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Finite groups in which nearly S-permutability is a transitive relation

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Abstract

A subgroup H of G is called nearly S-permutable in G if for every prime p such that $(p, |H|) = 1$ and for every subgroup K of G containing H the normalizer $N_K(H)$ contains some Sylow p-subgroup of K. A group G is called an $NSPT$-group if nearly S-permutability is a transitive relation in G. A number of new characterizations of finite solvable $NSPT$-groups are given.

1 Introduction

Throughout this paper, we assume that all groups considered are finite. Our notation is standard and consistent with [3]. We introduce and study the class $NSPT$-group; that is, the class of all finite groups in which nearly S-permutability is a transitive relation. More specifically, we are interested in

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subgroups, homomorphic images, and direct product of $NSPT$-groups and other related algebraic properties.

A subgroup H of a group G is said to *permute* with a subgroup K if HK is a subgroup of G. H is said to be *permutable* in G (or S-permutable) if it permutes with all the (Sylow) subgroups of G. One of the earliest results about permutable subgroups [7] states that every permutable subgroup... In [6] Kegel proved that S-permutable subgroups are necessarily subnormal. Actually a result stronger than permutable subgroups is that of subnormal subgroups. For a subgroup H of G, it is enough to know that H permutes with all of its conjugates to deduce that H is subnormal (see [5], p. 50). Nearly S-permutability have been introduced and studied in [2], while permutable and S-permutable subgroups are subnormal while a nearly S-permutable subgroup need not be subnormal in general. As an example one might consider D_{18} the dihedral group of order 18 which has three subgroups of order 6 and each of these subgroups will be nearly S-permutable but not subnormal. There has been several approaches to apply normality, permutibility and S-permutibility in the study of some finite group classes. To make our point clear we list some helping notations H s -per G denotes H is S-permutable in G, H per G denotes H is permutable in G, and H nsp G denotes H is nearly S-permutable in G. Let ρ be any of the properties {normality, permutability, Sylow permutability}. The property ρ is said to be transitive in a group G if for any two subgroups A and B in G the relations $A\ \rho\ B$, and $B\ \rho\ G$ always implies $A\ \rho\ G$. By T-groups, PT-groups, and PST-groups we denote the class of groups in which normality, (repct. permutability, and S-permutability) is transitive relation.

One of the generalizations for S-permutable subgroups we are interested in is the *nearly S-permutable* subgroups which was introduced in [2]. In this paper we introduce and study finite groups in which *nearly S-permutability is transitive* relation. Our main object is to determine some properties of finite solvable $NSPT$-groups.

We first consider the following definition:

Definition 1.1. *(Al-Sharo [2]) A subgroup H of G is called nearly S-permutable in G if for every prime p such that $(p, |H|) = 1$ and for every subgroup K of G containing H the normalizer $N_K(H)$ contains some Sylow p-subgroup of K. We shall write H nsp G to denote that H is nearly S-permutable in G.*

Motivated by the theory of *T-groups* (resp. *PT-groups*, *PST-groups*), we introduce the class of $NSPT$-groups.

Definition 1.2. *A group G is called $NSPT$-group if nearly S-permutablity is a transitive relation in G. That is, G is $NSPT$-group if for all subgroups H and K where H nsp K nsp G we have H nsp G.*

Since the concept of $NSPT$-groups relays on nearly S-permutability we fix some elementary properties of nearly S-permutable subgroups that follows directly from the definition.

Remark 1.3. *Let H be a nearly S-permutable subgroup of a group G. If K is any subgroup of G such that $H \leq K \leq G$. Then H is nearly S-permutable in K.*

Remark 1.4. *If G is p-group then every subgroup of G is nearly S-permutable in G.*

Remark 1.5. *Let us denote by $\pi(G)$ the set of all prime divisors of the group G. If H is subgroup of G such that $\pi(H) = \pi(G)$then H is nearly S-permutable in G.*

It is clear that the class $\mathfrak{A}$ of abelian groups is an example of $NSPT$-groups. Moreover, $\mathfrak{N}_p$-the class of all p-groups is another example of $NSPT$-group. One of the nice facts about the class of $NSPT$-groups is the following:

Theorem A. *Every nilpotent group is $NSPT$-group.*

As an example of *non $NSPT$-group* we have the following:

Example 1. Consider the alternating group $A_4 =< (1,2,3),(1,2)(3,4) >$ of order 12. If we take $H =< (1,3)(2,4),(1,2)(3,4) >$ to be the 2-Sylow subgroup of A_4 then H is normal in A_4. Hence, H is nearly S-permutable in A_4. Now H being a group of order 4 then H must abelian and every subgroup of H would be norm in H. Let us take $K =< (1,3)(2,4) >$. Then K nsp H nsp A_4 but K is not nsp A_4. To see this note that $K \leq A_4$, and $g,c,d,(3,|K|) = 1$, and $N_{A_4}(K) = H$ which doesn't contain any 3-Sylow subgroup of A_4. Therefor, K is not nsp A_4 and A_4 is not $NSPT$-group.

In the years 1953, 1964, and 1975, Gaschütz, Zacher, and Agrawal, respectively, proved the following definitive results on solvable T-groups, PT-groups, and PST-groups.

Theorem 1. *(Gaschütz [4], Zacher [10], Agrawal [1]) A solvable T-group (PT-group, PST-group) is supersolvable.*

The next theorem gives a similar result for $NSPT$-groups.

Theorem B. *A solvable $NSPT$-group is supersolvable.*

Theorem C. *If G_1 and G_2 are two $NSPT$-groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is also a $NSPT$-group.*

Remark 1.6. *In Theorem C., the condition that $(|G_1|, |G_2|) = 1$ is necessary. The following example shows this.*

Example 2. Let $C_3 =< c : c^3 = e >$ be the cyclic group of order 3 and $S_3 =< a, b : a^3 = b^2 = (ba)^2 = 1 >$ be the symmetric group on 3 letters. Then C_3 -being abelian group- must be $NSPT$-group. In the group S_3 the only nearly S-permutable subgroups are $< e >$, $A_3 =< a >$, and S_3 it self. Hence S_3 is also a $NSPT$-group. Let $D = S_3 \times C_3$. We show that D is not $NSPT$-group. Let $B = \langle (a, e), (e, c) \rangle \simeq A_3 \times C_3$, $B \in Syl_3(D)$. Then B has order 3^2 and index 2 in D. Therefore, B is abelian normal subgroup in D. The normality of B in D implies that B is nearly S-permutable in D. The fact that B is abelian implies that every subgroup of B is nearly S-permutable in B. In particular if we pick $A = \langle (a, c) \rangle$ then A nsp B, and B nsp D, but A is not nsp D. To see this we consider $A \leq G$ with $p = 2$ for which $(2, |A|) = 1$. Then the 2-Sylow subgroup of D has order 2 and is not contained in $N_D(A)$ and A is not nsp D. Hence D is not $NSPT$-group.

2 Preliminaries

In this section we list some results that are interesting in their own and some of which will be used in the proofs of the given theorem.

Lemma 2.1. *(Frattini Argument, see [5, Lemma 1.13.]) Let $N \trianglelefteq G$ and suppose that $P \in Sul_p(N)$. Then $G = N_G(P)N$.*

Lemma 2.2. *Let G_1 and G_2 be two groups such that $(|G_1|, |G_2|) = 1$, and $G = G_1 \times G_2$. Then the following statements are true:*

1) every p-Sylow subgroup of G is isomorphic to a p-Sylow either of G_1 or G_2.

2) for any subgroup $H \leq G$ we have and $N_G(H) \simeq N_{G_1}(H_1) \times N_{G_2}(H_2)$. Where $H_i \leq G_i$, for $i \in \{1, 2\}$.

Proof. 1)Since $G_1 \cap G_2 = 1$, and $G_i \trianglelefteq G$. Then $G \simeq G_1G_2$. Now if $P \in Syl_p(G)$ and $P_i \in Syl_p(G_i)$ then $P_i = P \cap G_i$, for $i \in \{1,2\}$ and 1) follows.

2) The second part of this statement follows from the first, and the first follows from $(|G_1|, |G_2|) = 1$. □

Lemma 2.3. *Let H be a subgroup of G. If H is S-permutable in G then H is nearly S-permutable in G.*

Proof. Let H be nearly S-permutable in G. If $\pi(G) = \pi(H)$ then by Remark 1.5 H will be nearly S-permutable in G. So we may assume that $\pi(G) \backslash \pi(H) \neq \phi$. Let p be any prime in $\pi(G) \backslash \pi(H)$. For any subgroup K of G such that $H \leq K \leq G$ we pick $P_K \in Syl_p(K)$. Since H s-per G, then H s-per K and therefore HP_K is a subgroup of K. Therefor, H is S-permutable in HP_K. So, H is subnormal in HP_K. Note that $g.c.d.(|H|, |HP_K : H|) = 1$. That is H is a subnormal Hall subgroup of HP_K. Hence, H is normal in HP_K. Therefore, $P_K \leq N_K(H)$ and H is nearly S-permutable in G. □

Lemma 2.4. *([2, Lemma 2.2])Let G be a group, $H \leq G$ and N be a normal subgroup of G.*

(1) *If H is nearly S-permutable in G, then HN is nearly S-permutable in G.*

(2) *If H is nearly S-permutable in G and H is a group of prime power order, then $H \cap N$ is nearly S-permutable in G.*

(3) *If H is nearly S-permutable in G and H is a group of prime power order, then HN/N is nearly S-permutable in G/N for any normal subgroup N of G.*

(4) *If H is nearly S-permutable in G and $|H| = p^n$ for some prime p, then $H \leq O_p(G)$.*

3 The Proofs

Proof of Theorem A. Let G be a nilpotent group then every p-Sylow subgroup of G is normal in G. Therefore, every p-Sylow subgroup permutes with every subgroup of G. That means every subgroup of G is S-permutable in G. By Lemma 2.3 every subgroup of G is nearly s-permutable in G. Hence, G is $NSPT$-group. □

Proof of Theorem B. Let G be a solvable group we prove that if G is not supersolvable then G is not $NSPT$-group. Since G is solvable and not supersolvable group then G has a chief factor A/B which is elementary abelian of order p^k for some prime p and an integer $k > 1$. Let P be a p-Sylow subgroup of A. From the Frattini argument (see Lemma 2.1) we get: $G = AN_G(P)$. Since P/B is normal in A/B then P is normal in A, consequently P is characteristic in A. Now$P\ char\ A \trianglelefteq G$ implies P normal in G. Hence $N_G(P) = G$. Let Q be a p-subgroup such that $PB \leq Q \leq P$ such that Q/B is in the center of a p-Sylow subgroup of G/B. First,we show that Q is not nearly S-permutable in G.

Since A/B is a chief factor of G, Q can not be normal in G. Since, $N_G(Q) \geq P \in Syl_p(G)$ and $N_G(Q) \neq G$ then there is a prime $r \neq p$ and an r-element $g \in G$ such that $g \notin N_G(Q)$. Let $K = \langle P, g\rangle$. So, $|K| = |P|\, r^t$ where g has order r^t. So, $N_K(Q)$ does not contain an r-Sylow subgroup R of K. That is Q is not nearly S-permutable in G. Hence, we have: Q nsp P, P nsp G, but Q is not nsp G. Therefore G is not $NSPT$-group. Contradiction, and the theorem follows. □

Proof of Theorem C. Let $G_1$and G_2 be two $NSPT$-groups with $(|G_1|, |G_2|) = 1$, and set $G = G_1 \times G_2$. Let $A \leq B$ be two subgroups of G such that A is nearly S-permutable in B, and B is nearly S-permutable in G. Assume that A is not nsp G. Then there exists a subgroup K of G such that $A \leq K \leq G$ and for some prime number p with $(p, |A|) = 1$ some p-Sylow subgroup P of K is not in $N_K(A)$. From Lemma 2.2 (1) $P = P_1P_2$ where $P_i \leq G_i$ for $i \in \{1, 2\}$. From $(|G_1|, |G_2|) = 1$ one of the the P_i's must be 1. So, we may assume that $p \in \pi(G_1)$. It is clear that all the proof's assumptions may be translated to the group G_1. Again, by Lemma2.2 (2) and the assumption $(|G_1|, |G_2|) = 1$ we get: $A = A_1A_2$ nsp $B = B_1B_2$ nsp $G = G_1 \times G_2$ implies A_1 nsp B_1 nsp G_1 and A_1 is not nsp G_1. Contradiction with the assumption that G_1 is $NSPT$-group and the theorem is proved. □

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