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Determinants Order Decrease/Increase for k Orders, Interpretation with Computer Algorithms and Comparison

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Abstract

In this paper we present the possibility of decreasing/increasing the order of determinants for k orders. Also in this paper we have presented the computer interpretation by introducing appropriate algorithms for decreasing/increasing the order of determinant. The decrease algorithem is computationally compared with the Laplace method of determinant calculation.

1 Introduction

For the calculation of determinants, various forms and methods have been developed [1, 2, 3, 4, 5, 6, 8, 11, 12, 14, 15, 17], each of the methods has several advantages over the others, but the calculation of determinants manually requires time and resources, so different algorithms have been developed for

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their computational calculation, where one of the most commonly used tools is that of MATLAB [7, 9, 10, 13, 14, 16].

Lemma 1.1: [1] For the determinant $|A|_{n\times n}$ the following formula applies:

$$det\left[(a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}}\right]\cdot det\left[(a_{ij})_{\substack{2\le i\le n-1\\ 1\le j\le n-2}}\right]$$
$$= det\left[(a_{ij})_{\substack{i\ne n\\ j\ne n}}\right]\cdot det\left[(a_{ij})_{\substack{i\ne 1\\ j\ne n-1}}\right] - det\left[(a_{ij})_{\substack{i\ne n\\ j\ne n-1}}\right]\cdot det\left[(a_{ij})_{\substack{i\ne 1\\ j\ne n}}\right].$$

Lemma 1.1 is special case of theorem 3.2 in [1], for $k = 1$, $l = n - 1$ and $k' = l' = n$.

2 Determinants Order Decrease and Intepretation with Computer Algorithm

In this section we have researched the possibility of determinants order decrease for k orders, also we provide the computer algorithm to decrease the order of determinants.

Theorem 2.1: Let $|A|$ be a nth order, the order of determinant is decreased for k orders according to the formula below, n, k are positive integers and $1 \le k < n - 1$:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{n-k}|B_{i1}|_{k\times k}}\begin{vmatrix} |B_{11}|_{t\times t} & |B_{12}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t}\\ |B_{21}|_{t\times t} & |B_{22}|_{t\times t} & \cdots & |B_{2,n-k}|_{t\times t}\\ \vdots & \vdots & \ddots & \vdots\\ |B_{n-k,1}|_{t\times t} & |B_{n-k,2}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t}\end{vmatrix}_{(n-k)\times(n-k)},$$

where:

$$|B_{ij}|_{t\times t} = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{i,k} & a_{i,k+j}\\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,k} & a_{i+1,k+j}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{i+k-1,1} & a_{i+k-1,2} & \cdots & a_{i+k-1,k} & a_{i+k-1,k+j}\\ a_{i+k,1} & a_{i+k,2} & \cdots & a_{i+k,k} & a_{i+k,k+j}\end{vmatrix}_{t\times t},$$

$t = k+1$, $1 \le i, j \le n-k$ and $|B_{i1}|_{k\times k} \neq 0$ is $|B_{i1}|_{t\times t}$, removing last row and last column.

Proof: First we will decrease order of determinants for $k = 1$.

In the nth order determinant, let us multiply elements of every row expect the first row with the first element of previous row, as follows:

$$|A|_{n\times n} = \frac{1}{a_{11}\cdot a_{21}\cdot\dots\cdot a_{n-2,1}\cdot a_{n-1,1}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11}a_{21} & a_{11}a_{22} & \cdots & a_{11}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1}a_{n-1,1} & a_{n-2,1}a_{n-1,2} & \cdots & a_{n-2,1}a_{n-1,n} \\ a_{n-1,1}a_{n,1} & a_{n-1,1}a_{n,2} & \cdots & a_{n-1,1}a_{n,n} \end{vmatrix}_{n\times n}$$

Let us arrange the last determinant, that first column elements are equal to zero expect first element, than the determinant is expanded according to the first column (Laplace Method), as follows:

$$|A|_{n\times n} = \frac{1}{a_{11}\cdot a_{21}\cdot\dots\cdot a_{n-1,1}} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & \cdots & a_{11}a_{2n}-a_{1n}a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1,1}a_{n2}-a_{n1}a_{n-1,2} & \cdots & a_{n-1,1}a_{nn}-a_{n1}a_{n-1,n} \end{vmatrix}_{n\times n}$$

$$= \frac{a_{11}}{a_{11}\cdot a_{21}\cdot\dots\cdot a_{n-1,1}} \begin{vmatrix} a_{11}a_{22}-a_{12}a_{21} & \cdots & a_{11}a_{2n}-a_{1n}a_{21} \\ \vdots & \ddots & \vdots \\ a_{n-1,1}a_{n2}-a_{n1}a_{n-1,2} & \cdots & a_{n-1,1}a_{nn}-a_{n1}a_{n-1,n} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{1}{a_{21}\cdot a_{31}\cdot\dots\cdot a_{n-1,1}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{n-1,1} & a_{n-1,2} \\ a_{n1} & a_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{n-1,1} & a_{n-1,n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{1}{\prod_{i=2}^{n-1}|B_{i1}|_{1\times 1}} \begin{vmatrix} |B_{11}|_{2\times 2} & \cdots & |B_{1,n-1}|_{2\times 2} \\ \vdots & \ddots & \vdots \\ |B_{n-1,1}|_{2\times 2} & \cdots & |B_{n-1,n-1}|_{2\times 2} \end{vmatrix}_{(n-1)\times(n-1)}.$$

The last formula proofs that Theorem 2.1 stands for $k = 1$.

Now we will proof that Theorem 2.1 stands for $k = 2$. From the last formula let us multiply elements/blocks of every row expect the first row with the first element/block of previous row, as follows:

$$|A|_{n\times n}=\frac{1}{\prod_{i=2}^{v}|B_{i1}|_{1\times 1}}\cdot\frac{1}{\prod_{i=1}^{u}|B_{i1}|_{2\times 2}}$$
$$\cdot\begin{vmatrix} |B_{11}|_{2\times 2} & |B_{12}|_{2\times 2} & \cdots & |B_{1,v}|_{2\times 2}\\ |B_{11}|_{2\times 2}\,|B_{21}|_{2\times 2} & |B_{11}|_{2\times 2}\,|B_{22}|_{2\times 2} & \cdots & |B_{11}|_{2\times 2}\,|B_{2,v}|_{2\times 2}\\ \vdots & \vdots & \ddots & \vdots\\ |B_{u,1}|_{2\times 2}\,|B_{v,1}|_{2\times 2} & |B_{u,1}|_{2\times 2}\,|B_{v,2}|_{2\times 2} & \cdots & |B_{u,1}|_{2\times 2}\,|B_{v,v}|_{2\times 2}\end{vmatrix}_{v\times v},$$

where: $u=n-2$ and $v=n-1$.

Let us arrange the last determinant, that first column elements/blocks are equal to zero expect first element/block and remove first column, as follows:

$$|A|_{n\times n}=\frac{1}{\prod_{i=2}^{v}|B_{i1}|_{1\times 1}}\cdot\frac{1}{\prod_{i=2}^{u}|B_{i1}|_{2\times 2}}\cdot$$
$$\begin{vmatrix} |B_{11}|_{2\times 2}\cdot|B_{22}|_{2\times 2}-|B_{12}|_{2\times 2}\cdot|B_{21}|_{2\times 2} & \cdots & |B_{11}|_{2\times 2}\cdot|B_{2,v}|_{2\times 2}-|B_{1,v}|_{2\times 2}\cdot|B_{21}|_{2\times 2}\\ \vdots & \ddots & \vdots\\ |B_{u,1}|_{2\times 2}\cdot|B_{v,2}|_{2\times 2}-|B_{u,2}|_{2\times 2}\cdot|B_{v,1}|_{2\times 2} & \cdots & |B_{u,1}|_{2\times 2}\cdot|B_{v,v}|_{2\times 2}-|B_{u,v}|_{2\times 2}\cdot|B_{v,1}|_{2\times 2}\end{vmatrix}_{u\times u}.$$

According to Lemma 1.1, we have:

$$|B_{i,1}|_{2\times 2}\cdot|B_{i+1,j+1}|_{2\times 2}-|B_{i,j+1}|_{2\times 2}\cdot|B_{i+1,1}|_{2\times 2}$$
$$=\begin{vmatrix} a_{i,1} & a_{i,2}\\ a_{i+1,1} & a_{i+1,2}\end{vmatrix}\cdot\begin{vmatrix} a_{i+1,1} & a_{i+1,j+2}\\ a_{i+2,1} & a_{i+2,j+2}\end{vmatrix}-\begin{vmatrix} a_{i,1} & a_{i,j+2}\\ a_{i+1,1} & a_{i+1,j+2}\end{vmatrix}\cdot\begin{vmatrix} a_{i+1,1} & a_{i+1,2}\\ a_{i+2,1} & a_{i+2,2}\end{vmatrix}$$
$$=a_{i+1,1}\cdot\begin{vmatrix} a_{i,1} & a_{i,2} & a_{i,j+2}\\ a_{i+1,1} & a_{i+1,2} & a_{i+1,j+2}\\ a_{i+2,1} & a_{i+2,2} & a_{i+2,j+2}\end{vmatrix}=|B_{i+1,1}|_{1\times 1}\cdot|B_{ij}|_{3\times 3}.$$

Then we have:

$$|A|_{n\times n}=\frac{1}{\prod_{i=2}^{v}|B_{i1}|_{1\times 1}}\cdot\frac{1}{\prod_{i=2}^{u}|B_{i1}|_{2\times 2}}\cdot$$
$$\cdot\begin{vmatrix} |B_{21}|_{1\times 1}\cdot|B_{11}|_{3\times 3} & \cdots & |B_{21}|_{1\times 1}\cdot|B_{1,u}|_{3\times 3}\\ \vdots & \ddots & \vdots\\ |B_{v,1}|_{1\times 1}\cdot|B_{u,1}|_{3\times 3} & \cdots & |B_{v,1}|_{1\times 1}\cdot|B_{u,u}|_{3\times 3}\end{vmatrix}_{u\times u}$$
$$=\frac{1}{\prod_{i=2}^{u}|B_{i1}|_{2\times 2}}\cdot\begin{vmatrix} |B_{11}|_{3\times 3} & \cdots & |B_{1,u}|_{3\times 3}\\ \vdots & \ddots & \vdots\\ |B_{u,1}|_{3\times 3} & \cdots & |B_{u,u}|_{3\times 3}\end{vmatrix}_{u\times u}.$$

This proofs that Theorem 2.1 stands for $k = 2$.

Let us suppose that Theorem 2.1 stands for $k-1$:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}} \cdot \begin{vmatrix} |B_{11}|_{k\times k} & |B_{12}|_{k\times k} & \cdots & |B_{1,t_1}|_{k\times k} \\ |B_{21}|_{k\times k} & |B_{22}|_{k\times k} & \cdots & |B_{2,t_1}|_{k\times k} \\ \vdots & \vdots & \ddots & \vdots \\ |B_{t_1 1}|_{k\times k} & |B_{t_1 2}|_{k\times k} & \cdots & |B_{t_1 t_1}|_{k\times k} \end{vmatrix}_{t_1\times t_1},$$

where $t_1 = n-(k-1)$.

We will proof that Theorem 2.1 stands for k. From the previous formula, we will multiply block-elements of every row expect the first row with the first block-element of previous row:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}} \cdot \frac{1}{\prod_{i=1}^{n-k} |B_{i1}|_{k\times k}} \cdot$$

$$\cdot \begin{vmatrix} |B_{11}|_{k\times k} & |B_{12}|_{k\times k} & \cdots & |B_{1,t_1}|_{k\times k} \\ |B_{11}|_{k\times k}\cdot|B_{21}|_{k\times k} & |B_{11}|_{k\times k}\cdot|B_{22}|_{k\times k} & \cdots & |B_{11}|_{k\times k}\cdot|B_{2,t_1}|_{k\times k} \\ \vdots & \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{k\times k}\cdot|B_{t_1,1}|_{k\times k} & |B_{n-k,1}|_{k\times k}\cdot|B_{t_1,2}|_{k\times k} & \cdots & |B_{n-k,1}|_{k\times k}\cdot|B_{t_1,t_1}|_{k\times k} \end{vmatrix}_{t_1\times t_1}.$$

Let us arrange the last determinant, that first column elements are equal to zero expect first element and calculate determinant using minors of first column, as follows:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}} \cdot \frac{|B_{11}|_{k\times k}}{\prod_{i=1}^{n-k} |B_{i1}|_{k\times k}} \cdot \begin{vmatrix} |B_{11}|_{k\times k}\cdot|B_{22}|_{k\times k} - |B_{12}|_{k\times k}\cdot|B_{21}|_{k\times k} & \cdots & |B_{11}|_{k\times k}\cdot|B_{2,t_1}|_{k\times k} - |B_{1,t_1}|_{k\times k}\cdot|B_{21}|_{k\times k} \\ \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{k\times k}\cdot|B_{t_1,2}|_{k\times k} - |B_{n-k,2}|_{k\times k}\cdot|B_{t_1,1}|_{k\times k} & \cdots & |B_{n-k,1}|_{k\times k}\cdot|B_{t_1,t_1}|_{k\times k} - |B_{n-k,t_1}|_{k\times k}\cdot|B_{t_1,1}|_{k\times k} \end{vmatrix}_{(n-k)\times(n-k)}.$$

According to Lemma 1.1 we have:

$$|B_{i,1}|_{k\times k}\cdot|B_{i+1,j+1}|_{k\times k} - |B_{i,j+1}|_{k\times k}\cdot|B_{i+1,1}|_{k\times k}$$

$$= \begin{vmatrix} a_{i1} & \cdots & a_{i,k-1} & a_{ik} \\ a_{i+1,1} & \cdots & a_{i+1,k-1} & a_{i+1,k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i+k-2,1} & \cdots & a_{i+k-2,k-1} & a_{i+k-2,k} \\ a_{i+k-1,1} & \cdots & a_{i+k-1,k-1} & a_{i+k-1,k} \end{vmatrix} \begin{vmatrix} a_{i+1,1} & \cdots & a_{i+1,k-1} & a_{i+1,k+j} \\ a_{i+2,1} & \cdots & a_{i+2,k-1} & a_{i+2,k+j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i+k-1,1} & \cdots & a_{i+k-1,k-1} & a_{i+k-1,k+j} \\ a_{i+k,1} & \cdots & a_{i+k,k-1} & a_{i+k,k+j} \end{vmatrix}$$

$$-\begin{vmatrix} a_{i1} & \cdots & a_{i,k-1} & a_{i,k+j} \\ a_{i+1,1} & \cdots & a_{i+1,k-1} & a_{i+1,k+j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i+k-2,1} & \cdots & a_{i+k-2,k-1} & a_{i+k-2,k+j} \\ a_{i+k-1,1} & \cdots & a_{i+k-1,k-1} & a_{i+k-1,k+j} \end{vmatrix} \begin{vmatrix} a_{i+1,1} & \cdots & a_{i+1,k-1} & a_{i+1,k} \\ a_{i+2,1} & \cdots & a_{i+2,k-1} & a_{i+2,k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i+k-1,1} & \cdots & a_{i+k-1,k-1} & a_{i+k-1,k} \\ a_{i+k,1} & \cdots & a_{i+k,k-1} & a_{i+k,k} \end{vmatrix}$$

$$= \begin{vmatrix} a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,k-2} & a_{i+1,k-1} \\ a_{i+2,1} & a_{i+2,2} & \cdots & a_{i+2,k-2} & a_{i+2,k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+k-2,1} & a_{i+k-2,2} & \cdots & a_{i+k-2,k-2} & a_{i+k-2,k-1} \\ a_{i+k-1,1} & a_{i+k-1,2} & \cdots & a_{i+k-1,k-2} & a_{i+k-1,k-1} \end{vmatrix}_{(k-1)\times(k-1)}$$

$$\cdot \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{i,k-2} & a_{i,k-1} & a_{i,k} & a_{i,k+j} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,k-2} & a_{i+1,k-1} & a_{i+1,k} & a_{i+1,k+j} \\ a_{i+2,1} & a_{i+2,2} & \cdots & a_{i+2,k-2} & a_{i+2,k-1} & a_{i+2,k} & a_{i+2,k+j} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{i+k-2,1} & a_{i+k-2,2} & \cdots & a_{i+k-2,k-2} & a_{i+k-2,k-1} & a_{i+k-2,k} & a_{i+k-2,k+j} \\ a_{i+k-1,1} & a_{i+k-1,2} & \cdots & a_{i+k-1,k-2} & a_{i+k-1,k-1} & a_{i+k-1,k} & a_{i+k-1,k+j} \\ a_{i+k,1} & a_{i+k,2} & \cdots & a_{i+k,k-2} & a_{i+k,k-1} & a_{i+k,k} & a_{i+k,k+j} \end{vmatrix}_{t\times t}$$

$$= |B_{i+1,1}|_{(k-1)\times(k-1)} \cdot |B_{i,j}|_{t\times t}$$

Then we have:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}} \cdot \frac{1}{\prod_{i=2}^{n-k} |B_{i1}|_{k\times k}} \cdot$$

$$\cdot \begin{vmatrix} |B_{21}|_{(k-1)\times(k-1)} \cdot |B_{11}|_{t\times t} & \cdots & |B_{21}|_{(k-1)\times(k-1)} \cdot |B_{1,n-k}|_{t\times t} \\ \vdots & \ddots & \vdots \\ |B_{t_1,1}|_{(k-1)\times(k-1)} \cdot |B_{n-k,1}|_{t\times t} & \cdots & |B_{t_1,1}|_{(k-1)\times(k-1)} \cdot |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)}$$

$$= \frac{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}}{\prod_{i=2}^{t_1} |B_{i1}|_{(k-1)\times(k-1)}} \cdot \frac{1}{\prod_{i=2}^{n-k} |B_{i1}|_{k\times k}} \cdot \begin{vmatrix} |B_{11}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t} \\ \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)}$$

$$= \frac{1}{\prod_{i=2}^{n-k} |B_{i1}|_{k\times k}} \cdot \begin{vmatrix} |B_{11}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t} \\ \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)}.$$

This proofs that Theorem 2.1 stands for k.

In the following we provide the computer algorithm to reduce the order of determinant:

Algorithm 2.1: Decrease order of determinants

Step 1: Insert order of determinant

Step 2: Insert the determinant

Step 3: Insert the order to reduce the determinant

Step 4: Calculating submatrices

Create Loop for i from 1 to $n-k$

Create Loop for j from 1 to $n-k$

$B(i,j) = det(A([i:i+k-1 \;\; i+k],[1:k \;\; j+k]));$

$N(i+1,1) = det(A([i+1:i+k],[1:k]));$

Step 5: Display determinant of submatrices

B

$C = N(2:n-k)$

Step 6: Calculate the decreased determinant

A_reduced = det of matrix B/product of submatrices $C = (N(2:n-k))$

Example 2.1: Let $|A|$ be determinant of 6th order, according to the schema/formula given above we will decrease the order of determinant for 3 orders, $(n=6, k=3)$:

$$|A| = \begin{vmatrix} 3 & 5 & -1 & 0 & 5 & -8 \\ -1 & 4 & -2 & 4 & 0 & 0 \\ 1 & 7 & 0 & 5 & 6 & -3 \\ 2 & -2 & -5 & 3 & 0 & -4 \\ -4 & 2 & 0 & 1 & -5 & 3 \\ 5 & -6 & 2 & 0 & 7 & 1 \end{vmatrix} = \frac{1}{\begin{vmatrix} -1 & 4 & -2 \\ 1 & 7 & 0 \\ 2 & -2 & -5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 7 & 0 \\ 2 & -2 & -5 \\ -4 & 2 & 0 \end{vmatrix}} .$$

$$\left|\begin{matrix} \begin{vmatrix} 3 & 5 & -1 & 0 \\ -1 & 4 & -2 & 4 \\ 1 & 7 & 0 & 5 \\ 2 & -2 & -5 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 5 & -1 & 5 \\ -1 & 4 & -2 & 0 \\ 1 & 7 & 0 & 6 \\ 2 & -2 & -5 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 5 & -1 & -8 \\ -1 & 4 & -2 & 0 \\ 1 & 7 & 0 & -3 \\ 2 & -2 & -5 & -4 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & 4 & -2 & 4 \\ 1 & 7 & 0 & 5 \\ 2 & -2 & -5 & 3 \\ -4 & 2 & 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 4 & -2 & 0 \\ 1 & 7 & 0 & 6 \\ 2 & -2 & -5 & 0 \\ -4 & 2 & 0 & -5 \end{vmatrix} & \begin{vmatrix} -1 & 4 & -2 & 0 \\ 1 & 7 & 0 & -3 \\ 2 & -2 & -5 & -4 \\ -4 & 2 & 0 & 3 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 7 & 0 & 5 \\ 2 & -2 & -5 & 3 \\ -4 & 2 & 0 & 1 \\ 5 & -6 & 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 7 & 0 & 6 \\ 2 & -2 & -5 & 0 \\ -4 & 2 & 0 & -5 \\ 5 & -6 & 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 7 & 0 & -3 \\ 2 & -2 & -5 & -4 \\ -4 & 2 & 0 & 3 \\ 5 & -6 & 2 & 1 \end{vmatrix} \end{matrix}\right|$$

$$= \frac{1}{87 \cdot 150} \cdot \begin{vmatrix} 428 & 231 & 191 \\ 57 & 33 & -213 \\ 807 & 333 & 387 \end{vmatrix} = \frac{1}{87 \cdot 150} \cdot (-10440000) = -800$$

Corollary 2.1: Based on the determinant transpose and Theorem 2.1, we conclude the following cases:

1. Up Side:

$$|A|_{n\times n} = \frac{1}{\prod_{j=2}^{n-k} |B_{1,j}|_{k\times k}} \begin{vmatrix} |B_{11}|_{t\times t} & |B_{12}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t} \\ |B_{21}|_{t\times t} & |B_{22}|_{t\times t} & \cdots & |B_{2,n-k}|_{t\times t} \\ \vdots & \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{t\times t} & |B_{n-k,2}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)},$$

where:

$$|B_{ij}|_{t\times t} = \begin{vmatrix} a_{1,j} & a_{1,j+1} & \cdots & a_{1,j+k-1} & a_{i,j+k} \\ a_{2,j} & a_{2,j+1} & \cdots & a_{2,j+k-1} & a_{2,j+k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k,j} & a_{k,j+1} & \cdots & a_{k,j+k-1} & a_{k,j+k} \\ a_{k+i,j} & a_{k+i,j+1} & \cdots & a_{k+i,j+k-1} & a_{k+i,j+k} \end{vmatrix}_{t\times t},$$

$t = k + 1$, $1 \leq i, j \leq n - k$ and $|B_{1,j}|_{k\times k} \neq 0$ is $|B_{1,j}|_{t\times t}$, removing last row and last column.

In this case the algorithm 2.1 changes in Step 4: Calculating submatrices, as follows:

Step 4: Calculating submatrices
Create Loop for j from 1 to $n-k$
Create Loop for i from 1 to $n-k$
$B(i,j) = det(A([1:k \;\; i+k],[j:j+k-1 \;\; j+k]));$
$N(j+1,1) = det(A([1:k],[j+1:j+k]));$

Example 2.2: Let us calculate the same example as provided in 2.1, but using up side:

$$|A| = \begin{vmatrix} 3 & 5 & -1 & 0 & 5 & -8 \\ -1 & 4 & -2 & 4 & 0 & 0 \\ 1 & 7 & 0 & 5 & 6 & -3 \\ 2 & -2 & -5 & 3 & 0 & -4 \\ -4 & 2 & 0 & 1 & -5 & 3 \\ 5 & -6 & 2 & 0 & 7 & 1 \end{vmatrix} = \frac{1}{\begin{vmatrix} 5 & -1 & 0 \\ 4 & -2 & 4 \\ 7 & 0 & 5 \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 & 5 \\ -2 & 4 & 0 \\ 0 & 5 & 6 \end{vmatrix}} \cdot \begin{vmatrix} 428 & -650 & -166 \\ -267 & 404 & 304 \\ 482 & -690 & -454 \end{vmatrix}$$

$$= \frac{1}{-58 \cdot (-74)} \cdot \begin{vmatrix} 428 & -650 & -166 \\ -267 & 404 & 304 \\ 482 & -690 & -454 \end{vmatrix} = \frac{1}{58 \cdot 74} \cdot (-3433600) = -800$$

2. Down Side:

$$|A|_{n\times n} = \frac{1}{\prod_{j=2}^{n-k} |B_{n-k,j}|_{k\times k}} \begin{vmatrix} |B_{11}|_{t\times t} & |B_{12}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t} \\ |B_{21}|_{t\times t} & |B_{22}|_{t\times t} & \cdots & |B_{2,n-k}|_{t\times t} \\ \vdots & \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{t\times t} & |B_{n-k,2}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)},$$

where:

$$|B_{ij}|_{t\times t} = \begin{vmatrix} a_{i,j} & a_{i,j+1} & \cdots & a_{i,j+k-1} & a_{i,j+k} \\ a_{n-k+1,j} & a_{n-k+1,j+1} & \cdots & a_{n-k+1,j+k-1} & a_{n-k+1,j+k} \\ a_{n-k+2,j} & a_{n-k+2,j+1} & \cdots & a_{n-k+2,j+k-1} & a_{n-k+2,j+k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-k+k,j} & a_{n-k+k,j+1} & \cdots & a_{n-k+k,j+k-1} & a_{n-k+k,j+k} \end{vmatrix}_{t\times t},$$

$t = k+1$, $1 \le i,j \le n-k$ and $|B_{n-k,j}|_{k\times k} \neq 0$ is $|B_{n-k,j}|_{t\times t}$, removing first row and last column.

In this case the algorithm 2.1 changes in Step 4: Calculating submatrices, as follows:

Step 4: Calculating submatrices
Create Loop for j from 1 to $n-k$
Create Loop for i from 1 to $n-k$
$B(i,j) = det(A([i \;\; n-k+1:n],[j:j+k-1 \;\; j+k]));$
$N(j+1,1) = det(A([n-k+1:n],[j+1:j+k]));$

Example 2.3: Let us calculate the same example as provided in 2.1, but using down side:

$$|A| = \begin{vmatrix} 3 & 5 & -1 & 0 & 5 & -8 \\ -1 & 4 & -2 & 4 & 0 & 0 \\ 1 & 7 & 0 & 5 & 6 & -3 \\ 2 & -2 & -5 & 3 & 0 & -4 \\ -4 & 2 & 0 & 1 & -5 & 3 \\ 5 & -6 & 2 & 0 & 7 & 1 \end{vmatrix} = \frac{1}{\begin{vmatrix} -2 & -5 & 3 \\ 2 & 0 & 1 \\ -6 & 2 & 0 \end{vmatrix} \cdot \begin{vmatrix} -5 & 3 & 0 \\ 0 & 1 & -5 \\ 2 & 0 & 7 \end{vmatrix}} \cdot \begin{vmatrix} 359 & -521 & -309 \\ 390 & -592 & -148 \\ 807 & -1231 & -519 \end{vmatrix}$$

$$= \frac{1}{46 \cdot (-65)} \cdot \begin{vmatrix} 359 & -521 & -309 \\ 390 & -592 & -148 \\ 807 & -1231 & -519 \end{vmatrix} = \frac{1}{46 \cdot (-65)} \cdot 2392000 = -800$$

3. Right Side:

$$|A|_{n\times n} = \frac{1}{\prod_{i=2}^{n-k} |B_{i,n-k}|_{k\times k}} \begin{vmatrix} |B_{11}|_{t\times t} & |B_{12}|_{t\times t} & \cdots & |B_{1,n-k}|_{t\times t} \\ |B_{21}|_{t\times t} & |B_{22}|_{t\times t} & \cdots & |B_{2,n-k}|_{t\times t} \\ \vdots & \vdots & \ddots & \vdots \\ |B_{n-k,1}|_{t\times t} & |B_{n-k,2}|_{t\times t} & \cdots & |B_{n-k,n-k}|_{t\times t} \end{vmatrix}_{(n-k)\times(n-k)},$$

where:

$$|B_{ij}|_{t\times t} = \begin{vmatrix} a_{i,j} & a_{i,n-k+1} & a_{i,n-k+2} & \cdots & a_{i,n-k+k} \\ a_{i+1,j} & a_{i+1,n-k+1} & a_{i+1,n-k+2} & \cdots & a_{i+1,n-k+k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+k-1,j} & a_{i+k-1,n-k+1} & a_{i+k-1,n-k+2} & \cdots & a_{i+k-1,n-k+k} \\ a_{i+k,j} & a_{i+k,n-k+1} & a_{i+k,n-k+2} & \cdots & a_{i+k,n-k+k} \end{vmatrix}_{t\times t},$$

$t = k+1$, $1 \le i,j \le n-k$ and $|B_{i,n-k}|_{k\times k} \neq 0$ is $|B_{i,n-k}|_{t\times t}$, removing last row and first column.

In this case the algorithm 2.1 changes in Step 4: Calculating submatrices, as follows:

Step 4: Calculating submatrices

Create Loop for i from 1 to $n-k$

Create Loop for j from 1 to $n-k$

$B(i,j) = det(A([i : i+k-1 \;\; i+k], [j \;\; n-k+1 : n]));$

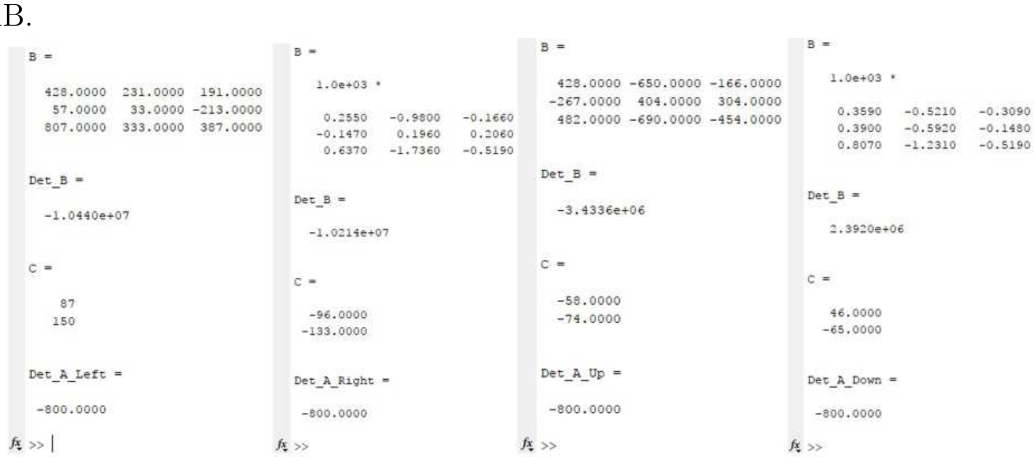
$N(i+1,1) = det(A([i+1 : i+k], [n-k+1 : n]));$

Example 2.4: Let us calculate the same example as provided in 2.1, but using right side:

$$|A| = \begin{vmatrix} 3 & 5 & -1 & 0 & 5 & -8 \\ -1 & 4 & -2 & 4 & 0 & 0 \\ 1 & 7 & 0 & 5 & 6 & -3 \\ 2 & -2 & -5 & 3 & 0 & -4 \\ -4 & 2 & 0 & 1 & -5 & 3 \\ 5 & -6 & 2 & 0 & 7 & 1 \end{vmatrix} = \frac{1}{\begin{vmatrix} 4 & 0 & 0 \\ 5 & 6 & -3 \\ 3 & 0 & -4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 6 & -3 \\ 3 & 0 & -4 \\ 1 & -5 & 3 \end{vmatrix}} \cdot \begin{vmatrix} 255 & -980 & -166 \\ -147 & 196 & 206 \\ 637 & -1736 & -519 \end{vmatrix}$$

$$= \frac{1}{-96 \cdot (-133)} \cdot \begin{vmatrix} 255 & -980 & -166 \\ -147 & 196 & 206 \\ 637 & -1736 & -519 \end{vmatrix} = \frac{1}{96 \cdot 133} \cdot (-10214400) = -800$$

Now we will calculate the same example using Algorithm 2.1 with MATLAB.



3 Determinants Order Increase and Intepretation with Computer Algorithm

In this section we have searched the possibility of determinants order increase for k orders, also we provide the computer algorithm to increase the order of determinants.

Theorem 3.1: If $|A| = (a_{i,j})_{n\times n}$ is a determinant of order $n \times n$, then:

$$|A|_{n\times n} =$$

$$= \begin{vmatrix} 1+b_{t,t} & b_{t,t} & c_{t,t-2} & \cdots & c_{t,n+1} & c_{t,1} & c_{t,2} & \cdots & c_{t,n} \\ 1+b_{t-1,t} & 1+b_{t-1,t} & b_{t-1,t} & \cdots & c_{t-1,n+1} & c_{t-1,1} & c_{t-1,2} & \cdots & c_{t-1,n} \\ 1+b_{t-2,t} & 1+b_{t-2,t} & 1+b_{t-2,t} & \cdots & c_{t-2,n+1} & c_{t-2,1} & c_{t-2,2} & \cdots & c_{t-2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+b_{n+1,1} & 1+b_{n+1,1} & 1+b_{n+1,1} & \cdots & 1+b_{n+1,1} & b_{n+1,1} & c_{n+1,2} & \cdots & c_{n+1,n} \\ a_{11} & a_{11} & a_{11} & \cdots & a_{11} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & a_{21} & \cdots & a_{21} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & a_{n1} & \cdots & a_{n1} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_{t\times t}$$

where c_{ij} and b_{ij} are random numbers, $t = n + k$ and n, k are positive integers.

In continues we will give only the description of proof of Theorem 3.1. Mathematical induction is used for the proof of this theorem, while in the steps of mathematical induction uses expansion of determinant based on Laplace (first row) and determinant properties.

In the following we provide the computer algorithm to increase the order of determinant:

Algorithm 3.1: Increase order of determinants

Step 1: Insert order of determinant
Step 2: Insert the order to increase the determinant
Step 3: Insert the determinant
Step 4: Insert the b and c elements
Step 5: Increasing the order of determinant
  Create Loop for i from 1 to $n + k$
  Create Loop for j from 1 to $n + k$
  Create Conditions:

```
        if $i = j + 1$ & $i > 1$ & $i < k + 2$ & $j < k + 2$;
            $D(i - 1, j + 1) = B(i - 1, j)$;
        elseif $i >= j$ & $i < k + 1$ & $j < k + 1$;
            Create Loop for $l$ from 1 to $j$
                $D(i, l) = 1 + B(i, j)$;
        elseif $j >= i + 1$ & $i < k + 1$ & $j > 1$ & $j < n + k$;
            $D(i, j + 1) = C(i, j - 1)$;
        elseif $i > k$ && $j > k$ && $j <= k + n$ && $i <= k + n$;
            $D(i, j) = A(i - k, j - k)$;
    Create Loop for $j$ from 1 to $k$
    Create Loop for $i$ from $k + 1$ to $k + n$
        $D(i, j) = A(i - k, 1)$;
    Step 6: Display the increased determinant
        D
    Step 7: Calculate the increased determinant
        A_increased = determinant of matrix $D$
```

Example 3.1: Let $|A|$ be a determinant of 3rd order, according to the schema/formula given above we will increase the order of determinant to the 7th order, $(n = 3, k = 4)$:

$$|A| = \begin{vmatrix} 5 & -2 & 3 \\ 1 & 0 & -2 \\ 7 & 4 & -4 \end{vmatrix} = 72$$

By applying Theorem 3.1, we can obtain the following determinant which has the same results with the above determinant:

$$|A| = \begin{vmatrix} 4 & 3 & 4 & 1 & -5 & 9 & -8 \\ 2 & 2 & 1 & -3 & 3 & 2 & 4 \\ -3 & -3 & -3 & -4 & -1 & 0 & 1 \\ 7 & 7 & 7 & 7 & 6 & 1 & 1 \\ 5 & 5 & 5 & 5 & 5 & -2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 \\ 7 & 7 & 7 & 7 & 7 & 4 & -4 \end{vmatrix} = 72$$

Now we will calculate the same example using Algorithm 3.1 with MATLAB.

```
A =

     5    -2     3
     1     0    -2
     7     4    -4


D =

     4     3     4     1    -5     9    -8
     2     2     1    -3     3     2     4
    -3    -3    -3    -4    -1     0     1
     7     7     7     7     6     1     1
     5     5     5     5     5    -2     3
     1     1     1     1     1     0    -2
     7     7     7     7     7     4    -4


Det_Increase_Left =

   72.0000
```

4 Computer Comparison of Execution Time and Algorithm Calls

In this section we have compared the computer execution time and algorithm calls of theorem 2.1, with the well-known Laplace method, the algorithm of Laplace method is as follows, developed by Rezaifar [14]. For better comparison, the submatrices on theorem 2.1, we have calculated using Laplace method.

Algorithm 4.1: Laplace method to calculate determinants

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Calculate determinants using Laplace Method
Create Loop for i from 1 to n
$d = d + [(-1)\hat{}(i+j) * A(1,i)] * det\_Laplace(A(2:n, [1:i-1\ \ i+1:n]))$;
Step 4: Display the result of the determinant

For computer computation of the execution time and algorithm calls of the determinant calculation, is used a computer with the following characteristics:

Table1: Computer characteristics used to simulate the calculation of determinants.

Name:	Lenovo
Model:	Ideapad 700-15ISK
CPU:	Intel Core i7 6700HQ 2.6Ghz
RAM:	16 GB DDR4
GPU:	FULL HD Display 15.6” 1920x1080, nVidia GTX 950 4096 mb dedicated graphics
HDD:	256 GB SSD

While software used for this simulation are presented in table 2.

Table 2: Computer tools used for determinant calculation simulation:

OS	Windows 10 Pro 64-bit, Version 1703 (OS Build 15063.483
Software	MATLAB, Version 9.0.0321247 (R2016a), 64-bit (win64)

The execution time of determinant calculation using the algorithm 2.1 for theorem 2.1 and Rezaifar algorithm of Laplace method, is presented in table 3 in seconds.

Table 3: Execution time presented in seconds:

Order of Det.	Laplace	Theorem 2.1				
		$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
3×3	0.0003	0.0012	N/A	N/A	N/A	N/A
4×4	0.0004	0.0068	0.0056	N/A	N/A	N/A
5×5	0.0006	0.0076	0.0070	0.0056	N/A	N/A
6×6	0.0024	0.0087	0.0077	0.0077	0.0064	N/A
7×7	0.0108	0.0082	0.0081	0.0086	0.0098	0.0122
8×8	0.0768	0.0176	0.0094	0.0093	0.0120	0.0217
9×9	0.6638	0.0797	0.0179	0.0112	0.0147	0.0323
10×10	6.4455	0.6576	0.0808	0.0203	0.0193	0.0465
11×11	71.6222	6.4871	0.6712	0.0871	0.0304	0.0661
12×12	874.3416	72.9083	6.6806	0.6977	0.1046	0.0986

Order of Det.	Theorem 2.1				
	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
8×8	0.0510	N/A	N/A	N/A	N/A
9×9	0.1046	0.3457	N/A	N/A	N/A
10×10	0.1876	0.7523	2.9049	N/A	N/A
11×11	0.2810	1.3298	6.4730	28.3171	N/A
12×12	0.4064	2.1232	11.6624	65.4192	313.6849

The algorithm calls of determinant calculation using the algorithm 2.1 for theorem 2.1 and Rezaifar algorithm of Laplace method, is presented in table 3 in total calls.

Table 4: Execution calls:

Order of Det.	Laplace	Theorem 2.1				
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
3×3	5	11	N/A	N/A	N/A	N/A
4×4	18	24	23	N/A	N/A	N/A
5×5	87	51	51	87	N/A	N/A
6×6	518	138	99	195	415	N/A
7×7	3,621	591	213	355	933	2,415
8×8	28,962	3,720	699	613	1,667	5,433
9×9	260,651	29,091	3,867	1,275	2,663	9,667
10×10	2,606,502	260,814	29,283	4,651	4,227	15,163
11×11	28,671,513	2,606,703	261,057	30,307	8,669	22,227
12×12	344,058,146	28,671,756	2,607,003	262,353	35,555	33,169

Order of Det.	Theorem 2.1				
	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
8×8	16,551	N/A	N/A	N/A	N/A
9×9	37,239	130,327	N/A	N/A	N/A
10×10	66,211	293,235	1,158,447	N/A	N/A
11×11	103,513	521,315	2,606,505	11,468,607	N/A
12×12	149,451	814,613	4,633,795	25,804,365	125,112,055

5 Conclusion

In this paper we have presented the theorem which decreases and increases the order of determinant from order n to any order k, we have also parsed the corresponding algorithms for decreasing or increasing the order of determinant. At the end of the paper we have presented the computational calculation of the determinant according to decrease algorithms presented with the Laplace algorithm developed by Rezaifar, the comparison results are elaborated in the following.

From the table 3 we can see that Laplace method is executed faster for the order less than or equal to six, but for higher orders, the algorithm 2.1, is executed much faster. While Table 3 shows that starting from the seventh order of the determinant, if the order of the determinant is even then the fastest execution of the algorithm 2.1 is done when $k = \frac{n}{2} - 1$, whereas if the order of the determinant is odd then the fastest execution of algorithm 2.1 is done when $k = \frac{n}{2} - 1.5$.

From the table 4 we can see that for the order 3 and 4, the Laplace method executes less algorithm calls than algorithm 2.1, while for the fifth order determinant, if we reduce the order of determinant for $k = 1, 2$, there are executed less algorithm calls with algorithm 2.1, and for $k = 3$ there are executed same number of algorithm calls as Laplace method. For order greater or equal to 6, the algorithm 2.1, executes less algorithm calls than Laplace method. From Table 4 it can be seen that starting from the sixth order of the determinant, for the even order of the determinant, the algorithm 2.1 executes less algorithm calls when $k = \frac{n}{2} - 1$, whereas for odd order of determinantn, the algorithm 2.1 has less algorithm calls when $k = \frac{n}{2} - 1.5$.

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