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Analogs of Groebner Bases in a Polynomial Ring with Countably Infinite Indeterminates

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Abstract

The work on the theory of Groebner bases for ideals in a polynomial ring with countably infinite indeterminates over a field [5] has created impetus to develop the theory of Sagbi bases [6] and Sagbi Groebner bases [3] in the same polynomial ring. This paper demonstrates the construction of Sagbi basis and Sagbi Groebner basis using the technique of constructing these bases in a polynomial ring with finite indeterminates.

1 Introduction

In 1965, for the study of structure of ideal in a polynomial ring involving finitely many indeterminates through its basis, the idea of Groebner basis was presented by Bruno Buchberger in his PhD dissertation [1], which solved many problems including the Ideal membership problem. It played an important role in the field of Computational Algebraic Geometry and Computational Commutative Algebra. Later for subalgebras in polynomial rings, the concept of Sagbi (Subalgebra Analog to Groebner Bases for Ideals) bases was introduced by Robianno and Sweedler in 1988 parallel to that of Groebner bases and hence played same computational role for subalgebras

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as Groebner bases did for ideals [6]. Furthermore, in 1988, a generalized type of Groebner basis was developed in the context of a valuation ring by Sweedler [7]. To complete Sweedler′s theory in a way for ideals of subalgebras of polynomial rings over field, Miller introduced bases for such ideals named as Sagbi Groebner bases in 1998 [3].

Later the study was made for Groebner basis for ideal in a polynomial ring with countably infinite indeterminates over a field in 2009 [5], where the Groebner bases were constructed by taking union of Groebner bases of partial ideals in polynomial ring with finite indeterminates. The purpose of this paper is to develop the theory of Sagbi bases and Sagbi Groebner bases in a polynomial ring with countably infinite indeterminates over field K. This idea is strongly motivated by the work [5] since the basic concept of Sagbi bases and Sagbi Groebner bases is parallel to Groebner bases.

This paper comprises of three main sections. In section 2, we will give related definitions and notations. In section 3, we will develop the theory of Sagbi basis in a polynomial ring with countably infinite indeterminates over field K where we will see result for construction of Sagbi bases (theorem 3.2) and reduced Sagbi bases (theorem 3.15) by the same technique. In section 4, we will develop the theory of Sagbi Groebner bases by using the same technique and we will see results for Sagbi Groebner bases (theorem 4.2) and for reduced Sagbi Greobner bases as well (theorem 4.15).

2 Notation and definition

This section introduces basic notations, terminologies and few definitions that will remain the same throughout this work. For their details, the reader may look at the references mentioned.

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Q}$ be the set of rational numbers. Let S be the polynomial ring with countably infinite indeterminates $K[x_1,x_2,...]$. We denote the set of all sequences $q = (q_1,q_2,...)$ of natural numbers by $\mathbb{N}^{(\infty)}$, where $q_i = 0$ for all i but finite number of natural numbers. By a monomial, we mean an element of the form $X^q = \prod_i x_i^{q_i}$ for some $q = (q_1,q_2,...) \in \mathbb{N}^{(\infty)}$. We denote the set of all monomials in S by MON(S). Now for total ordering $<$ on MON(S), a polynomial may be expressed as:

$$h = \alpha_q X^q + \alpha_r X^r + ... + \alpha_p X^p,$$

where $\alpha_q, \alpha_r, ..., \alpha_p \in K^*$, $X^q, X^r, ..., X^p \in \text{MON}(S)$, $X^p <... < X^r < X^q$, where $K^* = K - \{0\}$

Moreover, the leading monomial of h (i.e. X^q) is denoted by lm(h),
the leading coefficient of h i.e. α_q is denoted by lc(h),
the leading term of h (i.e. $\alpha_q X^q$) is denoted by lt(h),
and the support of h (i.e. $\{X^q,... , X^p\}$) is denoted by supp(h),
and the tail of h (i.e. h - lt(h)) is denoted by tail(h).// If F $\subset S$, then LM(F) = { lm(f) | f $\in$ F }, the set of leading monomials of elements of F. We denote $K[x_1,x_2,..., x_n]$ by $S^{\langle n \rangle}$. Hence, for any subset F of S, let us use the notation $\mathrm{F}^{\langle n \rangle}$ for F $\cap$ $K[x_1,x_2,..., x_n]$. The definition of monomial ordering on S is given below:

Definition 2.1. *A well ordering $<$ on $K[x_1,x_2,...]$ or MON(S) is said to be* ***monomial ordering****, if it is compatible with the multiplication of monomials; i.e., $X^q < X^r$ implies $X^q X^p < X^r X^p$ for all $X^q, X^r, X^p \in$ MON(S) (See([2], Chapter 15).*

Let us see some examples of monomial orderings on MON(S).

Example 2.2. *Let $p = (p_1,p_2,...)$, $q = (q_1,q_2,...) \in \mathbb{N}^{(\infty)}$.*
(1) The pure lexicographic order $>_{pl}$ is defined as:
$X^p >_{pl} X^q$ iff $p_i > q_i$ for the last index i with $p_i \neq q_i$.
(2) The homogeneous lexicographic order $>_{hl}$ is defined as:
$X^p >_{hl} X^q$ iff either $degX^p > degX^q$ or $degX^p = degX^q$ and $p_i > q_i$ for the last index i with $p_i \neq q_i$.
(3) The homogeneous reverse lexicographic order $>_{hrl}$ is defined as:
$X^p >_{hrl} X^q$ iff either $degX^p > degX^q$ or $degX^p = degX^q$ and $p_i < q_i$ for the first index i with $p_i \neq q_i$.

With $x_4 > x_3 > x_2 > x_1$, the above monomial orderings are all distinct as shown below:

$$x_1x_4 >_{pl} x_2x_3 >_{pl} x_2^3 >_{pl} x_1x_2$$
$$x_2^3 >_{hl} x_1x_4 >_{hl} x_2x_3 >_{hl} x_1x_2$$
$$x_2^3 >_{hrl} x_2x_3 >_{hrl} x_1x_4 >_{hrl} x_1x_2$$

Definition 2.3. *Let $H \subseteq S$. An H-**power product** is a finite product of the form $h_1^{e_1}... h_p^{e_p}$ where $h_i \in H$ and $e_i \in \mathbb{N}$ for $1 \leq i \leq m$ abbreviated as $H^{e^{\rightarrow}}$, where $e^{\rightarrow} \in \oplus_H \mathbb{N}$ is the vector having all coordinates equal to 0 except for $e_1, e_2, ..., e_p$ in the positions corresponding to $h_1, h_2, ..., h_p$.*

In case of ideals, we define in(G) := $\langle$LM(G)$\rangle$ i.e. the ideal generated by leading monomials of subset G of $K[x_1, x_2, ...]$ and in case of subalgebras, for any subset H of $K[x_1, x_2,...]$, we define in(H) := K[LM(H)], the subalgebra generated by leading monomials.

3 Sagbi Basis

The definition of Sagbi basis of a subalgebra in a polynomial ring with countably infinite indeterminates is similar to the case of subalgebra in a polynomial ring with finite indeterminates.

Definition 3.1. *Let A be a K-sublagebra of S. A subset $H \subseteq A$ is called a Sagbi basis for A if LM(H) generates in(A); i.e.,*

$$in(A) = in(H)$$

Now, we construct a Sagbi basis for any subalgebra A of S by using Sagbi basis for partial subalgebras; i.e., $A^{\langle n \rangle}$ in $S^{\langle n \rangle}$.

Theorem 3.2. *Let A be a K-subalgebra of S. If C is any arbitrary infinite subset of $\mathbb{N}$. For every $n \in C$, take a Sagbi basis H_n for subalgebra $A^{\langle n \rangle}$. The set $\cup_n H_n$ is a Sagbi basis for A.*

To prove the above theorem, we need a few results:

Lemma 3.3. *For a subset H of a K-subalgebra A of S, the following conditions are equivalent:*
(1) H is a Sagbi basis for A .
(2) $in(H) \cap S^{\langle n \rangle}$ generates the $in(A^{\langle n \rangle})$ for all $n \in \mathbb{N}$
(3) $in(H) \cap S^{\langle n \rangle}$ generates the $in(A^{\langle n \rangle})$ for infinitely many $n \in \mathbb{N}$.

Proof 3.4. (1) $\Longrightarrow$ (2)
Let $f \in A^{\langle n \rangle}$ which implies $f \in A$ and $f \in S^{\langle n \rangle}$ but since H is a Sagbi basis which implies $lm(f) \in in(H)$ also $lm(f) \in S^{\langle n \rangle}$ and hence $lm(f) \in in(H) \cap S^{\langle n \rangle}$
(2) $\Longrightarrow$ (3)
Trivial.
(3) $\Longrightarrow$ (1)
Let $f \in A$. Choose n so that $f \in S^{\langle n \rangle}$. By (3), there is integer $m \geq n$ so that $in(A^{\langle m \rangle})$ is generated by $in(H) \cap S^{\langle m \rangle}$. Since $f \in A^{\langle m \rangle}$, the $lm(f) \in in(H) \cap S^{\langle m \rangle}$ implies $lm(f) \in in(H)$. Hence, H is a Sagbi basis for A.

Corollary 3.5. *Let H be a subset of a subalgebra A of S. If $H^{\langle n \rangle}$ is a Sagbi basis for a subalgebra $A^{\langle n \rangle}$ for infinitely many integers n. Then H is a Sagbi basis for A.*

Proof 3.6. *Let us first prove the inclusion in($H^{\langle n\rangle}$) $\subseteq$ in(H) $\cap$ $S^{\langle n\rangle}$. For this, let $m \in$ in($H^{\langle n\rangle}$) implies there exist some $h \in H^{\langle n\rangle}$ so that lm(h) $= m$. Now $h \in H$ and $S^{\langle n\rangle}$ as well. Since $h \in S^{\langle n\rangle}$ implies $m \in S^{\langle n\rangle}$ for any monomial ordering, hence $m \in$ in(H) $\cap$ $S^{\langle n\rangle}$.*

Now, as in($H^{\langle n\rangle}$) generates in($A^{\langle n\rangle}$) for infinitely many integers n, above proved inclusion implies in(H) $\cap$ $S^{\langle n\rangle}$ generates in($A^{\langle n\rangle}$) for infinitely many integers n. Hence, H is a Sagbi basis by lemma 3.3.

Now, we can prove theorem 3.2.

Proof 3.7. *Let $H = \cup_n H_n$. Since $H^{\langle n\rangle}$ contains H_n for each $n \in C$, therefore $H^{\langle n\rangle}$ and H_n, both are Sagbi basis for $A^{\langle n\rangle}$ for such n. Hence by corollary 3.5, H is a Sagbi basis for A .*

This can be demonstrated by the following example.

Example 3.8. *Let $A = \{\, f \in K[x_1,\, x_2, ...] | \sigma(f) = f \ \forall \sigma \in S_n$ for some $n \geq 2\}$ where S_n is a group of permutation. We can see the subalgebra $A^{\langle 1\rangle}$ has Sagbi basis $H_1 = \emptyset$. The subalgebra $A^{\langle 2\rangle}$ has Sagbi basis $H_2 = \{x_1{+}x_2, x_1x_2\}$. The subalgebra $A^{\langle 3\rangle}$ has Sagbi basis $H_3 = \{x_1{+}x_2, x_1x_2, x_3, x_1x_3{+}x_2x_3\}$. Similarly we can see that $A^{\langle 4\rangle}$ has Sagbi basis $H_4 = \{x_1{+}x_2, x_1x_2, x_3, x_4, \sum_{i=3}^{4}(x_1x_i{+}x_2x_i), \sum_{i=3}^{4}(x_1x_ix_j{+}x_2x_ix_j)\}$. By theorem 3.2, $H = \cup_n H_n = \{x_1{+}x_2, x_1x_2, \sum_{i=3}^{4}(x_1x_i{+}x_2x_i),$
$\sum_{i=3}^{4}(x_1x_ix_j + x_2x_ix_j), .., x_3, x_4, x_5, x_6..\}$ is the Sagbi basis for A.*

Observe the inclusion in($H^{\langle n\rangle}$) $\subseteq$ in(H) $\cap$ $S^{\langle n\rangle}$, which is strict in general that is why if H is a Sagbi basis for A, then $H^{\langle n\rangle}$ need not be Sagbi basis for $A^{\langle n\rangle}$. We can see this through the example below.

Example 3.9. *Let $H = \{x_2^2 + x_3, x_1x_2^2 + x_1\}$ be the subset of $K[x_1, x_2, x_3]$ using homogeneous lexicographic ordering with $x_1 < x_2 < x_3$. We can clearly see even for finite set H, in(H) $\cap$ $S^{\langle 2\rangle} = K[x_2^2, x_1x_2^2]$ is not equal to in($H^{\langle 2\rangle}$) $= K[x_1x_2^2]$.*

In general, equality does not hold but it holds for some monomial orderings.

Corollary 3.10. *For pure lexicographic order, if H is a Sagbi basis for subalgerba A of S, then $H^{\langle n\rangle}$ is a Sagbi basis for $A^{\langle n\rangle}$ for all $n \in \mathbb{N}$.*

Proof 3.11. *We first prove the inclusion in($H^{\langle n\rangle}$) $\supset$ in(H) $\cap$ $S^{\langle n\rangle}$ for such ordering. For this, let $m \in$ in(H) $\cap$ $S^{\langle n\rangle}$; i.e., $m \in$ in(H) and $m \in S^{\langle n\rangle}$ as* *well. Now $m \in in(H)$ implies there exist some $h \in H$ so that $lm(h) = m$, since ordering is pure lexicographic, so $lm(h) \in S^{\langle n \rangle}$ implies $h \in S^{\langle n \rangle}$. Hence $m \in in(H^{\langle n \rangle})$.*

As other inclusion can be found in the proof of corollary 3.5, we get $in(H^{\langle n \rangle}) = in(H) \cap S^{\langle n \rangle}$ i.e. $in(H^{\langle n \rangle})$ is generating the $in(A^{\langle n \rangle})$ for each n. Now apply lemma 3.3.

Now, we will compute reduced Sagbi basis. For this, we have to define it first:

Definition 3.12. *Sagbi basis for subalgebra of S is said to be reduced if $\forall$ $h \in H$, $lm(h)$ does not belong to $in(H \setminus \{h\})$, $lc(h) = 1$ and no term from tail(h) is contained in $in(H)$.*

Like in the case of subalgebra in $S^{\langle n \rangle}$, the reduced Sagbi basis is unique.

Proposition 3.13. *For an arbitrary subalgebra A of S, there exist a unique reduced Sagbi basis for A.*

Proof 3.14. *We first prove that minimal generating set[1] for in(A) is unique. For this, let $M=\{\mu_\lambda \mid \lambda \in \Lambda\}$ and $M'=\{\nu_\lambda \mid \lambda \in \Lambda\}$ be both minimal generating sets for in(A), implies $in(M) = in(A) = in(M')$. Let $\mu_{\lambda_i} \in M$, since M' is a generating set, implies $\mu_{\lambda_i} = \nu_{\lambda_1}^{\alpha_1} .... \nu_{\lambda_s}^{\alpha_s}$. But as $\nu_\lambda \in in(A)$ and M is a generating set, therefore*

$\nu_{\lambda_1} = \mu_{\lambda_1}^{\beta_1^1} .... \mu_{\lambda_t}^{\beta_t^1}$

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$\nu_{\lambda_s} = \mu_{\lambda_1}^{\beta_1^s} \mu_{\lambda_2}^{\beta_2^s} ... \mu_{\lambda_p}^{\beta_p^s}$

Now $\mu_{\lambda_i} = \nu_{\lambda_1}^{\alpha_1} \nu_{\lambda_2}^{\alpha_2} .... \nu_{\lambda_s}^{\alpha_s} = \mu_{\lambda_1}^{\alpha_1\beta_1^1 + ... + \alpha_s\beta_1^s} .... \mu_{\lambda_t}^{\alpha_1\beta_t^1 + ... + \alpha_s\beta_t^s} \mu_{\lambda_p}^{\alpha_1\beta_p^1 + ... + \alpha_s\beta_p^s}$

But M is a minimal generating set, therefore all powers must be zero but $\alpha_1\beta_i^1 + ... + \alpha_s\beta_i^s = 1$. As all numbers are non-negative integers, implies $\alpha_k\beta_i^k=1$ for some k, implies $\alpha_k=1$ and $\beta_i^k = 1$. On the other hand, all zero powers force to $\alpha_k\beta_m^k=0$ implies $\beta_m^k = 0$ for all m other than i. Hence $\nu_{\lambda_k}=\mu_{\lambda_i}$.

To show the existence of a reduced Sagbi basis, let $\{ \mu_\lambda \mid \lambda \in \Lambda \}$ be a minimal generating set of in(A). Let $h_\lambda \in A$ such that $lm(h_\lambda) = \mu_\lambda$ and set

[1] A generating set for A where A is a subalgebra, is said to be minimal if all leading coefficients of the polynomials in it are 1, and if no leading monomial of one of the polynomials in it is a power product of the leading monomials of other polynomials in it.

$H = \{ h_\lambda \mid \lambda \in \Lambda \}$. Then clearly H is a Sagbi basis for A. Replacing h_λ with its S-NF[2] *w.r.t $H \setminus \{h_\lambda \}$, we see that H is a reduced Sagbi basis.*

Now we show uniqueness. Let H and H' be reduced Sagbi bases for A. Let $h \in H$. Since the minimal generating set is unique (i.e. $LM(H) = LM(H')$), there exists $h' \in H'$ such that $lm(h) = lm(h')$. Clearly $h - h' \in A$. If $h \neq h'$ then $lm(h - h') < lm(h)(=lm(h'))$. If $lm(h - h') \in supp(h)$, then $lm(h - h')$ does not belong to $in(H)$, since H is reduced Sagbi basis. If $lm(h - h') \in Supp(h')$, then by the same argument, $lm(h - h')$ doesn't belong to $in(H')$, which contradicts $h - h' \in A$.

The next theorem shows computation of reduced Sagbi basis for subalgebra A in S using reduced Sagbi basis for partial subaglebras $A^{\langle n \rangle}$ in $S^{\langle n \rangle}$.

Theorem 3.15. *Let A be a subalgebra of S. Consider a reduced Sagbi basis H_n for $A^{\langle n \rangle}$ inside the polynomial ring $S^{\langle n \rangle}$ for every $n \in \mathbb{N}$. Then the following set of polynomial in S:*

$$\tilde{H} = \cup_{m=1}^{\infty} (\cap_{n=m}^{\infty} H_n)$$

is a reduced Sagbi basis for A .

Proof 3.16. *Let H be a unique reduced Sagbi basis for A . First we prove that $H \subseteq \tilde{H}$. Let $h \in H$ and pick an integer n so that $h \in S^{\langle n \rangle}$. Note that $lm(h)$ cannot be written as power product of elements of minimal generating set of $in(A)$ excluding $lm(h)$ and so for lower terms of h, implies h is an element of reduced Sagbi basis for $A^{\langle n \rangle}$, so $h \in H_n$ for such n. Therefore, $h \in \tilde{H}$ i.e. $H \subseteq \tilde{H}$. Since $\tilde{H}$ contains a Sagbi basis for A and also $\tilde{H} \subseteq A$ imlpies $\tilde{H}$ is a Sagbi basis. Now to prove that $\tilde{H}$ is reduced Sagbi basis for A. Take an element h of $\tilde{H}$. Pick an integer m so that h belongs to $\cap_n H_n$ for $n \geq m$. Specifically h belongs to H_m. Since H_m is reduced Sagbi basis, we see that any term of h cannot be written as power product of leading monomials of other elements of H_m. Hence $\tilde{H}$ is a reduced Sagbi basis for A.*

4 Sagbi Groebner basis

In this section first, we define Sagbi Groebner basis of an ideal of a subalgebra in a polynomial ring with countably infinite indeterminates.

Definition 4.1. *Let I be an ideal of K-subalgebra A of S. A subset G of I is called a Sagbi Groebner basis for I if $LM(G)$ generates the monoid ideal $in(I)$ of the multiplicative monoid in(A;, i.e.,*

[2] We use "S-NF" for Sagbi Normal Form.

$$in(G)_{in(A)} = in(I)_{in(A)},$$

where $in(I)_{in(A)}$ is defined in section 2.

Now we consider our main theorem which shows the construction of Sagbi Groebner basis for any ideal I of subalgebra A in S by using Sagbi Groebner basis for partial ideals; i.e, $I \cap A^{\langle n \rangle}$.

Theorem 4.2. *Let I be an ideal of K-subalgebra A of S. Let C be an arbitrary infinite subset of $\mathbb{N}$. For each $n \in C$, take a Sagbi Groebner basis SG_n for ideal $I \cap A^{\langle n \rangle}$. Then the set $\cup_n SG_n$ is a Sagbi Groebner basis for I.*

We need a sequence of results for proving above theorem, that are proved below:

Lemma 4.3. *The following conditions are equivalent for a subset G of an ideal I of K-subalgebra A of S :*
(1) G is a Sagbi Groebner basis for I.
(2) $in(G) \cap in(A^{\langle n \rangle})$ generates the ideal $in(I \cap A^{\langle n \rangle})$ for all $n \in \mathbb{N}$
(3) $in(G) \cap in(A^{\langle n \rangle})$ generates the $in(I \cap A^{\langle n \rangle})$ for infinitely many $n \in \mathbb{N}$

Proof 4.4. (1) $\Longrightarrow$ (2)
Let $g \in I \cap A^{\langle n \rangle}$ which implies $g \in I$ and $g \in A^{\langle n \rangle}$ but since G is a Sagbi Groebner basis, there exist $a \in A$ and $l \in G$ such that $lm(g) = lm(a)lm(l)$ also $lm(g) \in in(A^{\langle n \rangle})$. Hence, $lm(g) \in in(G) \cap in(A^{\langle n \rangle})$
(2) $\Longrightarrow$ (3)
Obvious.
(3) $\Longrightarrow$ (1)
Let $g \in I$. Take n so that $g \in A^{\langle n \rangle}$. Then by (3) , there is integer $m \geq n$ such that $in(I \cap A^{\langle m \rangle})$ is generated by $in(G) \cap in(A^{\langle m \rangle})$. Since $g \in I \cap A^{\langle m \rangle}$, the $lm(g) \in in(G) \cap in(A^{\langle m \rangle})$ implies $lm(g) \in in(G)$. Hence, G is Sagbi Groebner basis for I.

Corollary 4.5. *Let G be a subset of an ideal I of a subalgebra A of S. Assume that $G \cap A^{\langle n \rangle}$ is a Sagbi Groebner basis for ideal $I \cap A^{\langle n \rangle}$ for infinitely many integers n. Then G is a Sagbi Groebner basis for I.*

Proof 4.6. *We first prove the inclusion $in(G \cap A^{\langle n \rangle}) \subseteq in(G) \cap in(A^{\langle n \rangle})$. Let $m \in in(G \cap A^{\langle n \rangle})$ implies there exist some $l \in G \cap A^{\langle n \rangle}$ so that $lm(l) = m$. Now $l \in G$ and $A^{\langle n \rangle}$ as well. Since $l \in A^{\langle n \rangle}$ implies $m \in in(A^{\langle n \rangle})$ for any monomial ordering, hence $m \in in(G) \cap in(A^{\langle n \rangle})$.*

Since $in(G \cap A^{\langle n \rangle})$ generates $in(I \cap A^{\langle n \rangle})$ for infinitely many integers n, above proved inclusion implies $in(G) \cap in(A^{\langle n \rangle})$ generates $in(I \cap A^{\langle n \rangle})$ for infinitely many integers n. Hence, G is a Sagbi Groebner basis by lemma 4.3.

Now, we prove theorem 4.2.

Proof 4.7. *Let $G = \cup_n SG_n$. Since $G \cap A^{\langle n \rangle}$ contains SG_n for each $n \in C$. Hence $G \cap A^{\langle n \rangle}$ and SG_n, both are Sagbi Groebner basis for $A^{\langle n \rangle}$ for such n. Hence by corollary 4.5, G is a Sagbi Groebner basis for A.*

Let us see this result through an example.

Example 4.8. *Consider the ideal $I = \langle \Sigma_{i<j}^{n} x_i x_j | n \geq 1 \text{ and } i, j \in \{1, 2, ..., n\}\rangle_A$ of the subalgebra defined in example3.8. Since $I^{\langle 1 \rangle}$ in $A^{\langle 1 \rangle} = \emptyset$, hence it's Sagbi Groebner basis $SG_1 = \emptyset$.*
We can see the Sagbi Groebner basis for $I^{\langle 2 \rangle}$ in $A^{\langle 2 \rangle}$ is the set $SG_2 = \{x_1 x_2\}$.
The Sagbi Groebner basis for $I^{\langle 3 \rangle}$ in $A^{\langle 3 \rangle}$ is the set $SG_3 = \{x_1x_2, x_1x_3 + x_2x_3\}$.
Hence by theorem 4.2, the set $G = \cup_n SG_n = \{x_1x_2, \Sigma_{i=1}^{j-1} x_i x_j | j \in \{3, 4, ..., n\}, n \geq 1\}$ is the Sagbi Groebner basis for ideal I.

Note that the inclusion $\text{in}(G \cap A^{\langle n \rangle}) \subseteq \text{in}(G) \cap \text{in}(A^{\langle n \rangle})$ is strict in general. From this we can see, if G is a Sagbi Groebner basis for I in A then $G \cap A^{\langle n \rangle}$ need not be Sagbi Groebner basis for $I \cap A^{\langle n \rangle}$.

Example 4.9. *Let $G = \{x_1x_2^2 + 2x_3, x_1^3 + x_2^2\}$ be the subset of subalgebra $A = K[x_1x_2^2 + 2x_3, x_1^3 + x_2^2, x_1^3 + x_3^2, 2x_1x_2^2 + x_1x_2]$ in $K[x_1, x_2, x_3]$ using homogeneous lexicographic ordering with $x_1 < x_2 < x_3$. Now, even for finite set G, in($G \cap A^{\langle 2 \rangle}) = \langle x_1^3 \rangle_{K[x_1^3, x_1x_2^2]}$ is not equal to in(G) $\cap$ in($A^{\langle 2 \rangle}$) $= \langle x_1^3, x_1x_2^2 \rangle_{K[x_1^3, x_1x_2^2]}$.*

This example shows that the equality does not always hold but it holds for some monomial orderings as shown below.

Corollary 4.10. *For pure lexicographic order, if G is a Sagbi Groebner basis for ideal I of subalgerba A in S, then $G \cap A^{\langle n \rangle}$ is a Sagbi Groebner basis for $I \cap A^{\langle n \rangle}$ for all $n \in \mathbb{N}$.*

Proof 4.11. *We first prove the inclusion in($G \cap A^{\langle n \rangle}$) $\supset$ in(G) $\cap$ in($A^{\langle n \rangle}$) for such ordering. For this, let $m \in$ in(G) $\cap$ in($A^{\langle n \rangle}$) i.e. $m \in$ in(G) and $m \in$ in($A^{\langle n \rangle}$) as well. Now, $m \in$ in(G) implies there exist some $l \in G$ so that lm(l) $= m$, since ordering is lexicographic, so $m \in$ in($A^{\langle n \rangle}$) implies $l \in A^{\langle n \rangle}$. Hence, $m \in$ in($G \cap A^{\langle n \rangle}$).*

Since other inclusion is proved in Corollary 4.5, we get in($G \cap A^{\langle n \rangle}$) $=$ in(G) $\cap$ in($A^{\langle n \rangle}$). Now, as in($G \cap A^{\langle n \rangle}$) is a generating set of in($I \cap A^{\langle n \rangle}$) for each n. Hence, by lemma 4.3, the proof is done.

Now, we will see how to compute a reduced Sagbi Groebner basis in S. For this, we need to define it first:

Definition 4.12. *A Sagbi Groebner basis G for a non zero ideal I of subalgebra A of S is called reduced Sagbi Groebner basis if every $l \in G$ is a monic polynomial i.e. lc(l) = 1 and lm(l) does not belong to in($G \setminus \{l\}$) and no term from tail(l) is contained in in(G).*

Likewise reduced Sagbi Groebner basis in $S^{\langle n\rangle}$, reduced Sagbi Groebner basis is unique in our case as well.

Proposition 4.13. *For an arbitrary ideal I of subalgebra A of S, there exist a unique reduced Sagbi Groebner basis for I.*

Proof 4.14. *First, we prove that minimal generating set[3] for in(I) in in(A) is unique . For this, let $M = \{\mu_\lambda \mid \lambda \in \Lambda\}$ and $M' = \{\nu_\lambda \mid \lambda \in \Lambda\}$ be both minimal generating sets for in(I), implies in(M) = in(I) = in(M'). Let $\mu_{\lambda_i} \in M$, since M' is a generating set , implies $\nu_{\lambda_j} \mid_{in(A)} \mu_{\lambda_i}$ for some $\nu_{\lambda_j} \in M'$. But $\nu_{\lambda_j} \in$ in(I) and also M is a generating set, implies $\mu_{\lambda_k} \mid_{in(A)} \nu_{\lambda_j}$ for some $\mu_{\lambda_k} \in M$. We get $\mu_{\lambda_k} \mid_{in(A)} \nu_{\lambda_j} \mid_{in(A)} \mu_{\lambda_i}$ but since M is minimal generating set, $\mu_{\lambda_i} = \mu_{\lambda_k}$, so $\mu_{\lambda_i} \mid_{in(A)} \nu_{\lambda_j} \mid_{in(A)} \mu_{\lambda_i}$. Hence $\mu_{\lambda_i} = \nu_{\lambda_j}$.*

Next, we show the existence of a reduced Sagbi Groebner basis. Let $\{ \mu_\lambda \mid \lambda \in \Lambda \}$ be a minimal generating set of in(I). Let $l_\lambda \in I$ such that lm(l_λ) = μ_λ and set $G = \{ l_\lambda \mid \lambda \in \Lambda \}$. Then clearly G is a Sagbi Groebner basis for I. Replacing l_λ with its SI-reductum[4] w.r.t. $G \setminus \{l_\lambda\}$, we see that G is a reduced Sagbi Groebner basis.

Now we prove uniqueness. Let G and G' be reduced Sagbi Groebner bases for I. Let $l \in G$. Since the minimal generating set is unique, LM(G) = LM(G'.) So there exists $l' \in G'$ such that lm(l) = lm(l'). Clearly $l - l' \in I$. If $l \neq l'$ then lm($l - l'$) < lm(l)(=lm(l')). If lm($l - l'$) $\in$ supp(l), then lm($l - l'$) does not belong to in(G), since G is reduced Sagbi Groebner basis. If lm($l - l'$) $\in$ Supp(l'), then by the same argument, lm($l - l'$) does not belong to in(G'), contradiction to $l - l' \in I$.

The next theorem shows computation of a reduced Sagbi Groebner basis for I in S using a reduced Sagbi Groebner basis for $I \cap A^{\langle n\rangle}$ in $S^{\langle n\rangle}$.

Theorem 4.15. *Let I be an ideal of subalgebra A of S. Take a reduced Sagbi Groebner basis SG_n for $I \cap A^{\langle n\rangle}$ inside $A^{\langle n\rangle}$ for each n and consider the following set of polynomials in A:*

[3] A generating set for I where I is an ideal of subalgebra A, is said to be minimal if all leading coefficients of the polynomials in it are 1, and if no leading monomial of one of the polynomials in it divide in $in(A)$ the leading monomial of any other polynomial in it.

[4] We use "SI-reductum" for Normal Form. The "SI-" prefix indicates both the subalgebra and ideal components of this operation.

$$\tilde{G} = \cup_{m=1}^{\infty}(\cap_{n=m}^{\infty} SG_n)$$

Then $\tilde{G}$ is a reduced Sagbi Groebner basis for I.

Proof 4.16. *Let G be unique reduced Sagbi Groebner basis for I. We first prove that $G \subseteq \tilde{G}$. Let $l \in G$ and take an integer n so that $l \in A^{\langle n \rangle}$. Note that lm(l) is not divisible (in in(A)) by any monomial of minimal generating set of in(I) excluding itself and so for lower terms of l, implies l is a member of reduced Sagbi Groebner basis for $I \cap A^{\langle n \rangle}$, hence $l \in SG_n$ for such n. Therefore $l \in \tilde{G}$; i.e., $G \subseteq \tilde{G}$. Since $\tilde{G}$ contains a Sagbi Groebner basis for I and also $\tilde{G} \subseteq I$, $\tilde{G}$ is a Sagbi Groebner basis.*

To show that $\tilde{G}$ is a reduced Sagbi Groebner basis for I. Let l, l' be distinct elements of $\tilde{G}$. Take an integer m so that l, l' belong to $\cap_n SG_n$ for $n \geq m$. In particular l,l' belongs to SG_m. Since SG_m is a reduced Sagbi Groebner basis, we see that lm(l) does not divide (in in(A)) any term of l', same for lm(l'). Hence $\tilde{G}$ is reduced Sagbi Groebner basis.

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