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Direct Solution of Second-Order System of ODEs Using Bernstein Polynomials on an Arbitrary Interval

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Abstract

In this paper, we propose a direct method based on the Bernstein polynomials for solving systems of second-order ordinary differential equations (ODEs) on an arbitrary interval. This method gives numerical solutions by converting the ODE system into a system of algebraic equations which can be solved easily. The approximate solutions are given in series form. Some numerical examples are given to show the applicability of the method.

1 Introduction

In recent years, many mathematical models appeared in problems in many fields of science and engineering. Some types of these models usually can be formulated in the form of differential equations, either as a first-order ODEs or higher-order ODEs. Finding accurate numerical and/or approximate solutions to these equations is very important.

Most of the existing purely numerical methods first reduce the second-order ODEs to a system of first-order ODEs. In 2009, Majid, Azmi and

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Suleiman [11] proposed a two-point four-step direct implicit block method for solving systems of second-order ODEs. Later, in 2011, Awoyemi et al [3] incorporated power series to create a modification of the block method for the direct solution of second-order ODEs. In 2012, Waeleh et al[18] developed a code for two-point block method for solving higher-order initial value problems for ODEs directly. Also, in 2012, Mukhtar et al [13] introduced a four-point direct block one-step method for solving directly second-order nonstiff initial value problems of ODEs. Another two-point, one-step block method for solving directly a general second-order ODEs was given by Abdul Majid et al [2]. In 2015, Kuboye and Omar [8] proposed a new six-step block method with power series as an approximate solution to solve second-order initial value problems of ODEs. A year later, Abdelrahim and Omar[1] presented a new hybrid block method based on collocation and interpolation techniques to solve second-order ODEs directly. Recently, a direct multistep block method for solving delay differential equations (DDEs) of second order was presented by Seong and Majid in [16].

There are papers discussing operational matrices using different type of polynomials. Recently, YüzbaşI and Ismailov [20] introduced an operational matrices using Taylor polynomials to solve linear Fredholm-Volterra integro-differential equations. In 2016, Gholami et al [9] used a pseudospectral integration matrix for solving single and multiterm time fractional diffusion equations. In addition, some special functions play an important role in many branches of physics and mathematics [17].

In this paper, we introduce a method for solving directly systems of second-order ODEs by means of operational matrices based on Bernstein polynomials which were introduced, in 1912, by Sergei Natanovich Bernstein [10] and have been used to solve many types of linear and non-linear equations due to the many interesting properties they possess, like continuity and unity partition (see [14]). Bernstein operational matrix of differentiation was proposed by [5]. Pandey and Kumar used Bernstein operational matrix to solve Emden-type equations. In 2007, Bhatti and Bracken [5] introduced a general form of Bernstein polynomials that have the same properties as the classical Bernstein polynomials defined on $[0,R,]$ where R is the maximum range over which the Bernstein polynomials form a complete basis. Furthermore, Jafarian et al [7] and Maleknejad et al [12] used Bernstein operational matrices to obtain an approximate solution for a system of Volterra-Fredholm integro-differential equations. Bataineh at al [4] applied operational matrices of Bernstein to solve delay differential equations by reducing it to a set of algebraic equations.

In this paper, we focus on the following system of second-order ODEs

$$\begin{aligned} y_1''(x) &= f_1'(x, y_1, y_1'), \quad y_1(a) = a_0, \quad y_1'(a) = b_0, \quad a \le x \le b, \\ &\vdots \\ y_m''(x) &= f_m(x, y_m, y_m'), \quad y_m(a) = a_m, \quad y_m'(a) = b_m. \end{aligned} \tag{1.1}$$

The capability of the method shall be tested on several test examples.

2 Definitions and Methodology

The Bernstein polynomials of degree n are defined as

$$\hat{B}_{v,n}(t) = \binom{n}{v} t^v (1-t)^{n-v}, \quad 0 \le v \le n \tag{2.2}$$

where

$$\binom{n}{v} = \frac{n!}{v!(n-v)!}, \quad v = 0, \dots, n. \tag{2.3}$$

These polynomials (2.2) form a complete basis for the vector space $\prod_n$ of polynomials of degree at most n. For convenience, $\hat{B}_{v,n}(t) = 0$ if $v < 0$, $n < v$ and $t \in [0, 1]$. These polynomials have many properties which make them important and useful, like continuity and unity partition (see [6]).

We note that [19] introduced an operational matrix of Bernstein polynomials on the interval $[0, 1]$. The Bernstein polynomials can be defined on an arbitrary interval $[a, b]$ (see [5]), using a normalization for $t = \frac{x-a}{b-a}$ on $[a, b]$, where the general formula will be

$$B_{v,n}(x) = \binom{n}{v} \frac{(x-a)^v (b-x)^{n-v}}{(b-a)^m}, \quad 0 \le v \le n \tag{2.4}$$

This general formula also forms a basis on $[a, b]$, and it has the same properties of the classical Bernstein $\hat{B}$. It can be easily shown that each Bernstein polynomial $B_{v,n}$ is positive and the sum of all Bernstein polynomials of degree n is $1, \forall x \in [a, b] \subseteq \Re$. To avoid confusion, we denoted by $\hat{B}$ the classical Bernstein polynomial and B the general Bernstein polynomials.

To apply our method on any arbitrary interval $[a, b]$, first of all, we should divide the interval $[a, b]$ into subintervals $[x_k, x_{k+1}]$ of length size $h = x_{k+1} - x_k$. Without loss of generality we take $h = 1$ and apply our method for each subinterval $[x_k, x_{k+1}]$.

As a result, any polynomial of degree n can be approximated by a linear combination of $B_{v,n}(x)$, $(v = 0, \ldots, n)$ as given below,

$$y(x) = \sum_{v=1}^{n} C_v B_{v,n} = C^T \Phi(x), \tag{2.5}$$

where $C^T = [C_0, C_1, \ldots, C_n]$ and $\Phi(x) = [B_{0,n}, B_{1,n}, \ldots, B_{n,n}]^T$.

We can also make the decomposition of the vector $\Phi(x)$ as a product of a square matrix of size $(n+1) \times (n+1)$ and a vector of size $(n+1) \times 1$; i.e., $\Phi(x) = AX$, where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ b_0 & b_1 & b_2 & \ldots & b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_0 & k_1 & k_2 & \ldots & k_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ x \\ x_2 \\ \vdots \\ x^n \end{bmatrix}. \tag{2.6}$$

We now have $y_i(x) = \sum_{v=1}^{n} C_{i,v} B_{v,n} = C_i^T \Phi(x)$, where $C_i^T = [C_{i,0}, C_{i,1}, \ldots, C_{i,n}]$ $(i = 1, \ldots, m)$. The derivative of the vector $\Phi(x)$, denoted by D', is

$$\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = D' \Phi(x), \tag{2.7}$$

where D' is the $(n+1) \times (n+1)$ operational matrix of the derivative that is given as $D' = A \sigma A^{-1}$, where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ b_0 & b_1 & b_2 & \ldots & b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_0 & k_1 & k_2 & \ldots & k_n \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{2.8}$$

Thus D' will take the following form:

$$A = \begin{bmatrix} \frac{-n}{h} & \frac{-1}{h} & 0 & 0 & \ldots & 0 \\ \frac{n}{h} & \frac{2-n}{h} & \frac{-2}{h} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \frac{-n}{h} \\ 0 & 0 & \ldots & 0 & \frac{1}{h} & \frac{n}{h} \end{bmatrix}. \tag{2.9}$$

Hence from (2.7) we have

$$\frac{\mathrm{d}^2\Phi(x)}{\mathrm{d}x} = (D')^2 \Phi(x), \ \ldots, \ \frac{\mathrm{d}^n\Phi(x)}{\mathrm{d}x} = (D')^n \Phi(x). \tag{2.10}$$

To use this operational matrix for solving a system of equations, we will approximate $y(x)$ by Bernstein polynomials as $y(x) = C^T\Phi(x)$ together with

$$y'(x) = C^T D'\Phi(x),$$
$$y''(x) = C^T (D')^2\Phi(x).$$

Hence system (1.1) can be written in the form

$$\begin{aligned} C_1^T(D')^2\Phi(x) &= f_1(x, C_1^T\Phi(x), C_1^T D'\Phi(x)), \quad C_1^T\Phi(x_k) = a_0, \quad C_1^T D'\Phi(x_k) = b_0 \\ &\vdots \\ C_m^T(D')^2\Phi(x) &= f_m(x, C_m^T\Phi(x), C_m^T D'\Phi(x)), \quad C_m^T\Phi(x_k) = a_m, \quad C_m^T D'\Phi(x_k) = b_m, \end{aligned} \tag{2.11}$$

We substitute the collocation nodes $x_k \leq x_0 < x_1 < \cdots < x_m \leq x_{k+1}$ in (2.11) to get a system of algebraic equations which will be solved by a computer algebra system like Maple. We use the Chebyshev roots for an arbitrary intervals $[x_k, x_{k+1}]$ as (2.12), and the intersection point of $\hat{B}_{v,n}(t), \hat{B}_{v,n-1}(t)$ as the collocation nodes

$$x_i = \frac{x_{k+1} + x_k}{2} + \frac{x_{k+1} - x_k}{2} \cos\left[(2i+1)\frac{\pi}{2n}\right], \quad i = 0, 1, \ldots, n-2. \tag{2.12}$$

The operational matrices discussed in this section are more general than those of Yousefi and Behroozifar [19], since we use the general Bernstein polynomials B with any step size h on $[a, b]$.

3 Test examples

In this section, we test the applicability of the proposed method on several systems of second-order ODEs. We apply the method with the number of Bernstein terms $n = 14$.

3.1 Example 1

First, Consider the following system [1].

$$y_1'' = -y_2', \quad y_1(0) = 0, \quad y_1'(0) = \frac{1}{1 - \mathrm{e}^{-1}}, \quad 0 \leq x \leq 10,$$
$$y_2'' = -y_1', \quad y_2(0) = 1, \quad y_2'(0) = \frac{1}{1 - \mathrm{e}^{-1}}.$$

This system admits the exact solution $y_1 = (1 - \mathrm{e}^{-x})/(1 - \mathrm{e}^{-1})$, $y_2 = (2 - \mathrm{e}^{-1} - \mathrm{e}^{-x})/(1 - \mathrm{e}^{-1})$. Rewrite the approximate solution of the system using Bernstein polynomials as

$$\begin{aligned} y_1(x) &= \sum_{v=0}^{14} C_{1,v} B_{v,14} = C_1^T \Phi(x), \\ y_2(x) &= \sum_{v=0}^{14} C_{2,v} B_{v,14} = C_2^T \Phi(x). \end{aligned}$$

The system above can be written as

$$C_1^T (D')^2 \Phi(x) + C_2^T D' \Phi(x) = 0, \tag{3.13}$$
$$C_2^T (D')^2 \Phi(x) + C_1^T D' \Phi(x) = 0, \tag{3.14}$$

with the initial conditions

$$C_1^T \Phi(0) = 0, \quad C_1^T D' \Phi(0) = \frac{1}{1 - \mathrm{e}^{-1}}, \tag{3.15}$$
$$C_2^T \Phi(0) = 1, \quad C_2^T D' \Phi(0) = \frac{1}{1 - \mathrm{e}^{-1}}. \tag{3.16}$$

To find the unknowns $C_i^T = [C_{i,0}, C_{i,1}, \ldots, C_{i,n}]$ $(i = 1, 2)$, we substitute the collocation points in equations (3.13)–(3.14). We have $n + 1 = 14 + 1 = 15$ algebraic equations which are solved using the Maple software with 30 digits. The following are the 14-term approximate solutions on $[0, 1]$:

$$\begin{aligned} y_1(x) \approx {} & -1.103918419260167200 \times 10^{-11} x^{14} + 2.318228680452035952 \times 10^{-10} x^{13} \\ & -3.258859166868980 0829 \times 10^{-9} x^{12} + 3.957271553457763 6723 \times 10^{-8} x^{11} \\ & -4.3589305524750579983 \times 10^{-7} x^{10} + 4.3594627941622729 6649 \times 10^{-6} x^{9} \\ & -3.923551161128930342863 \times 10^{-5} x^{8} + 3.13884258549681041 59657 \times 10^{-4} x^{7} \\ & -2.1971898680756482449 8355 \times 10^{-3} x^{6} + 1.318313922335099491 5301011 \times 10^{-2} x^{5} \\ & -6.591569611946838092 5755925 \times 10^{-2} x^{4} + 2.636627844782118715 37707584 \times 10^{-1} x^{3} \\ & -7.909883534346626059 437282814 \times 10^{-1} x^{2} + 1.581976706869326424 38500200512 x, \end{aligned}$$

Table 1: Absolute Errors for Example 1.

x	Error (y_1)	Error (y_2)
0.0	0.0	0.0
1.0	1.37×10^{-17}	1.37×10^{-17}
2.0	1.02×10^{-17}	1.05×10^{-17}
3.0	4.32×10^{-17}	1.47×10^{-16}
4.0	3.40×10^{-15}	3.01×10^{-14}
5.0	9.68×10^{-14}	2.95×10^{-13}
6.0	2.09×10^{-13}	3.57×10^{-12}
7.0	3.87×10^{-11}	2.57×10^{-11}
8.0	3.67×10^{-10}	6.11×10^{-10}
9.0	6.98×10^{-10}	3.28×10^{-9}
10.0	2.12×10^{-4}	4.58×10^{-4}

$$
\begin{aligned}
y_2(x) \approx & -1.103918419260708224 \times 10^{-11} x^{14} + 2.31822868045223196 \times 10^{-10} x^{13} \\
& -3.258859166868981 \times 10^{-9} x^{12} + 3.9572715534577521635 \times 10^{-8} x^{11} \\
& -4.3589305524750579983 \times 10^{-7} x^{10} + 4.3594627941622729664 9 \times 10^{-6} x^{9} \\
& -3.9235511611289303093 12 \times 10^{-5} x^{8} + 3.1388425854968104140 42 \times 10^{-4} x^{7} \\
& -2.197189868075648244907 17 \times 10^{-3} x^{6} + 1.318313922335099491528007 \times 10^{-2} x^{5} \\
& -6.591569611946838092575208 \times 10^{-2} x^{4} + 2.636627844782118715377071 2 \times 10^{-1} x^{3} \\
& -7.909883534346626059437282 47 \times 10^{-1} x^{2} + 1.5819767068693264243850020051 x + 1.
\end{aligned}
$$

In Table 1, we depict the absolute errors which show the good accuracy of the method. In Figure 1, we obtain a good approximation with high accuracy for $n = 10, n = 14$.

3.2 Example 2

Next we consider the following problem [11]

$$
\begin{aligned}
& y_1'' = -e^{-x} y_2', \quad y_1(0) = 1,\ y_1'(0) = 0, \quad 0 \le x \le 10, \\
& y_2'' = 2e^{x} y_1', \quad y_2(0) = 1,\ y_2'(0) = 1.
\end{aligned}
$$

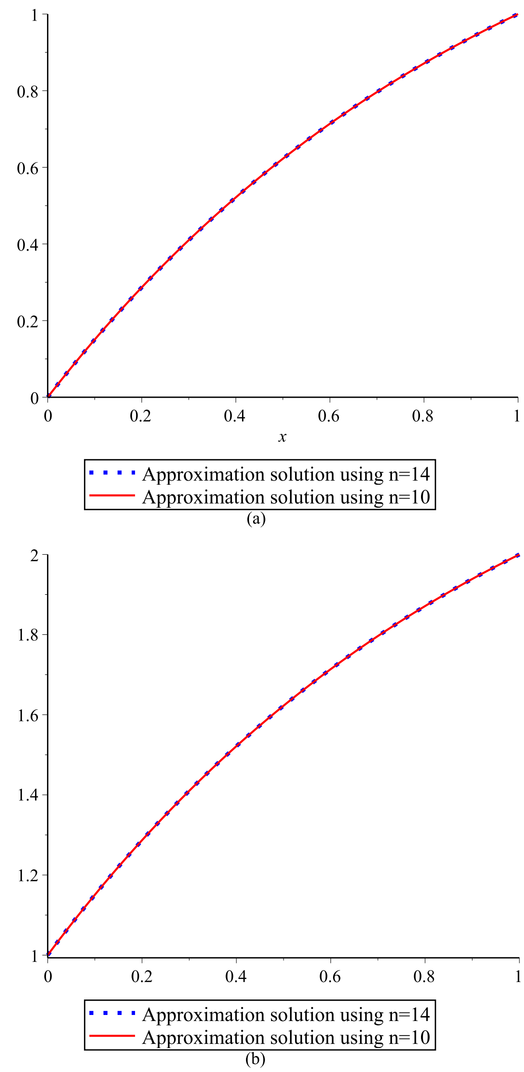

Figure 1: a: Approximate solution for example 1 (y_1) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$. b: Approximate solution for example 1 (y_2) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$.

The exact solutions are $y_1 = \cos x$, $y_2 = \mathrm{e}^x \cos x$. The approximate

Table 2: Absolute Errors for Example 2.

x	Error (y_1)	Error (y_2)
0.0	0.0	0.0
1.0	7.58×10^{-20}	4.56×10^{-18}
2.0	3.47×10^{-17}	8.62×10^{-16}
3.0	1.35×10^{-16}	1.50×10^{-15}
4.0	5.94×10^{-15}	2.01×10^{-13}
5.0	2.35×10^{-14}	1.04×10^{-12}
6.0	1.47×10^{-12}	2.83×10^{-12}
7.0	1.58×10^{-11}	9.17×10^{-9}
8.0	4.19×10^{-11}	2.16×10^{-8}
9.0	1.03×10^{-11}	4.25×10^{-6}
10.0	6.74×10^{-10}	1.17×10^{-5}

solutions on $[0, 1]$ obtained are:

$$
\begin{aligned}
y_1(x) \approx\ & -1.0234972746860626752 \times 10^{-11}x^{14} - 5.04025334921230635 \times 10^{-12}x^{13} \\
& +2.097879796630596528 \times 10^{-9}x^{12} - 1.2781026833196923 \times 10^{-11}x^{11} \\
& -2.7556249256651378334 7\times 10^{-7}x^{10} - 6.17635097418135 \times 10^{-12}x^{9} \\
& +2.4801589777250521570 48 \times 10^{-5}x^{8} - 6.813775547299438 \times 10^{-13}x^{7} \\
& -1.3888888887639286743 4819 \times 10^{-3}x^{6} - 1.444225682898 \times 10^{-14}x^{5} \\
& +4.1666666666667617999322452 \times 10^{-2}x^{4} - 2.8992398057 \times 10^{-17}x^{3} \\
& -0.49999999999999999977914844 45x^{2} + 1,
\end{aligned}
$$

$$
\begin{aligned}
y_2(x) \approx\ & 1.0551025072190085465 3 \times 10^{-9}x^{14} - 1.3947182593043668159 5 \times 10^{-8}x^{13} \\
& -1.3947182593043668159 5 \times 10^{-7}x^{12} - 8.1028207546174917323 9 \times 10^{-7}x^{11} \\
& +7.1317247053942047 \times 10^{-9}x^{10} + 4.4087611611826399006 31 \times 10^{-5}x^{9} \\
& +3.9682704029662207032316 \times 10^{-4}x^{8} + 1.5873011332405570296 2709 \times 10^{-3}x^{7} \\
& +8.38629255412292 \times 10^{-11}x^{6} - 3.3333333343127840092 9235 \times 10^{-2}x^{5} \\
& -0.1666666666660120811709 8182x^{4} - 0.3333333333333537627 6482211x^{3} \\
& +1.56788354802 \times 10^{-16}x^{2} + x + 1.
\end{aligned}
$$

Again, the absolute errors in Table 2 clearly demonstrate the accuracy of the method. We can see the approximate solution for example 2 with $n = 10, n = 14$ in Figure 2.

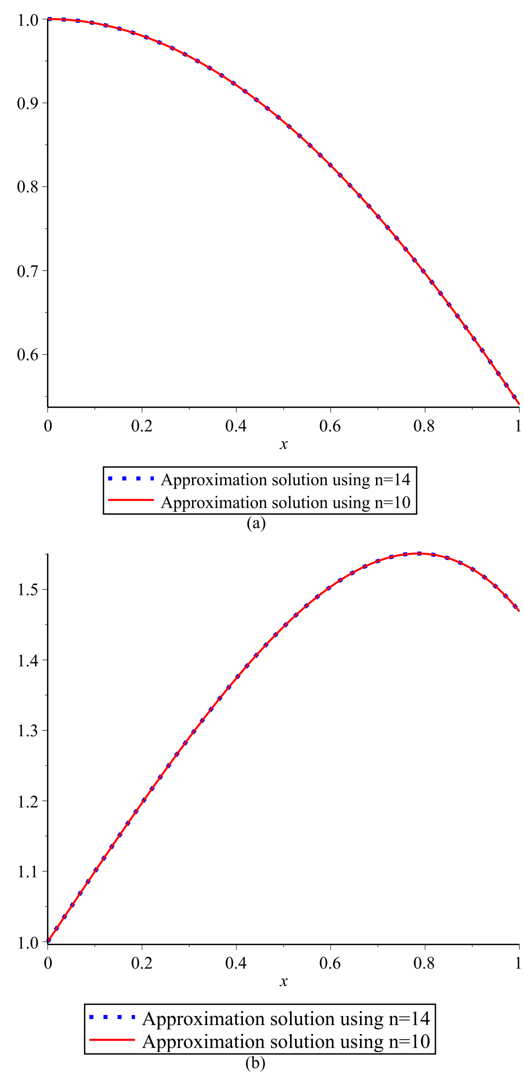

Figure 2: a: Approximate solution for example 2 (y_1) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$. b: Approximate solution for example 2 (y_2) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$.

3.3 Example 3:

Finely, we consider this problem obtained in [11] and [2].

$$y_1'' = -y_2' + \sin \pi x, \quad y_1(0) = 0, \quad y_1'(0) = -1, \quad 0 \leq x \leq 10$$
$$y_2'' = -y_1' + 1 - \pi^2 \sin \pi x, \quad y_2(0) = 1, \quad y_2'(0) = 1 + \pi.$$

Table 3: Absolute Errors for Example 3.

x	Error (y_1)	Error (y_2)
0.0	0.0	0.0
1.0	1.13×10^{-11}	7.15×10^{-11}
2.0	9.52×10^{-11}	1.73×10^{-7}
3.0	2.93×10^{-7}	6.18×10^{-7}
4.0	1.19×10^{-6}	1.40×10^{-6}
5.0	3.60×10^{-6}	3.51×10^{-6}
6.0	9.83×10^{-6}	9.51×10^{-6}
7.0	2.64×10^{-5}	2.61×10^{-5}
8.0	7.11×10^{-5}	7.19×10^{-5}
9.0	1.94×10^{-4}	2.02×10^{-4}
10.0	4.59×10^{-4}	6.31×10^{-4}

The exact solutions for this problem are $y_1 = 1 - \mathrm{e}^x, y_2 = \mathrm{e}^x + \sin \pi x$. Our method yields the following approximate solutions,

$$
\begin{aligned}
y_1(x) \approx & -1.2840530012435965634 \times 10^{-10}x^{14} + 6.3327908648376695 95 \times 10^{-10}x^{13} \\
& -4.582210629629255708 \times 10^{-9}x^{12} - 2.0316992703545796832 \times 10^{-8}x^{11} \\
& -2.8160983681921394035 \times 10^{-7}x^{10} - 2.7502815350537585 0881 \times 10^{-6}x^{9} \\
& -2.4805170692506560692 52 \times 10^{-5}x^{8} - 1.9841096133764485355316 \times 10^{-4}x^{7} \\
& -1.3888895103730817124 8676 \times 10^{-3}x^{6} - 8.3333331707961683701 52726 \times 10^{-3}x^{5} \\
& -4.1666666697068693605613072 \times 10^{-2}x^{4} - 0.16666666666509237203551936957x^{3} \\
& -0.50000000000083938312508451 18x^{2} + x,
\end{aligned}
$$

$$
\begin{aligned}
y_2(x) \approx & -1.0203536365052516037145122 \times 10^{-4}x^{14} + 7.1424781111169049015 93217 \times 10^{-4}x^{13} \\
& -3.9233039353241496298 1338 \times 10^{-4}x^{12} - 6.9311996056291278587 4603 \times 10^{-3}x^{11} \\
& -3.6209294373767244938772 \times 10^{-4}x^{10} + 8.2373726931420970456 89259 \times 10^{-2}x^{9} \\
& -8.1458103813750856914 8 \times 10^{-5}x^{8} - 0.5990279657167460055 6880494x^{7} \\
& +1.3785480212634055504 6994 \times 10^{-3}x^{6} + 2.5584994553535974772 480161x^{5} \\
& +4.1666363683212874131 05674 \times 10^{-2}x^{4} - 5.0010460829088733529 62920284x^{3} \\
& +0.4999999980330965078 6257049x^{2} + 4.1415926535897932384 626433832x + 1.
\end{aligned}
$$

In Table 3 we present the absolute error which clearly justifies good approximations. A good approximate solution has been obtained for $n = 10, n = 14$, see Figure 3.

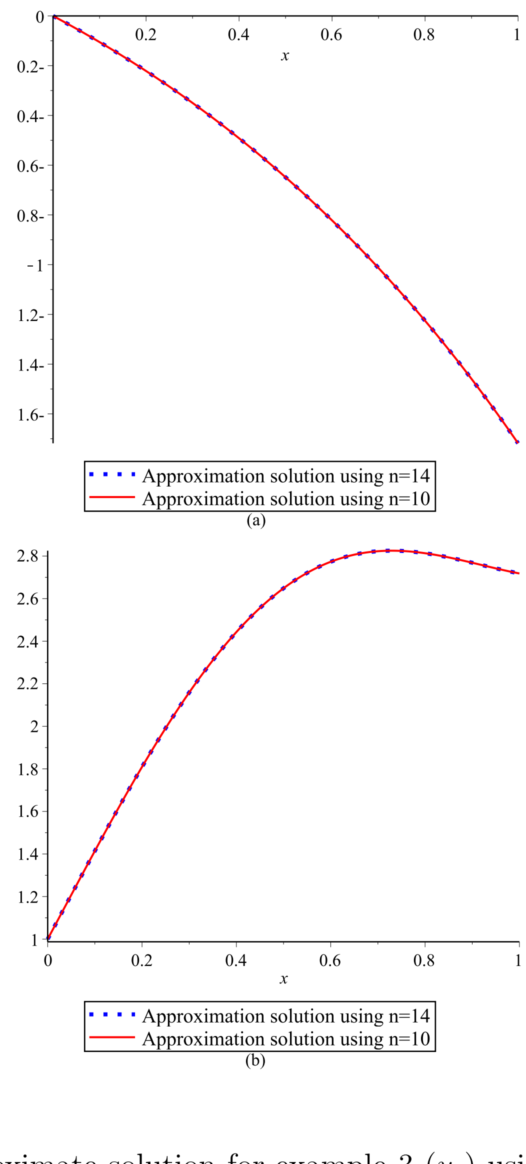

Figure 3: a: Approximate solution for example 3 (y_1) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$. b: Approximate solution for example 3 (y_2) using Bernstein polynomials with different numbers of terms $n = 10, n = 14$.

4 Conclusion

In this paper, we have constructed a new method which is efficient and suitable for solving a system of second-order ODEs directly on an arbitrary interval $[a, b]$. The test examples have demonstrated the capability of the method to solve nonlinear systems of second-order ODEs directly.

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