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On the properties of solutions for a system of non-linear differential equations of second order

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Abstract

In this paper, we relate with the stability, uniform stability, asymptotic stability, integrability and boundedness of solutions of a system of nonlinear differential equations of second order by the second method of Lyapunov. We obtain new sufficient conditions under which these properties of solutions of the equations considered are verified. Two examples are given to show the applicability of the results obtained and for illustrations. In the particular cases, the behaviors of the paths of that equations are exhibited by MATLAB-Simulink. The results of this paper extend and improve some results that can be found in the literature.

1 Introduction

In this paper, we consider the following system of nonlinear differential equations of second order (DE)

$$\ddot{X} + F\big(t, X, \dot{X}\big)\dot{X} + b(t)H(X) = P\big(t, X, \dot{X}\big), \tag{1.1}$$

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where $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $X \in \mathbb{R}^n$; F is a continuous symmetric $n \times n-$ matrix function for the arguments displayed, $b : \mathbb{R}^+ \to (0, \infty)$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions for the arguments displayed, and the functions $b(t)$ and H are also differentiable with $H(0) = 0$. Thus, the existence of the solutions of DE (1.1) is guaranteed by these basic assumptions. Moreover, we assume that the functions F, H and P satisfy the Lipschitz condition with respect to their respective arguments, except t. Thus, the uniqueness of the solutions of DE (1.1) is guaranteed by this assumption.

It follows from DE (1.1) that

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -F\big(t, X, Y\big)Y - b(t)H(X) + P\big(t, X, Y\big), \end{aligned} \tag{1.2}$$

which is obtained by putting $\dot{X} = Y$. Here, for brevity in notation, if a function is written without its argument, we mean that the argument is always t. For example X and Y represent $X(t)$ and $Y(t)$, respectively.

Denote the Jacobian matrix of $H(X)$ by

$$J_h(X) = \left(\frac{\partial h_i}{\partial x_j}\right), \quad (i, j = 1, 2, ..., n),$$

where $(x_1, x_2, ..., x_n)$ and $(h_1, h_2, ..., h_n)$ are components of X and H, respectively. We assume that the Jacobian matrix $J_h(X)$ exists and is continuous.

The symbol $\big\langle X, Y \big\rangle$ corresponding to any pair X, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum\limits_{i=1}^{n} x_i y_i$, that is, $\big\langle X, Y \big\rangle = \sum\limits_{i=1}^{n} x_i y_i$; thus $\big\langle X, Y \big\rangle = \|X\|^2$, and $\lambda_i(A), (i = 1, 2, ..., n)$ are the eigenvalues of the real symmetric $n \times n-$ matrix A. The matrix A is said to be negative-definite, when $\big\langle AX, X \big\rangle \le 0$ for all nonzero X in $\mathbb{R}^n$.

The motivation of the results of this paper has been inspired by the results in the papers and books in the references of this paper and more can be found in other sources such that all the ordinary differential equations, which are without delay, discussed in [1]-[35], are special cases of DE (1.1). Here, we would not like to give the details and comparisons with the results of this paper and the related ones that obtained in [1]-[35]. However, it is well known that the ordinary differential equations of second order have important and effective applications in control theory, sciences, engineering and more. Therefore, it deserves to investigate that equations in more general form than that discussed in the relevant literature. The aim of this paper is to do some new contributions to the relevant literature.

2 Qualitative results of solutions

Before giving the main results, we need the following lemma, which is known from the theory of the matrices.

Lemma 2.1. *If Λ is a real symmetric $n \times n-$ matrix and*

$$\sigma_2 \geq \lambda_i(\Lambda) \geq \sigma_1 > 0, \quad (i = 1, 2, ..., n),$$

where σ_1 and σ_2 are constants, then

$$\sigma_2 \|X\|^2 \geq \langle \Lambda X, X \rangle \geq \sigma_1 \|X\|^2.$$

Assumptions

(A1) There are positive constants b_1 and a_2 such that $1 \leq b(t) \leq b_1$, $b'(t) \geq 0$ for all $t \in \mathbb{R}^+$ and the matrix F is symmetric, and $\lambda_i(F(.)) \geq a_2$ for all $t \in \mathbb{R}^+$ and $X, Y \in \mathbb{R}^n$.

(A2) There are positive constants a_0 and a_1 such that $H(0) = 0$, $H(X) \neq 0, (X \neq 0)$, $J_h(X)$ is symmetric and $a_0 \leq \lambda_i(J_h(X)) \leq a_1$ for all $X \in \mathbb{R}^n$.

(A3) There is a continuous function $q(t)$ such that $\|P(t, X, Y)\| \leq |q(t)| \|Y\|$ for all $t \geq t_0$ and $X, Y \in \mathbb{R}^n$, where $|q(t)| \in L^1(0, \infty)$, $L^1(0, \infty)$ is space of integrable Lebesgue functions.

Let $P(.) \equiv 0$.

Our first result is given by the following theorem.

Theorem 2.2. *If assumptions (A1) and (A2) hold, then the zero solution of DE* (1.1) *is asymptotically stable, when $P(.) \equiv 0$.*

Proof. We define a continuous differentiable Lyapunov function $W(.) = W(t, X, Y)$ by

$$W(.) = \int_0^1 \langle H(\sigma X), X \rangle d\sigma + \frac{1}{2b(t)} \langle Y, Y \rangle.$$

It is clear that $W(t,0,0)=0$. On the other hand, since $H(0)=0$, $\frac{\partial}{\partial\sigma}H(\sigma X)=J_h(\sigma X)X$ and $\lambda_i\left(J_h(X)\right)\geq a_0$, then

$$H(X)=\int_0^1 J_h(\sigma X)X\,d\sigma$$

so that

$$\begin{aligned}
\int_0^1 \big\langle H(\sigma X),X\big\rangle d\sigma &= \int_0^1\int_0^1 \big\langle \sigma_1 J_h(\sigma_1\sigma_2 X)X,X\big\rangle d\sigma_2 d\sigma_1 \\
&\geq \int_0^1\int_0^1 \big\langle \sigma_1 a_0 X,X\big\rangle d\sigma_2 d\sigma_1 \\
&\geq \frac{a_0}{2}\left\|X\right\|^2.
\end{aligned}$$

Hence, it is clear that

$$W(.)\geq K_0\left(\left\|X\right\|^2+\left\|Y\right\|^2\right),$$

where $K_0=\min\{a_0,b_1^{-1}\}$.

The calculation of the time derivative of the Lyapunov function $W(.)$ along any solution (X,Y) of non-linear differential system (DS) (1.2) gives

$$\frac{d}{dt}W(.)=-\big\langle H(X),Y\big\rangle-\frac{1}{b(t)}\big\langle F(t,X,Y)Y,Y\big\rangle-\frac{b'(t)}{2b^2(t)}\big\langle Y,Y\big\rangle+\frac{d}{dt}\int_0^1 \big\langle H(\sigma X),X\big\rangle d\sigma.$$

Further, it follows that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \big\langle H(\sigma X),X\big\rangle d\sigma &= \int_0^1 \sigma\big\langle J_h(\sigma X)Y,X\big\rangle d\sigma+\int_0^1 \big\langle H(\sigma X),Y\big\rangle d\sigma \\
&= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\big\langle H(\sigma X),Y\big\rangle d\sigma+\int_0^1 \big\langle H(\sigma X),Y\big\rangle d\sigma \\
&= \sigma\big\langle H(\sigma X),Y\big\rangle\Big|_0^1=\big\langle H(X),Y\big\rangle.
\end{aligned}$$

Hence, we can conclude

$$\dot{W}(.) = -\frac{1}{b(t)}\big\langle F(t,X,Y)Y,Y\big\rangle - \frac{b'(t)}{2b^2(t)}\big\langle Y,Y\big\rangle.$$

By the assumptions $\lambda_i(F(t,X,Y)) \geq a_2$, $b'(t) \geq 0$ and $b(t) > 0$, we have

$$-\frac{1}{b(t)}\big\langle F(t,X,Y)Y,Y\big\rangle \leq -\frac{a_2}{b_1}\big\|Y\big\|^2$$

and

$$-\frac{b'(t)}{2b^2(t)}\big\|Y\big\| \leq 0.$$

Then, it is clear that

$$\dot{W}(.) \leq -\frac{a_2}{b_1}\big\|Y\big\|^2 \leq 0.$$

This inequality shows that the time derivative of the Lyapunov function $W(.)$ is negative semidefinite. Hence, we can conclude that the zero solution of DE (1.1) is stable.

We now consider the set defined by

$$I_s \equiv \{(t,X,Y) : \dot{W}(t,X,Y) = 0\}.$$

If we apply the LaSalle's invariance principle, then we observe that $(t,X,Y) \in I_s$ implies that $Y = 0$. Hence, DS (1.2) and together $Y = 0$, necessarily, implies

$$H(X) = 0.$$

Since, $Y = 0 \Rightarrow \dot{X} = 0$, then $X = \xi$, $\xi(\neq 0)$ is a vector. This equality can be hold if and only if

$$H(\xi) = 0.$$

Hence,

$$H(\xi) = 0 \Leftrightarrow \xi = 0$$

so that $H(X) = 0 \Leftrightarrow X = 0$. Therefore, we have $X = Y = 0$. In fact, this result shows that the largest invariant set contained in the set I_s is $(t,0,0) \in I_s$. Therefore, we can conclude that the zero solution of DS (1.2) is asymptotically stable. This completes the proof of Theorem 2.2. □

Corollary 2.3. *It can be proved that under the assumptions of Theorem 2.2, the zero solution of DE* (1.1) *is uniformly stable.*

Theorem 2.4. *If assumptions (A1) and (A2) hold, then the first derivatives of the solutions of DE* (1.1) *are square integrable when* $P(.) \equiv 0$.

Proof. We now give our attention to the Lyapunov function $W(t) = W(.) = W(t, X, Y)$ that is used in the proof of Theorem 2.2.

It is known from Theorem 2.2 that

$$\dot{W}(.) \leq -\frac{a_2}{b_1}\left\|Y\right\|^2 \leq 0.$$

Integrating this inequality from t_0 to t, we have

$$W(t, X(t), Y(t)) - W(t_0, X(t_0), Y(t_0)) \leq -\frac{a_2}{b_1}\int_{t_0}^{t} \left\|Y(s)\right\|^2 ds \leq 0.$$

Hence, it is clear that

$$W(t, X(t), Y(t)) + \frac{a_2}{b_1}\int_{t_0}^{t} \left\|Y(s)\right\|^2 ds \leq W(t_0, X(t_0), Y(t_0)).$$

We know the Lyapunov function $W(t, X, Y)$ is positive definite. Let

$$W(t_0, X(t_0), Y(t_0)) = K_1, \quad K_1 \in \mathbb{R}, \quad K_1 > 0.$$

Hence

$$\frac{a_2}{b_1}\int_{t_0}^{t} \left\|Y(s)\right\|^2 ds \leq W(t, X(t), Y(t)) + \frac{a_2}{b_1}\int_{t_0}^{t} \left\|Y(s)\right\|^2 ds \leq K_1.$$

This leads that

$$\int_{t_0}^{\infty} \left\|Y(s)\right\|^2 ds = \int_{t_0}^{\infty} \left\|\dot{X}(s)\right\|^2 ds \leq K_1 b_1 a_2^{-1}.$$

This result completes the proof of Theorem 2.4. $\square$

Theorem 2.5. *If the assumptions of Theorem 2.4 hold, then all solutions of DE* (1.1) *and their first order derivatives are bounded as* $t \to \infty$ *when* $P(.) \equiv 0$.

Proof. We consider the current inequalities

$$W(t,X,Y)+\frac{a_2}{b_1}\int_{t_0}^{t}\|Y(s)\|^2ds\le K_1$$

and

$$\|X\|^2+\|Y\|^2\le K_0^{-1}W(t,X,Y),$$

which can be found in the proofs of Theorem 2.4 and Theorem 2.2, respectively.

Then, in view of these inequalities, we can conclude that

$$\|X\|^2+\|Y\|^2\le K_2,\quad K_2=K_0^{-1}K_1.$$

This inequality competes the proof of Theorem 2.5. □

Let $P(.)\neq 0$.

Our fourth and the last result is the following theorem.

Theorem 2.6. *If assumptions* $(A1)-(A3)$ *hold, then there exists a positive constant* K_3 *such that all solutions DE* (1.1) *satisfies the inequalities*

$$\|X(t)\|\le K_3,\qquad \|\dot{X}(t)\|\le K_3$$

as $t\to+\infty$ *when* $P(.)\neq 0$.

Proof. We again consider the Lyapunov function, which is defined in Theorem 2.2. It is clear that

$$W(.)\ge K_0\left(\|X\|^2+\|Y\|^2\right).$$

Since $P(.)\neq 0$, the time derivative of the function $W(.)$ can be revised as follows

$$\begin{aligned}\dot{W}(.)&\le\frac{1}{b(t)}\langle Y,P(t,X,Y)\rangle\\&\le\|Y\|\|P(t,X,Y)\|\\&\le|q(t)|\|Y\|^2.\end{aligned}$$

Hence, it is clear that

$$\dot{W}(.)\le K_0^{-1}|q(t)|W(.).$$

Integrating the last estimate from 0 to t, $(t \geq 0)$, we have

$$W(t, X(t), Y(t)) \leq W(0, X(0), Y(0)) \exp\left(K_0^{-1} \int_0^t \left|q(s)\right| ds\right).$$

Let

$$K_2 = W(0, X(0), Y(0)).$$

Then, it is obvious that

$$W(t, X, Y) \leq K_2 \exp\left(K_0^{-1} \int_0^\infty \left|q(s)\right| ds\right).$$

Let

$$K_4 = K_2 \exp\left(K_0^{-1} \int_0^\infty \left|q(s)\right| ds\right).$$

Then, we can conclude that

$$\left\|X\right\|^2 + \left\|Y\right\|^2 \leq K_3 \quad \text{as} \quad t \to +\infty,$$

where $K_3 = K_0^{-1} K_4$. This completes the idea of Theorem 2.6. □

Remark 2.7. *In the proof of Theorem 2.6, without using the Gronwall-Bellman [1] inequality, we proved the bounded result. In fact, in Theorem 2.6, we have weaker conditions than that can found in literature and delete some reasonless of the conditions therein. This the positive effect of assumption (A3).*

Corollary 2.8. *Let us modify assumption (A3) as $\|P(t, X, Y)\| \leq \left|r(t)\right|$ for all $t \geq t_0$, where $\left|r(t)\right| \in L^1(0, \infty)$. In this case, in the light of the related assumptions, the time derivative of function $W(.)$ can give the following inequality*

$$\dot{W}(.) \leq \left|r(t)\right| \left\|Y\right\| \leq \sqrt{K_0} \left|r(t)\right| \sqrt{W(.)}.$$

The calculation of the integral of the last inequality from 0 to t, $(t \geq 0)$, implies that

$$\sqrt{W(.)} \leq \frac{1}{2}\sqrt{K_2} + \frac{1}{2}\sqrt{K_0} \int_0^t \left|r(s)\right| ds$$

so that

$$\sqrt{K_0}\left(\sqrt{\|X\|^2+\|Y\|^2}\right) \le \sqrt{W(.)} \le \frac{1}{2}\sqrt{K_2}+\frac{1}{2}\sqrt{K_0}\int_0^\infty |r(s)|ds.$$

This inequality guarantees the boundedness all solution of DE (1.2) *as* $t \to \infty$.

Example 2.1. In particular case of DE (1.1), when $P(.) \equiv 0$, we consider the following LDE

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &+ \begin{bmatrix} 9+\exp(-t)+(x_1^2+x_1'^2)^{-1} & 0 \\ 0 & 6+\exp(-t)(x_2^2+x_2'^2)^{-1} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ (2-\exp(-t)) \times \begin{bmatrix} x_1+\arctan x_1 \\ x_2+\arctan x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \ge 0 \qquad (2.1) \end{aligned}$$

When we compare LDE (2.1) with DE (1.1), it is easy to see following findings.

$$F(t,X,X') = \begin{bmatrix} 9+\exp(-t)+(x_1^2+x_1'^2)^{-1} & 0 \\ 0 & 6+\exp(-t)(x_2^2+x_2'^2)^{-1} \end{bmatrix}.$$

The eigenvalues of $F(.)$ are

$$\lambda_1(F(.)) = 9+\exp(-t)+(x_1^2+x_1'^2)^{-1}$$

and

$$\lambda_2(F(.)) = 6+\exp(-t)(x_2^2+x_2'^2)^{-1}.$$

Hence,

$$\lambda_i(F(.)) \ge 6 = a_2 > 0, \quad (i=1,2).$$

Next,

$$\begin{aligned} b(t) &= 2-\exp(-t), \quad 1 \le 2-\exp(-t) \le 2 = b_1, \\ b'(t) &= \exp(-t) \ge 0, \quad t \ge 0. \end{aligned}$$

Further,

$$H(X) = \begin{bmatrix} x_1+\arctan x_1 \\ x_2+\arctan x_2 \end{bmatrix}.$$

The Jacobian matrix of $H(X)$ is given by

$$J_h(X) = \begin{bmatrix} 1+(1+x_1^2)^{-1} & 0 \\ 0 & 1+(1+x_2^2)^{-1} \end{bmatrix}.$$

Then, the eigenvalues of $J_h(X)$ are

$$\lambda_1(J_h(X)) = 1+(1+x_1^2)^{-1}$$

and

$$\lambda_2(J_h(X)) = 1+(1+x_2^2)^{-1}.$$

From this point, we have

$$a_0 = 1 \leq \lambda_i(J_h(X)) \leq 2 = a_1, \quad (i=1,2).$$

Thus, all the assumptions of Theorem 2.2, Theorem 2.4 and Theorem 2.5 are satisfied. Therefore, we can conclude that the zero solution of DE (2.1) is asymptotically stable, the first derivatives of the solutions of DE (2.1) are square integrable and all solutions of DE (2.1) and their first order derivatives are bounded as $t \to \infty$ when $P(.) \equiv 0$ (see, Figure 1 and Figure 2).

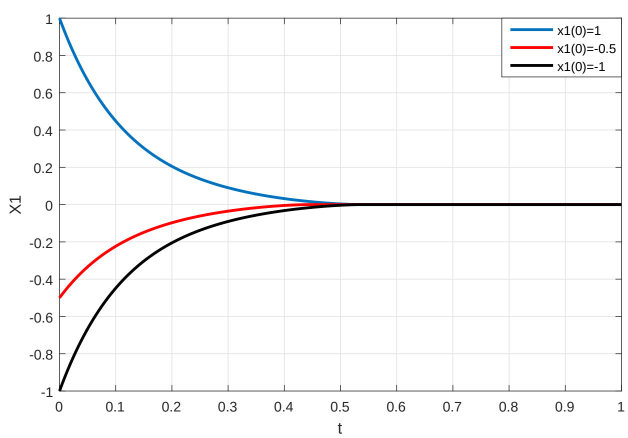

Figure 1: Paths of $x_1(t)$ for Example 2.1

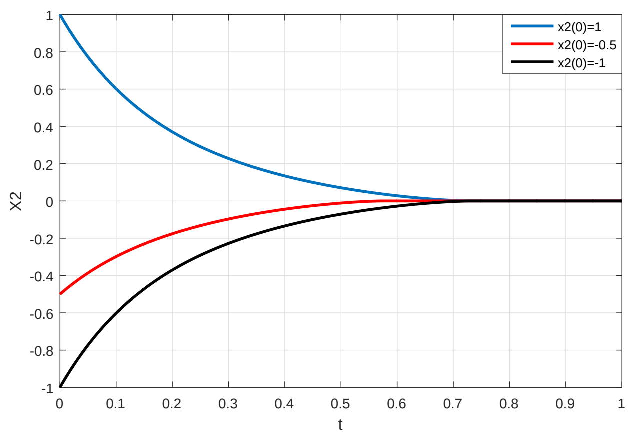

Figure 2: Paths of $x_2(t)$ for Example 2.1

Example 2.2. In a particular case of LDE (1.1), when $P(.) \neq 0$, we consider the following LDE

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} 9+\exp(-t)+(x_1^2+x_1'^2)^{-1} & 0 \\ 0 & 6+\exp(-t)(x_2^2+x_2'^2)^{-1} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+(2-\exp(-t)) \times \begin{bmatrix} x_1+\arctan x_1 \\ x_2+\arctan x_2 \end{bmatrix} = \begin{bmatrix} (\cos t)x_1'(1+t^2)^{-1}(2+x_1^2+x_1'^2)^{-1} \\ (\sin t)x_2'(1+t^2)^{-1}(1+x_2^2+x_2'^2)^{-1} \end{bmatrix}. \tag{2.2}$$

The comparison done for DE (2.1) is also valid for DE (2.2). In addition, if we compare DE (2.2) with DE (1.1), it is clear that

$$P(t,X,X') = \begin{bmatrix} \frac{(\cos t)x_1'}{(1+t^2)(2+x_1^2+x_1'^2)} \\ \frac{(\sin t)x_2'}{(1+t^2)(1+x_2^2+x_2'^2)} \end{bmatrix}.$$

Further, it follows that

$$\|P(t,X,X')\| \leq \frac{1}{1+t^2}\|X'\| = g(t)\|Y\|.$$

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2},$$

that is, $q(t) \in L^1(0,\infty)$. Thus, all the assumptions of Theorem 2.6 hold. Then all solutions of LDE (2.2) and their first order derivatives are bounded as $t \to \infty$ (see, Figure 3 and Figure 4).

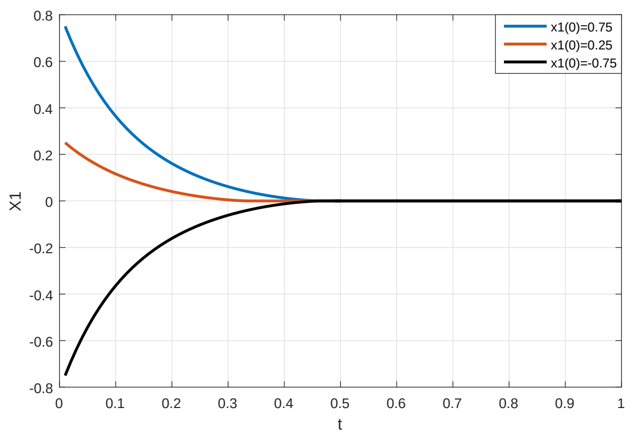

Figure 3: Paths of $x_1(t)$ for Example 2.2

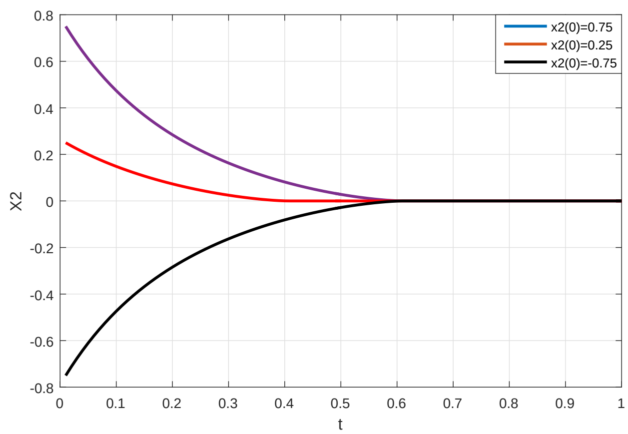

Figure 4: Paths of $x_2(t)$ for Example 2.2

3 Conclusion

A class of non-linear differential equations of second order is considered. We investigate the asymptotically stability, integrability and boundedness of solutions of this system are investigated by the second method of Lyapunov. Examples are given to show the applicability of the assumptions constructed. By means of MATLAB-Simulink, it is shown the behaviors of the paths of solutions of DE (2.1) and DE (2.2).

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