is not a fuzzy UP-subalgebra of B. Then there exist $x, y \in A$ and $a, b \in B$ such that

$$f(x \cdot y) < \min\{f(x), f(y)\}$$

and

$$g(a * b) < \min\{g(a), g(b)\}.$$

Thus $\min\{f(x \cdot y), g(a * b)\} < \min\{\min\{f(x), f(y)\}, \min\{g(a), g(b)\}\}$. Since

$f \cdot g$ is a fuzzy UP-subalgebra of $A \times B$, we have

$$\begin{aligned}
&\min\{f(x \cdot y), g(a * b)\} \\
&\qquad = (f \cdot g)(x \cdot y, a * b) \\
&\qquad = (f \cdot g)((x, a) \diamond (y, b)) \\
&\qquad \geq \min\{(f \cdot g)(x, a), (f \cdot g)(y, b)\} && \text{(Definition 1.11)} \\
&\qquad = \min\{\min\{f(x), g(a)\}, \min\{f(y), g(b)\}\} \\
&\qquad = \min\{\min\{f(x), f(y)\}, \min\{g(a), g(b)\}\}. && \text{(Lemma 2.23 2)}
\end{aligned}$$

Thus $\min\{f(x \cdot y), g(a * b)\} \not< \min\{\min\{f(x), f(y)\}, \min\{g(a), g(b)\}\}$ which is a contradiction. Hence, either f is a fuzzy UP-subalgebra of A or g is a fuzzy UP-subalgebra of B. □

Theorem 2.29. *Let f be a fuzzy set in A. Then the following statements hold:*

1. *f is a fuzzy UP-ideal of A if and only if β_f is a fuzzy UP-ideal of $A \times A$, and*

2. *f is a fuzzy UP-subalgebra of A if and only if β_f is a fuzzy UP-subalgebra of $A \times A$.*

Proof. 1 Assume that f is a fuzzy UP-ideal of A. By Theorem 2.241, we have $\beta_f = f \cdot f$ is a fuzzy UP-ideal of $A \times A$.

Conversely, assume that β_f is a fuzzy UP-ideal of $A \times A$. Since $f \cdot f = \beta_f$, it follows from Theorem 2.27 4 that f is a fuzzy UP-ideal of A.

2 Assume that f is a fuzzy UP-subalgebra of A. By Theorem 2.24 2, we have $\beta_f = f \cdot f$ is a fuzzy UP-subalgebra of $A \times A$.

Conversely, assume that β_f is a fuzzy UP-subalgebra of $A \times A$. Since $f \cdot f = \beta_f$, it follows from Theorem 2.28 that f is a fuzzy UP-subalgebra of A. □

3 Conclusions

In the present paper, we have studied the notions of fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras in term of upper t-(strong) level subsets and lower t-(strong) level subsets of a fuzzy set and investigated some of its essential properties. We think this work would enhance the scope for further study in this field of fuzzy sets. It is our hope that this work would serve as a foundation for the further study in this field of fuzzy sets in UP-algebras.

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