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New types of $(\alpha,\ \beta)$ - fuzzy subalgebras of BCK/BCI-algebras

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Abstract

Using the notion of $(\alpha,\ \beta)$-type fuzzy sets, conditions for a subset to be a subalgebra in BCK/BCI-algebras are provided. Given $\varepsilon, \delta \in [0,1]$ with $\varepsilon > \delta$, conditions for the $(\varepsilon,\ \delta)$-characteristic fuzzy set to be fuzzy subalgebras with the type $(\alpha,\ \beta)$ are discussed.

1 Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [11], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In BCK/BCI-algebras, the concept of (α, β)-fuzzy subalgebras, which is studied in the papers [3], [4], [5] and [12], is also important and useful generalization of the well-known concepts, called fuzzy subalgebras. Recently, Muhiuddin et al. studied the

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fuzzy set theoretical approach to the BCK/BCI-algebras on various aspects (see for e.g., [8], [9], [10]).

In this paper, using the notion of (α, β)-fuzzy subalgebra $\mu_S^{(\varepsilon,\delta)}$, we investigate conditions for the S to be a subalgebra of X where (α, β) is one of $(\in, \in\vee q_k)$, $(\in, q_k)$, $(q_k, \in)$, (q_k, q), (q, q_k), (q_k, q_k), $(\in, \in\wedge q_k)$, $(q, \in\vee q_k)$, $(q, \in\wedge q_k)$, $(q_k, \in\wedge q)$, $(q_k, \in\vee q)$, $(q_k, \in\wedge q_k)$, $(q_k, \in\vee q_k)$, $(\in\vee q, q_k)$, $(\in\wedge q, q_k)$ and $(\in\wedge q_k, q_k)$. Given $\varepsilon, \delta \in [0,1]$ with $\varepsilon > \delta$, we discuss conditions for the (ε, δ)-characteristic fuzzy set to be fuzzy subalgebras with the type (α, β).

2 Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [2] and [7] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [11] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t \, q \, \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee\, q\ \mu$ (resp. $x_t \in\wedge\, q\ \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in\vee\, q\,, \in\wedge\, q\,\}$.

A fuzzy set μ in X is said to be an (α, β)-*fuzzy subalgebra* of X, where $\alpha, \beta \in \{\in,\ q\,, \in\vee\, q\,, \in\wedge\, q\,\}$ and $\alpha \neq\ \in\wedge\, q\,$, (see [4]) if it satisfies the following condition:

$$x_{t_1} \alpha\, \mu,\ y_{t_2} \alpha\, \mu\ \Rightarrow\ (x * y)_{\min\{t_1, t_2\}}\, \beta\, \mu. \tag{2.1}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

3 $(\varepsilon,\,\delta)$-characteristic fuzzy sets

In what follows, let X denote a BCK/BCI-algebra, S a non-empty subset of X and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon > \delta$. Also, let k denote an arbitrary element of $[0, 1)$ unless otherwise specified.

To say that $x_t\ q_k\, \mu$, we mean $\mu(x) + t + k > 1$. To say that $x_t \in\vee\, q_k\ \mu$, we mean $x_t \in \mu$ or $x_t\ q_k\, \mu$.

Definition 3.1. *A fuzzy set* μ *in* X *is called an* $(\alpha,\,\beta)$*-fuzzy subalgebra of* X *if it satisfies:*

$$x_{t_1} \alpha \mu,\ \ y_{t_2} \alpha \mu\ \ \Rightarrow\ \ (x * y)_{\min\{t_1, t_2\}} \beta \mu \tag{3.1}$$

for all $x, y \in X$ *and* $t_1, t_2 \in (0, 1]$ *where* $(\alpha,\,\beta)$ *is any one of* $(\in,\, \in\vee\, q_k\,)$, $(\in,\, \in\wedge\, q_k)$, $(\in,\ q_k\,)$, $(q,\ q_k\,)$, $(q,\, \in\vee\, q_k\,)$, $(q,\, \in\wedge\, q_k)$, $(\,q_k\,,\, \in)$, $(\,q_k\,,\ q_k\,)$, $(\,q_k\,, \in\vee\, q\,)$, $(\,q_k\,, \in\wedge\, q\,)$, $(\,q_k\,, \in\vee\, q_k\,)$, $(\,q_k\,, \in\wedge\, q_k)$, $(\in\vee\, q\,,\ q_k\,)$ *and* $(\in\wedge\, q_k,\ q_k\,)$.

Lemma 3.2 ([6]). *A fuzzy set* μ *in* X *is an* $(\in, \in\vee\, q_k\,)$*-fuzzy subalgebra of* X *if and only if it satisfies:*

$$(\forall x, y \in X)\left(\mu(x * y) \geq \min\{\mu(x), \mu(y), \tfrac{1-k}{2}\}\right). \tag{3.2}$$

Corollary 3.3 ([4]). *A fuzzy set* μ *in* X *is an* $(\in, \in\vee\, q\,)$*-fuzzy subalgebra of* X *if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\}). \tag{3.3}$$

In what follows, let X denote a BCK/BCI-algebra, S a non-empty subset of X and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon > \delta$ unless otherwise specified.

Define an $(\varepsilon,\, \delta)$-*characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ in X as follows (see [8]):

$$\mu_S^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in S, \\ \delta & \text{otherwise.} \end{cases}$$

In particular, the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ in X with $\varepsilon = 1$ and $\delta = 0$ is the characteristic function χ_S of S in X.

Theorem 3.4. *If S is a subalgebra of X, then the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee\, q_k\,)$-fuzzy subalgebra of X.*

Proof. Assume that S is a subalgebra of X. For any $x, y \in X$, if $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), \tfrac{1-k}{2}\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \delta \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), \tfrac{1-k}{2}\right\}.$$

It follows from Lemma 3.2 that $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee\, q_k\,)$-fuzzy subalgebra of X. □

If we take $k = 1$ in Theorem 3.4, then we have the following corollary.

Corollary 3.5. *If S is a subalgebra of X, then the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee\, q\,)$-fuzzy subalgebra of X.*

The converse of Theorem 3.4 is not true in general as seen in the following example.

Example 3.6. *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the Cayley table which is given in Table 1. For a subset $S = \{0, c, d\}$ of X, consider an $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ in X with $\varepsilon = 0.7$ and $\delta = 0.4$. Then $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee\, q_k\,)$-fuzzy subalgebra of X for $k = 0.2$, but S is not a subalgebra of X since $d * c = b \notin S$.*

Theorem 3.7. *Let the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ be an $(\in, \in \vee\, q_k\,)$-fuzzy subalgebra of X. If $2\varepsilon + k \leq 1$, then S is a subalgebra of X.*

Table 1: Cayley table for the $*$-multiplication

$*$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	b	0	0	0
c	0	c	b	0	0
d	0	d	c	b	0

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Using Lemma 3.2, we have

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), \tfrac{1-k}{2}\right\} = \min\{\varepsilon, \tfrac{1-k}{2}\} = \varepsilon,$$

and so $x * y \in S$. Therefore S is a subalgebra of X. □

Taking $k = 0$ in Theorem 3.7 induces the following corollary.

Corollary 3.8. *Assume that $\varepsilon \leq 0.5$. If the $(\varepsilon,\ \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of X then S is a subalgebra of X.*

Corollary 3.9. *A non-empty subset S of X is a subalgebra of X if and only if the characteristic function χ_S of S is an $(\in, \in\vee q)$-fuzzy subalgebra of X.*

Proof. The necessity is by taking $\varepsilon = 1$ and $\delta = 0$ in Corollary 3.5.

Conversely, suppose that the characteristic function χ_S of S is an $(\in, \in\vee q)$-fuzzy subalgebra of X. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, which implies from (3.3) that

$$\chi_S(x * y) \geq \min\{\chi_S(x), \chi_S(y), 0.5\} = \min\{1, 0.5\} = 0.5.$$

Hence $x * y \in S$, and therefore S is a subalgebra of X. □

Theorem 3.10. *Let $\mu_S^{(\varepsilon,\delta)}$ be an $(\in,\ q_k)$-fuzzy subalgebra of X. If $2\delta + k \leq 1$ or $\varepsilon + \delta + k \leq 1$, then S is a subalgebra of X.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon > \delta$, that is, $x_\delta \in \mu_S^{(\varepsilon,\delta)}$ and $y_\delta \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\delta = (x * y)_{\min\{\delta,\delta\}}\ q_k\ \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x * y) + \delta + k > 1$. If $2\delta + k \leq 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) >$

$1-\delta-k \geq 1-\frac{1-k}{2}-k=\frac{1-k}{2} \geq \delta$. Thus $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$, and so $x*y \in S$. Therefore S is a subalgebra of X. Now, suppose that $\varepsilon+\delta+k \leq 1$. Since $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, we have $(x*y)_\varepsilon=(x*y)_{\min\{\varepsilon,\varepsilon\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x*y)+\varepsilon+k>1$. Since $\varepsilon+\delta+k \leq 1$, it follows that $\mu_S^{(\varepsilon,\delta)}(x*y)>1-\varepsilon-k \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$. Thus $x*y \in S$, and S is a subalgebra of X. □

If we take $k=0$ in Theorem 3.10, then we have the following corollary.

Corollary 3.11. *Let $\mu_S^{(\varepsilon,\delta)}$ be an $(\in,q)$-fuzzy subalgebra of X. If $\delta \leq 0.5$ or $\varepsilon+\delta \leq 1$, then S is a subalgebra of X.*

Theorem 3.12. *Assume that $2\varepsilon+k>1$. If the (ε,δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(q_k,\in)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x,y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x)=\varepsilon=\mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x)+\varepsilon+k=\varepsilon+\varepsilon+k>1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y)+\varepsilon+k=\varepsilon+\varepsilon+k>1,$$

that is, $x_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q_k,\in)$-fuzzy subalgebra of X, it follows that $(x*y)_\varepsilon=(x*y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$, that is, $x*y \in S$. Therefore S is a subalgebra of X. □

If we take $k=0$ in Theorem 3.12, then we have the following corollary.

Corollary 3.13. *Let $\varepsilon>0.5$. If the (ε,δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.14. *Assume that $2\varepsilon+k>1$ and $\varepsilon+\delta \leq 1$. If the (ε,δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a (q_k,q)-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x,y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x)=\varepsilon=\mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x)+\varepsilon+k=\varepsilon+\varepsilon+k>1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y)+\varepsilon+k=\varepsilon+\varepsilon+k>1,$$

that is, $x_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a (q_k,q)-fuzzy subalgebra of X, it follows that $(x*y)_\varepsilon=(x*y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y)>1-\varepsilon \geq \delta$, and therefore $\mu_S^{(\varepsilon,\delta)}(x*y)=\varepsilon$. This proves that $x*y \in S$, and S is a subalgebra of X. □

Theorem 3.15. *Assume that $\varepsilon>0.5$ and $\varepsilon+\delta+k \leq 1$. If the (ε,δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a (q,q_k)-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q,\, q_k\,)$-fuzzy subalgebra of X, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \, q_k \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon - k \geq \delta$, and therefore $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. This proves that $x * y \in S$, and S is a subalgebra of X. □

Combining Theorems 3.14 and 3.15, we have the following theorem.

Theorem 3.16. *Assume that $2\varepsilon + k > 1$ and $\varepsilon + \delta + k \leq 1$. If the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k\,,\, q_k\,)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

If we take $k = 0$ in Theorem 3.16, then we have the following corollary.

Corollary 3.17. *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta \leq 1$. If the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a (q, q)-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.18. *Assume that $\varepsilon + \delta + k \leq 1$. If the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in,\, \in \wedge\, q_k)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Assume that $\varepsilon + \delta + k \leq 1$ and the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \wedge\, q_k)$-fuzzy subalgebra of X. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \wedge\, q_k \mu_S^{(\varepsilon,\delta)}$, that is, $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \, q_k \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon + k > 1$. If $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and thus $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon + k > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon - k \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, which shows that $x * y \in S$. Therefore S is a subalgebra of X. □

The following corollary is induced by taking $k = 0$ in Theorem 3.18.

Corollary 3.19. *Assume that $\varepsilon + \delta \leq 1$. If the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in,\, \in \wedge\, q\,)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.20. *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta + k \leq 1$. If the $(\varepsilon,\, \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge\, q_k)$-fuzzy subalgebra or a $(q, \in \vee\, q_k\,)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q_k)$-fuzzy subalgebra of X, then

$$(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge \, q_k \mu_S^{(\varepsilon,\delta)},$$

that is, $\mu_S^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon + k > 1$. If $\mu_S^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$, then $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon + k > 1$, then $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon - k \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x * y \in S$, and therefore S is a subalgebra of X.

If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \vee\, q_k)$-fuzzy subalgebra of X, then $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\vee\, q_k \, \mu_S^{(\varepsilon,\delta)}$, and so that $(x*y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ or $(x*y)_\varepsilon \; q_k \, \mu_S^{(\varepsilon,\delta)}$. If $(x*y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$ and so $x*y \in S$. If $(x*y)_\varepsilon \; q_k \, \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon + k > 1$. Since $\varepsilon + \delta + k \leq 1$, it follows that $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon - k \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x*y \in S$. Therefore S is a subalgebra of X. □

Theorem 3.21. *Assume that $2\varepsilon + k > 1$ and $\varepsilon + \delta \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k \,, \in\wedge\, q\,)$-fuzzy subalgebra or a $(\, q_k \,, \in\vee\, q\,)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon + k = \varepsilon + \varepsilon + k > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon + = \varepsilon + \varepsilon + k > 1,$$

that is, $x_\varepsilon \, q_k \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q_k \, \mu_S^{(\varepsilon,\delta)}$. If $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k \,, \in\wedge\, q\,)$-fuzzy subalgebra of X, then

$$(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge \, q \, \mu_S^{(\varepsilon,\delta)},$$

that is, $\mu_S^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$, then $x*y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x*y \in S$, and therefore S is a subalgebra of X.

If $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k \,, \in\vee\, q\,)$-fuzzy subalgebra of X, then $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\vee\, q \, \mu_S^{(\varepsilon,\delta)}$, and so that $(x*y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ or $(x*y)_\varepsilon \; q \, \mu_S^{(\varepsilon,\delta)}$. If $(x*y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$ and so $x*y \in S$. If $(x*y)_\varepsilon \; q \, \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$. Since $\varepsilon + \delta \leq 1$, it follows that $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x*y \in S$. Therefore S is a subalgebra of X. □

Combining Theorems 3.20 and 3.21 induces the following theorem.

Theorem 3.22. *Assume that $2\varepsilon + k > 1$ and $\varepsilon + \delta + k \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(q_k, \in \wedge q_k)$-fuzzy subalgebra or a $(q_k, \in \vee q_k)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

By taking $k = 0$ in Theorem 3.22, we have the following corollary.

Corollary 3.23. *Assume that $\varepsilon > 0.5$ and $\varepsilon+\delta \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge q)$-fuzzy subalgebra or a $(q, \in \vee q)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.24. *Assume that $\varepsilon+\delta+k \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in \vee q, q_k)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. If S is not a subalgebra of X, then there exists $a, b \in S$ such that $a * b \notin S$. Thus $\mu_S^{(\varepsilon,\delta)}(a) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(b)$ and $\mu_S^{(\varepsilon,\delta)}(a * b) = \delta$. Hence $a_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, which imply that $a_\varepsilon \in \vee q\, \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in \vee q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}(a * b) + \varepsilon + k = \delta + \varepsilon + k \leq 1$, we have $(a * b)_\varepsilon\, \overline{q_k}\, \mu_S^{(\varepsilon,\delta)}$. This is a contradiction, and so S is a subalgebra of X. □

If we take $k = 0$ in Theorem 3.24, then we have the following corollary.

Corollary 3.25. *Assume that $\varepsilon + \delta \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in \vee q, q)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.26. *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta + k \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in \wedge q, q_k)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, $\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \varepsilon > 1$. Thus $x_\varepsilon \in \wedge q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \wedge q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is an $(\in \wedge q, q_k)$-fuzzy subalgebra of X, we have $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon, \varepsilon\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. It follows from the condition $\varepsilon + \delta + k \leq 1$ that $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon - k \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Hence $x * y \in S$, and S is a subalgebra of X. □

If we take $k = 0$ in Theorem 3.26, then we have the following corollary.

Corollary 3.27. *Assume that $\varepsilon > 0.5$ and $\varepsilon+\delta \leq 1$. If the (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $(\in \wedge q, q)$-fuzzy subalgebra of X, then S is a subalgebra of X.*

Theorem 3.28. *Assume that* $1-\varepsilon < \varepsilon+k \leq 1-\delta$. *If the* (ε, δ)*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $(\in\wedge\, q_k,\ q_k\,)$*-fuzzy subalgebra of* X*, then* S *is a subalgebra of* X*.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, $\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon + k > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \varepsilon + k > 1$. Thus $x_\varepsilon \in\wedge\, q_k \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in\wedge\, q_k \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is an $(\in\wedge\, q_k,\ q_k\,)$-fuzzy subalgebra of X, we have $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\,\varepsilon\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. It follows from the condition $\varepsilon + k \leq 1 - \delta$ that $\mu_S^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon - k \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Hence $x*y \in S$, and S is a subalgebra of X. □

Corollary 3.29. *Assume that* $1-\varepsilon < \varepsilon \leq 1-\delta$. *If the* (ε, δ)*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $(\in\wedge\, q\,,\ q\,)$*-fuzzy subalgebra of* X*, then* S *is a subalgebra of* X*.*

Theorem 3.30. *Assume that* $\varepsilon+\delta \leq 1$ *and* $\varepsilon > \frac{1-k}{2}$. *If the* (ε, δ)*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is a* $(\,q_k\,,\ \in\vee\, q\,)$*-fuzzy subalgebra of* X*, then* S *is a subalgebra of* X*.*

Proof. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon + k = \varepsilon + \varepsilon + k > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon + k = \varepsilon + \varepsilon + k > 1,$$

that is, $x_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(\,q_k\,, \in\vee\, q\,)$-fuzzy subalgebra of X, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\,\varepsilon\}} \in\vee\, q\, \mu_S^{(\varepsilon,\delta)}$, that is, $\mu_S^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$ or $\mu_S^{(\varepsilon,\delta)} + \varepsilon > 1$.

$\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$, that is, $x*y \in S$. Therefore S is a subalgebra of X. □

4 $\mathcal{R}$-conditional (α, β)-fuzzy subalgebras

Let $\mathcal{R} := \{\rho \in (0,1] \mid \rho \text{ has relations to } \varepsilon \text{ and/or } \delta\}$. An (ε, δ)-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ in X is called an $\mathcal{R}$*-conditional fuzzy subalgebra* of X with the type (α, β) (briefly, $\mathcal{R}$*-conditional* (α, β)*-fuzzy subalgebra* of X), where $\alpha, \beta \in \{\in,\ q\,,\ \in\vee\, q\,,\ \in\wedge\, q\,\}$, $\alpha \neq\ \in\wedge\, q\,$, or (α, β) is any one of $(\in,\ q_k\,)$, $(\in,\ \in\wedge\, q_k)$, $(\,q\,,\ q_k\,)$, $(\,q\,,\ \in\wedge\, q_k)$, $(\,q_k\,,\ \in)$, $(\,q_k\,,\ q\,)$, $(\,q_k\,,\ \in\wedge\, q\,)$, $(\,q_k\,,\ q_k\,)$ and $(\,q_k\,,\ \in\wedge\, q_k)$ if it satisfies the following condition, for any $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$,

$$x_{\rho_1} \alpha\, \mu_S^{(\varepsilon,\delta)},\quad y_{\rho_2} \alpha\, \mu_S^{(\varepsilon,\delta)} \ \Rightarrow\ (x*y)_{\min\{\rho_1,\rho_2\}}\, \beta\, \mu_S^{(\varepsilon,\delta)}. \tag{4.1}$$

Table 2: Cayley table for the $*$-multiplication

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Example 4.1. *(1) Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 2.*
Then X is a BCK-algebra (see [7]). Consider a subset $S := \{0, 2, 4\}$ of X and take

$\mathcal{R}_1 = \{\rho \in (0, 1] \mid 0.3 < \rho \leq 0.7\}$,

$\mathcal{R}_2 = \{\rho \in (0, 1] \mid 0.2 < \rho < 0.3\}$,

Then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}_1$-conditional $(\in, \in\wedge\, q\,)$-fuzzy subalgebra of X where $\delta = 0.2$ and $\varepsilon = 0.7$. The $(0.8,\, 0.3)$-characteristic fuzzyset $\mu_S^{(0.8,\,0.3)}$ is an $\mathcal{R}_2$-conditional $(q_{0.4},\, q_{0.4})$-fuzzy subalgebra of X.

(2) Let $X = \{0, 1, 2, a, b\}$ be a be a set with the Cayley table which is given in Table 3.

Table 3: Cayley table for the $*$-multiplication

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	a	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then X is a BCI-algebra (see [2, 7]). Consider a subset $S := \{0, 1, 2\}$ of X and let

$\mathcal{R}_1 = \{\rho \in (0, 1] \mid 0.3 < \rho \leq 0.9\}$,

$\mathcal{R}_2 = \{\rho \in (0, 1] \mid \rho \leq 0.4\}$,

$\mathcal{R}_3 = \{\rho \in (0,1] \mid 0.3 < \rho \le 0.7\}$.
Then $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}_1$*-conditional* (q,q)*-fuzzy subalgebra of* X *where* $\delta = 0.1$ *and* $\varepsilon = 0.7$. *The* $(0.4,\, 0.2)$*-characteristic fuzzyset* $\mu_S^{(0.4,\,0.2)}$ *is an* $\mathcal{R}_2$*-conditional* $(\in,\, \in\wedge\, q_{0.6})$*-fuzzy subalgebra of* X. *The* $(\varepsilon,\ \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}_3$*-conditional* $(q,\, \in\wedge\, q_{0.2})$*-fuzzy subalgebra of* X *with* $\delta = 0.3$ *and* $\varepsilon = 0.5$.

Theorem 4.2. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \rho > \delta \text{ and } \varepsilon + \rho > 1-k\}$. *If* S *is a subalgebra of* X, *then* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\in,\ q_k)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \in \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \in \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \ge \rho_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \ge \rho_2 > \delta$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon$, that is, $x, y \in S$. Thus $x * y \in S$, and so $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x*y) + \min\{\rho_1,\rho_2\} + k = \varepsilon + \min\{\rho_1,\rho_2\} + k > 1,$$

that is, $(x*y)_{\min\{\rho_1,\rho_2\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in,\ q_k)$-fuzzy subalgebra of X. □

Corollary 4.3. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \rho > \delta \text{ and } \varepsilon + \rho > 1\}$. *If* S *is a subalgebra of* X, *then* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\in, q)$*-fuzzy subalgebra of* X.

Theorem 4.4. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \ge \rho > \delta \text{ and } \varepsilon + \rho > 1-k\}$. *If* S *is a subalgebra of* X, *then* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\in, \in\wedge\, q_k)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \in \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \in \mu_S^{(\varepsilon,\delta)}$. In the proof of Theorem 4.2, we know that $(x*y)_{\min\{\rho_1,\rho_2\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Now, $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon \ge \min\{\rho_1,\rho_2\}$, that is, $(x*y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_{\min\{\rho_1,\rho_2\}} \in\wedge\, q_k\, \mu_S^{(\varepsilon,\delta)}$, and so $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, \in\wedge\, q_k)$-fuzzy subalgebra of X. □

Corollary 4.5. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \delta < \rho \le \varepsilon \text{ and } 1-\rho < \varepsilon\}$. *If* S *is a subalgebra of* X, *then the* (ε,δ)*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\in, \in\wedge\, q\,)$*-fuzzy subalgebra of* X.

Theorem 4.6. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \ge \rho \text{ and } \delta + \rho \le 1+k\}$. *If* S *is a subalgebra of* X, *then* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\,q_k\,, \in)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 + k > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 + k > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 + k \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 + k \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since S is a subalgebra of X, we have $x * y \in S$. Thus $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\}$, and hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\, q_k\,, \in)$-fuzzy subalgebra of X. □

Corollary 4.7. *Let $\mathcal{R} := \{\rho \in (0, 1] \mid \varepsilon \geq \rho$ and $\delta \leq 1 - \rho\}$. If S is a subalgebra of X, then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in)$-fuzzy subalgebra of X.*

Theorem 4.8. *Let $\mathcal{R} := \{\rho \in (0, 1] \mid \delta \leq 1 - \rho < \varepsilon + k\}$. If S is a subalgebra of X, then the $(\varepsilon,\ \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q,\ q_k\,)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 \geq \delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$ and so that $x, y \in S$. Since S is a subalgebra of X, we have $x * y \in S$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} + k = \varepsilon + \min\{\rho_1, \rho_2\} + k > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a $(q,\ q_k\,)$-fuzzy subalgebra of X. □

Corollary 4.9. *Let $\mathcal{R} := \{\rho \in (0, 1] \mid \delta \leq 1 - \rho < \varepsilon\}$. If S is a subalgebra of X, then the $(\varepsilon,\ \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional (q, q)-fuzzy subalgebra of X.*

Theorem 4.10. *Let $\mathcal{R} := \{\rho \in (0, 1] \mid \delta \leq 1 - \rho < \varepsilon + k$ and $\varepsilon \geq \rho\}$. If S is a subalgebra of X, then the $(\varepsilon,\ \delta)$-characteristic fuzzyset $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in\!\wedge\, q_k)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. We can see that $(x * y)_{\min\{\rho_1,\rho_2\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$ in the proof of Theorem 4.8. Since $\varepsilon \geq \rho$, we have $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \rho$, that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in\!\wedge\, q_k \mu_S^{(\varepsilon,\delta)}$, and therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in\!\wedge\, q_k)$-fuzzy subalgebra of X. □

Theorem 4.11. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \delta + \rho + k < 1 < \varepsilon + \rho\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\, q_k\,,\ q\,)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 + k > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 + k > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 - k \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 - k \geq \delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$ and so that $x, y \in S$. Since S is a subalgebra of X, we have $x * y \in S$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k\,,\ q\,)$-fuzzy subalgebra of X. □

Corollary 4.12. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \delta + \rho < 1 < \varepsilon + \rho\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\, q\,,\ q\,)$*-fuzzy subalgebra of* X.

Proof. It is by taking $k = 0$ in Theorem 4.11. □

Theorem 4.13. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \delta + \rho + k < 1 < \varepsilon + \rho\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\, q_k\,,\ q_k\,)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. In the proof of Theorem 4.11, we can see that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Hence

$$\begin{aligned} \mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} + k &= \varepsilon + \min\{\rho_1, \rho_2\} + k \\ &\geq \varepsilon + \min\{\rho_1, \rho_2\} > 1, \end{aligned}$$

and so $(x * y)_{\min\{\rho_1,\rho_2\}}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\, q_k\,,\ q_k\,)$-fuzzy subalgebra of X. □

Theorem 4.14. *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \geq \rho$ *and* $\varepsilon + \rho > 1 \geq \delta + \rho + k\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\, q_k\,, \in\!\wedge\, q\,)$*-fuzzy subalgebra of* X.

Proof. Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q_k\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q_k\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 + k > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 + k > 1$, which imply that

$\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 - k \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 - k \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since S is a subalgebra of X, we have $x * y \in S$ and thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\},$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now,

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

and so $(x * y)_{\min\{\rho_1,\rho_2\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in\!\wedge\, q \, \mu_S^{(\varepsilon,\delta)}$, and $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k \,, \in\!\wedge\, q\,)$-fuzzy subalgebra of X. □

Theorem 4.15. *Let* $\mathcal{R} := \{\rho \in (0, 1] \mid \varepsilon \geq \rho$ *and* $\varepsilon + \rho + k > 1 \geq \delta + \rho + k\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\, q_k \,, \in\!\wedge\, q_k)$*-fuzzy subalgebra of* X*.*

Proof. For any $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ such that $x_{\rho_1} \, q_k \, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \, q_k \, \mu_S^{(\varepsilon,\delta)}$, we have $x * y \in S$ in the proof of Theorem 4.14 since $\delta + \rho + k \leq 1$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\},$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now,

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} + k = \varepsilon + \min\{\rho_1, \rho_2\} + k > 1,$$

and so $(x * y)_{\min\{\rho_1,\rho_2\}} \, q_k \, \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in\!\wedge\, q_k \mu_S^{(\varepsilon,\delta)}$, and $\mu_S^{(\varepsilon,\delta)}$ is a $(\, q_k \,, \in\!\wedge\, q_k)$-fuzzy subalgebra of X. □

Corollary 4.16. *Let* $\mathcal{R} := \{\rho \in (0, 1] \mid \varepsilon \geq \rho$ *and* $\varepsilon + \rho > 1 \geq \delta + \rho\}$. *If* S *is a subalgebra of* X*, then the* $(\varepsilon,\, \delta)$*-characteristic fuzzyset* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(q, \in\!\wedge\, q\,)$*-fuzzy subalgebra of* X*.*

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