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On Subclass of Harmonic Univalent Functions defined by a Generalised Operator

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Abstract

We introduce a new subclass for harmonic univalent in the unit disk $\mathbb{U}$ define by the constructed operator L_n^σ in [1]. Properties such as coefficient bounds, distortion bounds, extreme points, and convolution will be studied.

1 Introduction

Let $f = u + iv$ be a complex valued harmonic function in a complex domain $\mathbb{C}$ that is both u and v are real harmonic in $\mathbb{C}$. Let

$$f(z) = h + \overline{g} \tag{1.1}$$

where h and g are analytic in $\mathbb{D} \subset \mathbb{C}$ and $\mathbb{D}$ is any simply connected domain. Let $\mathcal{SH}$ be the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f^{'}(0) - 1 = 0$, h and g define as follows

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{1.2}$$

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In 1984 Clunie and Sheil-Small [8] introduced and investigated the class $\mathcal{SH}$ as well as its geometric subclasses and obtained some properties of this class and this motivated many researchers to introduce some subclasses of the class $\mathcal{SH}$, (see [3, 4, 6]). The importance of these functions is due to their use in the study of minimal surfaces as well as in various problems related to applied mathematics. Let D^n with ($n \in N_0 = 0, 1, 2, ...$), be the Salagean derivative operator defined as $D^n f(z) = D(D^{n-1}f(z)) = z[D^{n-1}f(z)]'$ with $D^0 f(z) = f(z)$ given as

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_n z^k. \tag{1.3}$$

Let I^σ one-parameter Jung-Kim-Srivastava integral operator defined as $I^\sigma f(z) = \frac{2^\sigma}{2\Gamma\sigma} \int_0^z (log\frac{z}{t})^{\sigma-1} f(t)dt$ given as

$$I^\sigma f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^\sigma a_k z^k. \tag{1.4}$$

The operator L_n^σ was define as follows in [1]

$$L_n^\sigma f(z) = z + \sum_{k=2}^{\infty} k^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k. \tag{1.5}$$

with $L_n^0 f(z) = D^n f(z)$ and $L_0^\sigma f(z) = I^\sigma f(z)$. We define the operator on f as follows

$$L_n^\sigma f(z) = L_n^\sigma h(z) + (-1)^n \overline{L_n^\sigma g(z)}. \tag{1.6}$$

where $L_n^\sigma h(z) = z + \sum_{k=2}^{\infty} k^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k$ and $L_n^\sigma g(z) = \sum_{k=1}^{\infty} k^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k$ and also

$$L_0^0 f(z) = h(z) + \overline{g(z)}. \tag{1.7}$$

The two operators have been used by researchers to generalised the concepts of starlikeness and convexity of functions in the unit disk. (see [9, 10, 11]). We define $M_\sigma^n(\beta)$ be the family of harmonic functions of the form (1) such that

$$Re\left(\frac{M_\sigma^{n+1} f(z)}{M_\sigma^n f(z)} >\right) \beta. \tag{1.8}$$

Clearly the class $M_\sigma^n(\beta)$ includes a variety of well-known subclasses of $\mathcal{SH}$. For example, $M_0^0(\beta) \equiv \mathcal{SH}(\beta)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order β in U and $M_0^1(\beta) \equiv \mathcal{KH}$ is the class of sense-preserving, harmonic univalent functions f which are convex of order β in U studied by Jahangiri [2], $M_0^n(\beta)$ is the class of Salagean-type harmonic univalent functions introduced by Jahangiri et al. [5, 7]. We let the subclass $\overline{M}_\sigma^n(\beta)$ consist of harmonic functions $f_n = h(z) + g_n(z)$ in the class $M_\sigma^n(\beta)$ where h and f are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k, |b_k| < 1. \tag{1.9}$$

In this work, we give the sufficient condition for functions in the class $M_\sigma^n(\beta)$ which is sufficient for the functions in the class $\overline{M}_\sigma^n(\beta)$. The distortion, extreme point and convolution for the functions in the class $\overline{M}_\sigma^n(\beta)$were also obtained.

2 Main Results

Theorem 2.1. *Let $f(z) = h(z) + \overline{g(z)}$, where h(z) and g(z) are given by* (1.2) *If*

$$\sum_{k=2}^{\infty} [(k-\beta)|a_k| + (k+\beta)|b_k|] k^n (2/k+1)^\sigma \leq 2(1-\beta). \tag{2.1}$$

where $a_1 = 1$, $0 \leq \beta < 1$, $\sigma, n \in \mathbb{N}_0$.then f is sense-preserving, harmonic univalent in U, and $f \in M_\sigma^n(\beta)$

Proof If $z_1 \neq z_2$. then

$$\left|\frac{f(z_1) - f(z_2))}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1^k - z_2^k) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}\right|$$

$$> 1 - \frac{\sum_{k=2}^{\infty} k b_k}{1 - \sum_{k=2}^{\infty} k a_k} \geq 1 - \frac{\sum_{k=1}^{\infty} k^n 2^\sigma/(k+1)^\sigma |b_k|}{\sum_{k=1}^{\infty} k^n 2^\sigma/(k+1)^\sigma |a_k||} \geq 0 \tag{2.2}$$

which proves univalence. Note that f is sense-preserving in $\mathbb{U}$, because

$$|h^{'}(z)| \geq \left(1 - \sum_{k=1}^{\infty} k|a_k||z|^{k-1}\right) > \left(1 - \sum_{k=1}^{\infty} k^n \left(\frac{2}{k+1}\right)^\sigma |a_k|\right)$$

$$\geq \left(\sum_{k=1}^{\infty} k^n \left(\frac{2}{k+1}\right)^\sigma |b_k|\right) \geq \left(\sum_{k=1}^{\infty} k|b_k||z|^{k-1}\right) \geq |g^{'}(z)| \tag{2.3}$$

by (1.8)

$$Re\left(\frac{L_\sigma^{n+1}f(z)}{L_\sigma^{n+1}f(z)}\right)=\left(\frac{L_\sigma^{n+1}h(z)+\overline{(-1)^{n+1}L_\sigma^{n+1}g(z)}}{L_\sigma^{n}h(z)+\overline{(-1)^{n}L_\sigma^{n}g(z)}}\right)>\beta \tag{2.4}$$

Using the fact that $Rew(z)>\beta$ if and only if $|1-\beta+w|\geq|1+\beta-w|$, it suffices to show that

$$\left|1-\beta+\frac{L_\sigma^{n+1}f(z)}{L_\sigma^{n}f(z)}\right|-\left|1+\beta-\frac{L_\sigma^{n+1}f(z)}{L_\sigma^{n}f(z)}\right|\geq 0 \tag{2.5}$$

$$|L_\sigma^{n+1}f(z)+(1-\beta)L_\sigma^{n}f(z)|-|L_\sigma^{n+1}f(z)-(1+\beta)L_\sigma^{n}f(z)|\geq 0 \tag{2.6}$$

substituting for $L_\sigma^{n+1}f(z)$, $L_\sigma^{n}f(z)$ in (2.6), we have that

$$\begin{aligned}
&|L_\sigma^{n+1}h(z)+\overline{(-1)^{n+1}L_\sigma^{n+1}g(z)}+(1-\beta)\left[L_\sigma^{n}h(z)+\overline{(-1)^{n}L_\sigma^{n+1}g(z)}\right]|\\
&-|L_\sigma^{n+1}h(z)+\overline{(-1)^{n+1}L_\sigma^{n+1}g(z)}-(1+\beta)\left[L_\sigma^{n}h(z)+\overline{(-1)^{n}L_\sigma^{n+1}g(z)}\right]|\\
&=\left|z+\sum_{k=1}^{\infty}k^{n+1}\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k+(-1)^{n+1}\sum_{k=1}^{\infty}k^{n+1}\left(\frac{2}{k+1}\right)^{\sigma}\overline{b_kz^k}\right|\\
&+(1-\beta)\left[z+\sum_{k=1}^{\infty}k^{n}\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k+(-1)^{n}\sum_{k=1}^{\infty}k^{n}\left(\frac{2}{k+1}\right)^{\sigma}\overline{b_kz^k}\right]\\
&-\left|z+\sum_{k=1}^{\infty}k^{n+1}\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k+(-1)^{n+1}\sum_{k=1}^{\infty}k^{n+1}\left(\frac{2}{k+1}\right)^{\sigma}\overline{b_kz^k}\right|\\
&-(1+\beta)\left[z+\sum_{k=1}^{\infty}k^{n}\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k+(-1)^{n}\sum_{k=1}^{\infty}k^{n}\left(\frac{2}{k+1}\right)^{\sigma}\overline{b_kz^k}\right]\\
&=\left|(2-\beta)z+\sum_{k=2}^{\infty}(k+1-\beta)k^n\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k-(-1)^n\sum_{k=2}^{\infty}(k+1-\beta)k^n\left(\frac{2}{k+1}\right)^{\sigma}b_kz^k\right|\\
&-\left|(-\beta)z+\sum_{k=2}^{\infty}(k-1-\beta)k^n\left(\frac{2}{k+1}\right)^{\sigma}a_kz^k-(-1)^n\sum_{k=2}^{\infty}(k+1+\beta)k^n\left(\frac{2}{k+1}\right)^{\sigma}b_kz^k\right|\\
&\geq 2(1-\beta)|z|-\sum_{k=2}^{\infty}2k^n(k-\beta)\left(\frac{2}{k+1}\right)^{\sigma}|a_k||z|^k-\sum_{k=2}^{\infty}2k^n(k-\beta)\left(\frac{2}{k+1}\right)^{\sigma}|b_k||z|^k
\end{aligned}$$

$$= 2(1-\beta)\left[1-\sum_{k=2}^{\infty}2k^n\frac{(k-\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|a_k|-\sum_{k=2}^{\infty}2k^n\frac{(k+\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|b_k|\right] \tag{2.7}$$

This last expression is nonnegative by (2.1), and so the proof is complete. The harmonic function

$$f(z) = z+\sum_{k=2}^{\infty}2k^n\frac{(k+\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}x_k z^k+\sum_{k=2}^{\infty}2k^n\frac{(k+\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}y_k z^k \tag{2.8}$$

where $n,\sigma \in N_0$, $0 \le \beta < 1$, and $\sum_{k=2}^{\infty} x_k + \sum_{k=2}^{\infty} y_k = 1$, shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.7)are in $M_{\sigma}^{n}(\beta)$ because

$$\sum_{k=2}^{\infty}\left[\frac{(k-\beta)}{1-\beta}|a_k|+\frac{(k+\beta)}{1-\beta}|b_k|\right]k^n\left(\frac{2}{k+1}\right)^{\sigma} = 1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=2}^{\infty}|y_k| = 1+1 = 2 \tag{2.9}$$

Theorem 2.2. *Let $f_n(z) = h(z)+\overline{g_n(z)}$, then $f \in \overline{M}_{\sigma}^{n}(\beta)$ if and only if*

$$\sum_{k=2}^{\infty}[(k-\beta)|a_k|+(k+\beta)|b_k|k^n(2/k+1)^{\sigma} \le 2(1-\beta) \tag{2.10}$$

where $a_1 = 1$ $0 \le \beta < 1$, $\sigma, n \in N_0$. and $f \in M_{\sigma}^{n}(\beta)$

Proof By condition (1.5) and since $\overline{M}_{\sigma}^{n}(\beta) \subset M_{\sigma}^{n}(\beta)$, it shows that (2.10) is true

Theorem 2.3. *Let $f_n \in \overline{M}_{\sigma}^{n}(\beta)$, then for $|z| = r < 1$, we have*

$$|f(z)| \le (1+|b_1|)r+\frac{1}{2^n}\left(\frac{1-\beta}{2-\beta}-\frac{1+\beta}{2-\beta}|b_1|\right)r^2$$

$$|f(z)| \ge (1+|b_1|)r+\frac{1}{2^n}\left(\frac{1-\beta}{2-\beta}-\frac{1+\beta}{2-\beta}|b_1|\right)r^2 \tag{2.11}$$

Proof Taking the absolute value of $f(z)$, we obtain

$$|f(z)| = \left|z-\sum_{k=2}^{\infty}a_k z^k+(-1)^n\sum_{k=1}^{\infty}b_k\overline{z^k}\right|$$

$$\le (1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{(2-\beta)2^n}\left(\sum_{k=2}^{\infty}\frac{(2-\beta)2^n}{1-\beta}|a_k| + \frac{(2-\beta)2^n}{1-\beta}|b_k|\right)r^2 \quad (2.12)$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{(2-\beta)2^n}\left(\sum_{k=2}^{\infty}k^n\frac{(k-\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|a_k| + k^n\frac{(k+\beta)}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|b_k|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{1-\beta}{(2-\beta)2^n}\left(1-\frac{1+\beta}{1-\beta}|b_1|\right)r^2$$

for $|b_1| < 1$. This shows that the bounds given in Theorem (2.3) are sharp. By following proof , the lower bound is achieved and the proof is omitted.
Corollary 2.1 If the function $f_n = h_n + \overline{g_n}$ in $f \in \overline{M}_n^n(\beta)$

$$\left[w : |w| < \frac{2^{n+1}-1-(2^n-1)\beta}{2^n(2-\beta)} - \frac{2^{n+1}+1}{2^n(2-\beta)}|b_1|\right] \subset f(U) \quad (2.13)$$

Theorem 2.4. *Let $f_n = h_n + \overline{g_n}$, where h and g are given by (1.8), $f \in \overline{M}_{\sigma}^n(\beta)$ if and only if*

$$f_n(z) = (X_k h_k(z) + Y_k g_{nk}(z)) \quad (2.14)$$

where $h_k(z) = z - (1-\beta)/(k-\beta)k^n(k+1/2)^{\sigma}z^k$, where $(k = 2, 3, ...)$, $g_{nk} = z - (-1)^n(1-\beta)/(k-\beta)k^n(k+1/2)^{\sigma}z^k$ and $(X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular the extreme points of $M_{\sigma}^n(\beta)$ are h_k and g_{nk}

Proof for functions $f_n = h + \overline{g}$, where h and g are given by (1.8), we have that

$$f_n(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_{nk}(z))$$

$$\sum_{k=1}^{\infty}(X_k+Y_k)z - \sum_{k=2}^{\infty}1-\beta/(k-\beta)k^n(k+1/2)^{\sigma}X_k z^k + (-1)^n\sum_{k=1}^{\infty}(1-\beta/)/(k+\beta)k^n(k+1/2)^{\sigma}Y_k z^k \quad (2.15)$$

then

$$\sum_{k=2}^{\infty}\frac{2^{\sigma}(k-\beta)k^n}{(1-\beta)(k+1)^{\sigma}}|a_k| + \sum_{k=2}^{\infty}\frac{2^{\sigma}(k-\beta)k^n}{(1-\beta)(k+1)^{\sigma}}|b_k| = \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k \quad (2.16)$$

and so f is in closed convex hulls of $M_\sigma^n(\beta)$. The converse is true and the proof is omitted In the next theorem, we show that the class $M_\sigma^n(\beta)$ is invariant under convolution
For harmonic function $f_n(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n \sum_{k=1}^{\infty}|b_k|z^k$ and $F_n(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n \sum_{k=1}^{\infty}|B_k|z^k$. The convolution $f(z)$ and $F(z)$ gives

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty}|a_k||A_k|z^k + (-1)^n \sum_{k=1}^{\infty}|b_k||B_k|z^k$$

.

Theorem 2.5. *For $0 \le \gamma \le \beta < 1$, let $f \in M_\sigma^n(\beta)$ and $F \in M_\sigma^n(\gamma)$. Then $f(z) * F(z) \in M_\sigma^n(\beta) \subset M_\sigma^n(\gamma)$*

Proof Let the functions $f_n(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n \sum_{k=1}^{\infty}|b_k|z^k$ be in the class $M_\sigma^n(\beta)$ and the functions $F_n(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n \sum_{k=1}^{\infty}|B_k|z^k$ be in the class $M_\sigma^n(\gamma)$. We need to show the convolution satisfies the required condition of theorem 2.1, note that $|A_k| \le 1$ and $|B_k| \le 1$. By the convolution, we obtain

$$\sum_{k=2}^{\infty}\frac{k-\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}k^n|a_k||A_k| + \sum_{k=1}^{\infty}\frac{k+\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|b_k||B_k| \tag{2.16}$$

$$\le \sum_{k=2}^{\infty}\frac{k-\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}k^n|a_k| + \sum_{k=1}^{\infty}\frac{k+\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|b_k|$$

$$\le \sum_{k=2}^{\infty}\frac{k-\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}k^n|a_k| + \sum_{k=1}^{\infty}\frac{k+\beta}{1-\beta}\left(\frac{2}{k+1}\right)^{\sigma}|b_k| \le 1 \tag{2.17}$$

since $0 \le \gamma \le \beta < 1$ and $f \in M_\sigma^n(\beta)$. Therefore $f(z) * F(z) \in M_\sigma^n(\beta) \subset M_\sigma^n(\gamma)$.

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