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A Note on Stability of Certain Lienard Fractional Equation

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Abstract

In this note we obtain sufficient conditions under which we can guarantee the stability of solutions of a Liénard fractional equation.

1 Introduction

The classical Liénard equation $x'' + f(x)x' + g(x) = 0$, and its equivalent system (with $F(x) = \int_0^x f(r)dr$)

$$x' = y - F(x),$$

$$y' = -g(x),$$

appears as a simplified model in many domains in science and engineering (cf. [19], [25] and [26]). It was intensively studied during the first half of 20th century as it can be used to model oscillating circuits or simple pendulums, in

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this case, f and g represents the friction and acceleration terms. The Liénard equation is often taken as the typical example of nonlinear self-excited vibration problem, it can also be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements, and model certain mechanical systems with nonlinear damping, restoring force or stiffness. Moreover, some nonlinear evolution equations (such as the Burgers-Korteweg-de Vries equation) which arise from various physical phenomena can also be transformed to Liénard equation. We recommend [12] for other references about more applications. One of the first models where this equation appears was introduced by Van der Pol (see [27,28]); he considered the equation

$$x'' + \mu(x^2 - 1)x' + x = 0,$$

for modeling the oscillations of a triode vacuum tube. Therefore, the study of these equations is of physical significance.

However, in the non-integer case, the care have received these equations is scarce.

Fractional calculus concerns the generalization of differentiation and integration to non-integer (fractional) orders. The subject has a long mathematical history being discussed for the first time already in the correspondence of Leibniz with L'Hopital when this replied "What does $\frac{d^n}{dx^n}f(x)$ mean if n=$\frac{1}{2}$?" in September 30 of 1695. Over the centuries many mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature (see [23]).

The nature of many systems makes that they can be more precisely modeled using fractional differential equations. The differentiation and integration of arbitrary orders have found applications in diverse fields of science and engineering like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals (see [11,20,24,31]).

So, non-integer order differential equations and systems (commonly called fractional) are a rapidly developing field in technical and mathematical sciences. Most focus is oriented on their asymptotic properties using Lyapunov theory analogous to ordinary case (see [1,4,16,17,18]) and applications (see for example, in addition to those listed above [8,10,15,19,30]). The goal of this paper is to highlight one of the interesting results from the first group.

In this way the Lyapunov Direct Method provides a way to analyze the stability of dynamical systems without solving the differential equations. It is especially advantageous when the solution is difficult or even impossible to find with classical methods.

It is interesting to investigate an extension of the method for non-integer order systems. Such extension is based on the concept of Caputo derivative which is presented bellow. Then we present basic results equivalent the ordinary case.

So, in order to prove the stability of fractional order nonlinear and time varying systems in the vector case, some other techniques must be applied. One of these techniques is the fractional-order extension of Lyapunov direct method, proposed by Li et al. [18]. Using this technique, however, is often a really hard task, since finding Lyapunov candidate function is more complex in the fractional order case.

Some authors have proposed Lyapunov functionals to prove the stability of fractional order systems. The two prominent works [16,17] can be cited, however the relation between the Lyapunov function and the fractional differential equation is not elementary nor simple, [4] proposes some other Lyapunov functionals, where the relation between them and the fractional differential system is more elementary, but these functionals are neither simple, and they are valid for fractional systems with specific characteristics.

Other work related to this issue are as follows. In [6] the authors focused to the existence of fractional order Lyapunov stability theorem for autonomous systems and introducing a fractional order Lyapunov function based on the definition of fractional calculation and integer order Lyapunov theory. [29] proved an elementary lemma which estimates fractional derivatives of Volterra-type Lyapunov functions in the sense Caputo when $\alpha \in (0, 1)$. Moreover, by using this result, the authors studied the uniform asymptotic stability of some Caputo-type epidemic systems with a pair of fractional-order differential equations. The difficulty of fractional direct Lyapunov stability theorem lies in that how to design a positive definite function V and easily as certain whether fractional derivative of the function V is less than zero. In view of this difficulty, in [13] the authors proposed a Lyapunov stability theorem for autonomous fractional system without delay and extend the obtained theorems to fractional system with delay. [9] presents two new lemmas related to the Caputo fractional derivatives, when $\alpha \in (0, 1]$, for the case of general quadratic forms and for the case where the trace of the product of a rectangular matrix and its transpose appear. Those two lemmas allow using general quadratic Lyapunov functions and the trace of a matrix inside a Lyapunov function respectively, in order to apply the fractional-order extension of Lyapunov direct method, to analyze the stability of fractional order systems. Besides, the paper presents a theorem for proving uniform stability in the sense of Lyapunov for fractional order systems. Baranowski and other in [3] presents certain properties of Lyapunov direct method for non-integer order systems. Mittag-Leffler stability is defined and its relationship with Lyapunov stability is investigated. General results for Lyapunov functions are presented and a new result allowing constructive stability analysis is proved. In [21] studied stability of fractional nonlinear systems and aims to give several simple criteria of Mittag-Leffler and asymptotic stability. [2] presents the analysis of three classes of fractional differential equations appearing in the field of fractional adaptive systems, for the case when the fractional order is in the interval $\alpha \in (0, 1]$, and the Caputo definition for fractional derivatives is used. The boundedness of the solutions is proved for all three cases, and the convergence to zero of the mean value of one of the variables is also proved. In [7] stability analysis of fractional-order nonlinear systems is studied. An extension of Lyapunov direct method for fractional-order systems using Bihari's and Bellman-Gronwall's inequality and a proof of comparison theorem for fractional-order systems are proposed.

This paper presents a new result on stability for systems of Liénard type with Caputo fractional derivatives and we present a simple Lyapunov candidate function for many fractional order systems, and the consequently stability proof for them, using the fractional-order extension of the Lyapunov direct method [18]. The fundamental difference of our work with the above, is the use of Lemma 1, which brings us closer to the classic version of the Second Method of Lyapunov.

2 Preliminaries

In this section, some basic definitions related to fractional calculus are presented. Some concepts and techniques related to the stability of fractional order systems are presented as well. In fractional calculus, the traditional definitions of the integral and derivative of a function are generalized from integer orders to real and complex orders. In the time domain, the fractional order derivative and fractional order integral operators are defined by a convolution operation. Several definitions exist regarding the fractional derivative of non integer order, but the Caputo definition is used the most in engineering applications, since this definition incorporates initial conditions for its integer order derivatives, i.e., initial conditions that are physically appealing in the traditional way.

Definition 2.1. **[Caputo fractional derivative]** *(see [5], [18] and [23]). The Caputo fractional derivative of order α on the half axis is defined as follows*

$${}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, n-1 < \alpha < n, n \in N, 0 < a < t < \infty.$$

In this paper, we consider the following system:

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha} x(t) &= y(t) - F(x(t)), \\ {}_{0}^{C}D_t^{\alpha} y(t) &= -g(x(t)). \end{aligned} \tag{2.1}$$

as a natural generalization of the classical Liénard system, with $F(x)$ as above, and f and g are continuous functions such that $f : \mathbb{R} \to \mathbb{R}_+$, and $g : \mathbb{R} \to \mathbb{R}$ with $xg(x) > 0$ for $x \neq 0$. The system (1) is equivalent to the equation ${}_{0}^{C}D_t^{\alpha}\left[{}_{0}^{C}D_t^{\alpha} x(t)\right] +{}_{0}^{C} D_t^{\alpha}\left[F\left[x(t)\right]\right] + g\left[x(t)\right] = 0$. We present some definitions and results for the system

$${}_{0}^{C}D_t^{\alpha} x(t) = f(t, x(t)) \tag{2.2}$$

and later we apply it to the system (1). We consider the system (2) with initial condition $x(0)$, $\alpha \in (0,1)$, $f : [0,+\infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,+\infty) \times \Omega$, and $\Omega \subseteq \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The equilibrium point of (2) is defined as follows:

Definition 2.2. *The solution of Caputo system (2) such that $x(t) = x_0 =$ const is called the equilibrium.*

Directly from the definition of Caputo derivative, we see that x_0 is an equilibrium point of Caputo fractional system (1), if and only if $f(t, x_0) = 0$, for t$\in [t_0, +\infty)$ (see [10] and [30]). It can noted be also that it is sufficient to analyze only the equilibrium point at the origin. We can always transform the systems using substitution $y = x - x_0$, by this we state all definitions and theorems for the case when the equilibrium point is the origin.

In this paper, we assume that the system satisfies a certain condition of existence and uniqueness (see [1], [4] or [23] for details).

Definition 2.3. *The trivial solution of (2) is said to be stable if, for any* $\varepsilon > 0$ *and any* $t_0 > 0$, *there exists a positive constant* $\delta = \delta(t_0, \epsilon)$ *such that, whenever* $\|x(t_0)\| < \delta$ *we have* $\|x(t)\| < \varepsilon$, *for* $t \geq t_0$.

Definition 2.4. *The solution* $x = 0$ *of (2) is said to be uniformly stable if the number* δ *in the previous definition is independent of* t_0.

With $C(\mathbb{R})$ and $CI(\mathbb{R})$ we respectively denote the families of continuous functions and increasing continuous functions defined on $\mathbb{R}$.

Definition 2.5. *(see [22]).* $CS(\mathbb{R} = \{h \in C(\mathbb{R}) : \ xh(x) > 0, \ x \neq 0\}$

Definition 2.6. *(see [22]).*$CC(\mathbb{R}) := CI(\mathbb{R}) \cap CS(\mathbb{R})$

Definition 2.7. *A continuous function* $\beta : [0, t) \to [0, +\infty)$ *is said to belong to class-K if it is strictly increasing and* $\beta(0) = 0$.

Remark 2.8. *It is clear that* $K \equiv CC([0, +\infty))$.

Definition 2.9. S_r *is the sphere* $S_r = \{x \in \mathbb{R}^n : \ \|x\| < r\}$.

Definition 2.10. [Riemann-Liouville fractional derivative] *(see [23]). The Riemann-Liouville fractional derivative of order* α *is defined as follows*

$$ {}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-s)^{m-\alpha} f(s) ds, \quad m \leq \alpha < m+1 $$

Some properties of the Caputo and Riemann-Liouville derivatives, are listed below (see [23]):

Property 1: When $0 < \alpha < 1$, we have

$$ {}_{t_0}^{C}D_t^\alpha x(t) = {}_{t_0}D_t^\alpha x(t) - \frac{x(t_0)}{\Gamma(1-\alpha)} (t-t_0)^{-\alpha}. $$

In particular, if $x(t_0) = 0$, we have

$$ {}_{t_0}^{C}D_t^\alpha x(t) = {}_{t_0}D_t^\alpha x(t). $$

Property 2: For any $\nu > -1$, we have

$$ {}_{t_0}D_t^{\alpha}(t-t_0)^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}(t-t_0)^{\nu-\alpha}. $$

If $0 < \alpha < 1$, taking into account the Property 1, we have

$$ {}_{t_0}^{C}D_t^{\alpha}(t-t_0)^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}(t-t_0)^{\nu-\alpha}. $$

Property 3:

$$ {}_{t_0}^{C}D_t^{\alpha}\left[ax(t)+by(t)\right] = a{}_{t_0}^{C}D_t^{\alpha}x(t) + b{}_{t_0}^{C}D_t^{\alpha}y(t), $$

with a and b arbitrary constants.

Property 4: From the definition of Caputo's derivative, when $0 < \alpha < 1$ we have

$$ I_{t_0}^{\alpha}{}_{t_0}^{C}D_t^{\alpha}x(t) = x(t) - x(t_0), $$

where

$$ \left(I_{t_0}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(s)}{(t-s)^{1-\alpha}}ds, t > t_0, \Re(\alpha) > 0. $$

Henceforth, in this paper, we will take for convenience set $t_0 = 0$.

Taking as a starting point Lemma 1 of [1] we have the following result.

Lemma 2.11. *Let $x(t) : [0, +\infty) \to \mathbb{R}$ be a continuous and derivable function, $F(x) = \int_0^x f(r)dr$ and $f : \mathbb{R} \to \mathbb{R}$ be a positive function satisfying a Lipschitz condition $|f(x(t)) - f(x(\tau))| \leq M\,|x(t) - x(\tau)|$ for some constant $M > 0$. Then, for any time instant $t \geq 0$ we have*

$$ {}_{0}^{C}D_t^{\alpha}F(x(t)) \leq f(x(t)){}_{0}^{C}D_t^{\alpha}x(t),\ \alpha \in (0,1) \tag{2.3} $$

Proof. Prove that (3) is true, it is equivalent to prove that

$$ f(x(t)){}_{0}^{C}D_t^{\alpha}x(t) -{}_{0}^{C}D_t^{\alpha}F(x(t)) \geq 0,\ \alpha \in (0,1) $$

From Definition 1 we have

$$
\begin{aligned}
f\left[x(t)\right]{}_0^C D_t^\alpha x(t) -{}_0^C D_t^\alpha F\left[x(t)\right] &\geq 0\\
f\left[x(t)\right]\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(\tau)}{(t-\tau)^\alpha}d\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{F'\left[x(t)\right]}{(t-\tau)^\alpha}d\tau &\geq 0\\
\frac{1}{\Gamma(1-\alpha)}\left[f\left[x(t)\right]\int_0^t \frac{x'(\tau)}{(t-\tau)^\alpha}d\tau - \int_0^t \frac{F'\left[x(t)\right]}{(t-\tau)^\alpha}d\tau\right] &\geq 0\\
\frac{1}{\Gamma(1-\alpha)}\left[\int_0^t \frac{f\left[x(t)\right]x'(\tau) - F'\left[x(t)\right]}{(t-\tau)^\alpha}d\tau\right] &\geq 0
\end{aligned}
$$

Using the Lipschitz condition for some constant $M > 0$, we have

$$\frac{M}{\Gamma(1-\alpha)}\int_0^t \frac{|x(t)-x(\tau)|}{(t-\tau)^\alpha}x'(\tau)d\tau \geq 0$$

Making a variable change $y(\tau) = x(t) - x(\tau)$, integrating by parts and eliminating the indetermination $\frac{0}{0}$ in $\tau = t$ using the L'Hopital rule conclude

$$\frac{M}{2\Gamma(1-\alpha)}\left[\frac{y^2(0)}{t^\alpha} + \int_0^t \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}}d\tau\right] \geq 0$$

A result that is clearly true. This completes the proof. □

Remark 2.12. *If $F(x) = \frac{x^2}{2}$ we obtain the Lemma 1 of [1].*

3 Main Results

We now present the results on stability of the system (2).

Theorem 3.1. *Suppose that there exists a Lyapunov function $V(t,x)$ defined on $[0,+\infty) \times S_H$ which satisfies the following conditions:*

(i) $V(t,x)$ is a definite positive function.

(ii) for $t \geq 0$, ${}_0^C D_t^\alpha V(t,x(t)) \leq 0, \alpha \in (0,1)$.

Then, the trivial solution of the system (2) is stable.

Proof. From i) we have that there exists a function $a(\|x\|)$ such that $a(\|x\|) \leq V(t,x)$. Corresponding to any $\varepsilon > 0$, $\epsilon < r$, we have $a(\varepsilon) \leq V(t,x)$ for $t \in [0,+\infty)$ and x such that $\|x(t)\| = \varepsilon$. For a fixed $t_0 \in [0,+\infty)$, we can choose a $\delta(t_0, \varepsilon) > 0$ such that $\|x(t_0)\| < \delta$ implies $V(t_0, x_0) < a(\varepsilon)$, because $V(t_0, 0) \equiv 0$ and $V(t, x)$ is continuous. Suppose that a solution $\overline{x}(t)$ of (2) with $\|\overline{x}(0)\| < \delta$ such that $\|\overline{x}(t_1)\| = \varepsilon$ at some t_1. From ii) it follows that $V(t_1, x_1) \leq V(t_0, x_0)$, because

$$ {}_0^C D_t^\alpha V\left(t, x(t)\right) \;\leq\; 0, \quad 0 < \alpha < 1 $$

$$ \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{V'\left(s, x\left(s\right)\right)}{\left(t-s\right)^\alpha} ds \;\leq\; 0 $$

then

$$ V'\left(s, x\left(s\right)\right) \;\leq\; 0, \quad \forall s \in \left(0, t\right). $$

Hence

$$ V\left(t_1, x(t_1)\right) \leq V\left(t_0, x(t_0)\right), \quad \forall t_0, t_1 \in \left(0, t\right). $$

Then

$$ a(\varepsilon) \leq V(t_1, x_1) \leq V(t_0, x_0) < a(\varepsilon). $$

This is a contradiction, and hence, if $\|x(t_0)\| < \delta(t_0, \varepsilon)$, then $\|x(t)\| < \varepsilon$ for all $t \geq 0$. That is, the stability of the trivial solution. □

Theorem 3.2. *Assume that there exists a function $V(t, x)$ satisfying the following conditions:*

(i) $V \in C([0, +\infty) \times S_r, \mathbb{R}_+)$, *$V(t, x)$ is positive definite, decreasing and locally lipschitzian in x;*

(ii) for $t \geq 0$, ${}_0^C D_t^\alpha V(t, x(t)) \leq 0, \alpha \in (0, 1)$.

Then the trivial solution of (2) is uniformly stable.

Proof. Since V is positive definite and decreasing, there exist functions $a, b \in K$ satisfying

$$b(\|x\|) \le V(t,x) \le a(\|x\|), \ (t,x) \in [0,+\infty)xS_r \tag{3.4}$$

Let $0 < \varepsilon < \rho$, $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon) \tag{3.5}$$

We claim that, if $\|x_0\| < \delta$ then $\|x(t)\| < \varepsilon$, $t \ge 0$. Suppose that this is not true. Then, there exists a solution $\overline{x}(t)$ of (2) with $\|\overline{x}(0)\| < \delta$ such that $\|\overline{x}(t_1)\| = \varepsilon$ and $\|\overline{x}(t)\| \le \varepsilon$ for $t \in [0, t_1]$ so that

$$V(t_1, x(t_1)) \ge b(\varepsilon) \tag{3.6}$$

taking into account (4). Furthermore, this means that $x(t_1) \in S_r$, $t \in [0, t_1]$. Hence the condition ii) and the inequality (4) gives the estimate

$$V(t, x(t)) \le V(0, x_0) \le a(\|x_0\|) \tag{3.7}$$

Now, (6), (7) and (5) lead to contradiction

$$b(\varepsilon) \le V(t_1, x(t_1)) \le a(\|x_0\|) \le a(\delta) < b(\varepsilon)$$

This proves that the trivial solution of (2) is uniformly stable. □

Theorem 3.3. *Suppose $f : \mathbb{R}^n \times C \to \mathbb{R}^n$ in (2) maps $\mathbb{R}^n \times A$, with A bounded sets in C, into bounded sets in $\mathbb{R}^n$ and $a, b, c : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous decreasing functions, where additionally $a(s)$, $b(s)$ are positive for $s > 0$ and $a(0) = b(0) = 0$. If there exists a continuously differentiable function*

(i) $V : [0,+\infty) \times S_r \to \mathbb{R}$, *such that* $a(\|x_0\|) \le V(t,x) \le b(\|x\|)$, *and*

(ii) ${}_0^C D_t^\alpha V(t,x) \le -c(\|x(t)\|)$, $\alpha \in (0,1)$.

then, the trivial solution of (2) is uniformly stable. If, in addition

$$\underset{s\to+\infty}{Lim}\ a(s) = \infty$$

then it is globally uniformly asymptotically stable.

Proof. For any $\varepsilon > 0$, since b is continuous and $b(0) = 0$ we can find a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that $b(\delta) < a(\varepsilon)$. Hence, for any initial time and any initial condition $x(0) = x_0$ with $\|x_0\| < \delta$, we have

$$ {}_0^C D_t^{\alpha} V(t, x(t)) \leq 0 $$

and therefore

$$ V(t, x(t)) \leq V(0, x_0) $$

for any $t \geq 0$. This implies a contradiction as in the previous theorem. This proves the uniform stability.

To prove uniform asymptotic stability, let $0 < \varepsilon < r$ and $\delta = \delta(\varepsilon) > 0$ correspond to uniform stability. Choose an $\varepsilon_0 \leq r$ and designate by $\delta_0 = \delta(\varepsilon_0) > 0$ where ε_0 is fixed. Let us now choose $\|x_0\| < \delta_0$, $T(\varepsilon) = \left[\frac{b(\delta_0)}{c(\delta(\varepsilon))}\Gamma(1+\alpha)\right]^{\frac{1}{\alpha}}$ and we would have $\|x(t)\| \geq \delta(\varepsilon)$ for all $t \geq 0$. Therefore

$$ {}_0^C D_t^{\alpha} V(t, x(t)) \leq -c\left(\delta(\varepsilon)\right), \ for \ t \geq 0 $$

and hence by properties 2 and 3 we conclude

$$ {}_0^C D_t^{\alpha} \left[V(t, x(t)) + c\left(\delta(\varepsilon)\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \right] \leq 0 $$

then by using the Property 4 we have

$$ V(t, x(t)) + c\left(\delta(\varepsilon)\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} \leq V(0, x_0) $$

As a result we obtain

$$ V(t, x(t)) \leq b(\|x_0\|) - c\left(\delta(\varepsilon)\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} $$

$$ V(t, x(t)) \leq b(\delta_0) - c\left(\delta(\varepsilon)\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)} $$

which for $t = t_0 + T(\varepsilon)$, reduces to

$$ 0 < a(\delta(\varepsilon)) \leq V(T, x) \leq b(\delta_0) - c\left(\delta(\varepsilon)\right) \frac{T^{\alpha}}{\Gamma(1+\alpha)} = 0. $$

This contradiction proves that there exists a $t_1 \in [0, T(\varepsilon)]$ such that $\|x(t_1)\| < \delta(\varepsilon)$. Thus, in any case we have $\|x(t)\| < \varepsilon$, for $t \geq T(\varepsilon)$, whenever

$\|x_0\| < \delta_0$, proving the uniform asymptotic stability of the trivial solution of (2).

Finally, if $\underset{s\to+\infty}{Lim}\ c(s) = \infty$, then δ_0 above may be arbitrary large, and ε can be chosen after δ_0 is given to satisfy $b(\delta_0) < a(\varepsilon)$, and therefore global asymptotic stability can be concluded. □

Remark 3.4. *We observe from the above proof that a, b, c and V need only to be defined in a neighborhood of zero except for the case of global stability. We also notice that the lower bound of V need only to be a positive function of x.*

4 An application to system (1)

Now we will give a result on the stability of the fractional system of Liénard type (1).

Theorem 4.1. *Let $x(t)$ and $y(t)$ continuous and derivable functions. If f and g are continuous functions satisfying:*

(i) $f : \mathbb{R} \to \mathbb{R}_+$ a Lipschitzian function,

(ii) $g : \mathbb{R} \to \mathbb{R}$ and $xg(x) > 0$ for $x \neq 0$,

then, the trivial solution of system (1) is stable.

Proof. We consider the following Lyapunov Function

$$V(x, y) = G(x) + \frac{y^2}{2}. \tag{4.8}$$

With $G(x) = \int_0^x g(s)ds$. We calculate the fractional derivative of (8) along the system (1):

$${}_0^C D_t^\alpha V(x(t), y(t)) = {}_0^C D_t^\alpha \left[G(x(t))\right] + {}_0^C D_t^\alpha \left[\frac{y^2(t)}{2}\right].$$

Using the Lemma 2.11 we have

$${}_0^C D_t^\alpha V(x(t), y(t)) \le g(x) \left[{}_0^C D_t^\alpha x(t)\right] + y(t) \left[{}_0^C D_t^\alpha y(t)\right],$$

$${}_0^C D_t^\alpha V(x(t), y(t)) \le g(x) \left[y - F(x)\right] + y(t) \left[-g(x)\right],$$

$$ {}_0^C D_t^\alpha V(x(t), y(t)) \leq -g(x)F(x). $$

Under conditions on f and g, we have that

$$ {}_0^C D_t^\alpha V(x(t), y(t)) \leq 0. \tag{4.9} $$

(8) and (9) completes the proof. □

References

[1] N. Aguila Camacho, M. A. Duarte Mermoud, J. A. Gallegos, Lyapunov functions for fractional order systems, Commun. Nonlinear Sci. Numer. Simulat., **19,** (2014), 2951–2957.

[2] N. Aguila-Camacho, M. A. Duarte Mermoud, Boundedness of the solutions for certain classes of fractional differential equations with application to adaptive system,s ISA Transactions, **60,** (2016), 82–88.

[3] J. Baranowski, M. Zagorowska, W. Bauer, T. Dziwinski, P. Piatek, Applications of Direct Lyapunov Method in Caputo Non-Integer Order Systems, Electronika IR Electrotechnika, ISSN 1392-1215, **21,** No. 2, 2015.

[4] T. A. Burton, Fractional differential equations and Lyapunov functionals, Nonlinear Anal., **74,** (2011), 5648–5662.

[5] Michele Caputo, Linear model of dissipation whose Q is almost frequency independent-II, Geophys. J. R. Astr. Soc. **13,** (1967), 529–539.

[6] D. Chen, R. Zhang, X. Liu, X. Ma, Fractional order Lyapunov stability theorem and its applications in synchronization of complex dynamical networks, Commun. Nonlinear Sci. Numer. Simulat., **19,** (2014), 4105–4121.

[7] H. Delavari, D. Baleanu, J. Sadati, Stability analysis of Caputo fractional order nonlinear systems revisited, Nonlinear. Dyn., **67,** (2012), 2433–2439.

[8] M. A. Duarte Mermoud, N. AguilaCamacho, J. A. Gallegos, R. Castro-Linares, Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, Commun. Nonlinear Sci. Numer. Simulat., **22,** (2015), 650–659.

[9] M. A. Duarte Mermoud, N. Aguila Camacho, Fractional order adaptive control of simple systems, In Fifteenth Yale workshop on adaptive and learning systems, K. Narendra (editor), (2011), 57–62.

[10] O. El Figuigui N. Elalami, Application of fractional adaptive High-Gain controller to a LEO (Low Earth Orbit) satellite, In International conference on computers, industrial engineering, (2009), 1850–1856.

[11] J. D. Gabano,T. Poinot, Fractional modeling and identification of thermal systems, Signal Process, **91,** 2011, 531–541.

[12] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Applied Mathematical Sciences, Vol. 42, Springer-Verlag, New York, 2002, Revised and corrected reprint of the 1983 original.

[13] J. B. Hu, Guo-Ping Lu, Shi-Bing Zhang, Ling-Dong Zhao, Lyapunov stability theorem about fractional system without and with delay, Commun. Nonlinear Sci. Numer. Simulat., **20,** (2015), 905–913.

[14] A. Kilbas, H. Srivastava, J. Trujillo, Theory and applications of fractional differential equations, Elsevier, 2006.

[15] S. Ladaci, J. J. Loiseau, A. Charef, Fractional order adaptive high-gain controllers for a class of linear systems, Commun. Nonlinear Sci. Numer. Simul., **13,** (2008), 707–714.

[16] V. Lakshmikantham, S. Leela, M. Sambandham, Lyapunov theory for fractional differential equations, Commun. Appl. Anal., **12,** (2008), 365–376.

[17] V. Lakshmikantham, S. Leela, S. J. Vasundara Devi, Theory of fractional dynamic systems, Cambridge Scientific Publishers, 2009.

[18] . Li, Y. Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag Leffler stability, Comput. Math. Appl., **59,** (2010), 1810–1821.

[19] A. Liénard, Étude des oscillations entretenues, Revue Génerale de l'Électricité, **23,** (1928), 901–912, 946–954.

[20] J. Lin, T. Poinot, J. C. Trigeassou, H. Kabbaj, J. Faucher, Modélisation et identification d'ordre non entire d'une machine asynchrone, In Conférence Internationale Francophone d'Automatique, 2000.

[21] S. Liu, W. Jiang, X. Li, X. F. Zhou, Lyapunov stability analysis of fractional nonlinear systems, Applied Mathematics Letters, **51,** (2016), 13–19.

[22] Juan E. Nápoles Valdes, A note on the asymptotic stability in the whole of nonautonomous systems, Revista Colombiana de Matemáticas, **33,** (1999), 1–8.

[23] I. Podlubny, Fractional differential equations, Academic Press, 1999.

[24] J. Sabatier, M. Aoun, A. Oustaloup, G. Gréegoire, F. Ragot, P. Roy, Fractional system identification for lead acid battery state of charge estimation, Signal Process, **86,**(2006), 2645–2657.

[25] Cemil Tunc, Some new stability and boundedness results of solutions of Lienard type equations with deviating argument, Nonlinear Anal. Hybrid Syst., **4,** (2010), 85–91.

[26] Cemil Tunc, New stability and boundedness results of solutions of Lienard type equations with multiple deviating arguments, J. Contemp. Math. Anal., **45,** (2010, 47–56.

[27] B. Van der Pol, On oscillation hysteresis in a triode generator with two degrees of freedom, Phil. Mag. **43,** no. 6, (1922), 700–719.

[28] B. Van der Pol, On relaxation-oscillations, Phil. Mag., **11,** no. 2, (1926), 978–992.

[29] C. Vargas-De-León, Volterra-type Lyapunov functions for fractional-order epidemic systems, Commun. Nonlinear Sci. Numer. Simulat. **24,** (2015), 75–85.

[30] H. Ya Li, G. Rui Kun, Application of fractional-order model reference adaptive control on industry boiler burning system, In International conference on intelligent computation technology and automation (ICICTA), **1,** (2010), 750–753.

[31] M. Zamani, M. Karimi Ghartemani, N. Sadati, M. Parniani, Design of a fractional order PID controller for an AVR using particle swarm optimization, Control Eng. Pract., **17,** (2009), 1380–1387.