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Formulas for finding UP-algebras

Akarachai Satirad, Phakawat Mosrijai, Aiyared Iampan

Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand

email: akarachai.sa@gmail.com, phakawat.mo@gmail.com,
aiyared.ia@up.ac.th

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Abstract

In this paper, we prove that every nonempty set and every nonempty totally ordered set can be a UP-algebra.

1 Introduction and Preliminaries

Iampan [1] introduced a new algebraic structure, called a UP-algebra, which is a generalization of a KU-algebra. Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Denote $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$ and $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting

$$A \cdot B = B \cap (A' \cup \Omega) \text{ for all } A, B \in \mathcal{P}_\Omega(X)$$

and a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting

$$A * B = B \cup (A' \cap \Omega) \text{ for all } A, B \in \mathcal{P}^\Omega(X).$$

Satirad et al. [3] proved that $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ and $(\mathcal{P}^\Omega(X), *, \Omega)$ are UP-algebras. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras.

In this paper, we prove that every nonempty set and every nonempty totally ordered set can be a UP-algebra.

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Now we will recall the definition of a UP-algebra from [1].

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

For a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [1, 2]).

$$(\forall x \in A)(x \cdot x = 0), \tag{1.1}$$
$$(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{1.2}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{1.3}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{1.4}$$
$$(\forall x, y \in A)(x \cdot (y \cdot x) = 0), \tag{1.5}$$
$$(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{1.6}$$
$$(\forall x, y \in A)(x \cdot (y \cdot y) = 0), \tag{1.7}$$
$$(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{1.8}$$
$$(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{1.9}$$
$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{1.10}$$
$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{1.11}$$
$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \text{ and} \tag{1.12}$$
$$(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{1.13}$$

2 Main Results

In this section, we prove three theorems which show that every nonempty set and every nonempty totally ordered set can be a UP-algebra.

Theorem 2.1. *Let X be a nonempty set and let arbitrary $t \in X$. Define a binary operation $\cdot$ on X by: for all $x, y \in X$,*

$$x \cdot y = \begin{cases} y & \text{if } x \neq y, \\ t & \text{otherwise.} \end{cases}$$

Then $(X, \cdot, t)$ *is a UP-algebra.*

Proof. For any $x \in X$, we have $t \cdot x = x$ and $x \cdot t = t$. Thus (UP-2) and (UP-3) hold.

Let $x, y, z \in X$.

Case 1: $x = y = z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot (t \cdot t) = t \cdot t = t.$$

Case 2: $x = y$ but $y \neq z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (t \cdot z) = z \cdot z = t.$$

Case 3: $x = z$ but $z \neq y$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot t) = z \cdot t = t.$$

Case 4: $y = z$ but $z \neq x$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot (y \cdot z) = t \cdot t = t.$$

Case 5: $x \neq y, x \neq z$ and $y \neq z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot z) = z \cdot z = t.$$

Thus (UP-1) holds.

Let $x, y \in X$ be such that $x \cdot y = t$ and $y \cdot x = t$. If $x \neq y$, then $x \cdot y = y$ and $y \cdot x = x$. Thus $x = t = y$, which is a contradiction. Hence, $x = y$. Thus (UP-4) holds.

Hence, $(X, \cdot, t)$ is a UP-algebra. □

Example 2.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ and choose $t = 2$. By Theorem 2.1, we have $(X, \cdot, 2)$ is a UP-algebra with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	2	0	1	3	4	5
2	2	0	1	3	4	5
0	2	2	1	3	4	5
1	2	0	2	3	4	5
3	2	0	1	2	4	5
4	2	0	1	3	2	5
5	2	0	1	3	4	2

Theorem 2.3. *Let X be a nonempty totally ordered set and let arbitrary $t \in X$. Define a binary operation $\cdot$ on X by: for all $x, y \in X$,*

$$x \cdot y = \begin{cases} y & \text{if } x > y \text{ or } x = t, \\ t & \text{otherwise.} \end{cases}$$

Then $(X, \cdot, t)$ is a UP-algebra.

Proof. For any $x \in X$, we have $t \cdot x = x$ and $x \cdot t = t$. Thus (UP-2) and (UP-3) hold.

Let $x, y, z \in X$.

Case 1: $x = y = z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot (t \cdot t) = t \cdot t = t.$$

Case 2: $x = y$ but $y \neq z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (t \cdot (y \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t.$$

Case 3: $x = z$ but $z \neq y$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot ((x \cdot y) \cdot t) = (y \cdot z) \cdot t = t.$$

Case 4: $y = z$ but $z \neq x$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot ((x \cdot y) \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = t.$$

Case 5: $x \neq y, x \neq z$ and $y \neq z$.

Case 5.1: $x > y > z$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot z) = z \cdot z = t$.

Case 5.2: $x > z > y$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$.

Case 5.3: $y > x > z$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot ((x \cdot y) \cdot z)$. If $x = t$, then $z \cdot ((x \cdot y) \cdot z) = z \cdot ((t \cdot y) \cdot z) = z \cdot (y \cdot z) = z \cdot z = t$. If $x \neq t$, then $z \cdot ((x \cdot y) \cdot z) = z \cdot (t \cdot z) = z \cdot z = t$.

Case 5.4: $y > z > x$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot ((x \cdot y) \cdot (x \cdot z))$. If $x = t$, then $z \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot z) = z \cdot z = t$. If $x \neq t$, then $z \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (t \cdot t) = z \cdot t = t$.

Case 5.5: $z > x > y$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot (x \cdot z))$. If $x = t$, then $(y \cdot z) \cdot (y \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$. If $x \neq t$, then $(y \cdot z) \cdot (y \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot t) = (y \cdot z) \cdot t = t$.

Case 5.6: $z > y > x$. If $x = t$, then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$. If $x \neq t$, then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (t \cdot t) = (y \cdot z) \cdot t = t$.

Thus (UP-1) holds.

Let $x, y \in X$ be such that $x \cdot y = t$ and $y \cdot x = t$. We see that if $x = t$ or $y = t$, then $x = t = y$. Let $x \neq t$ and $y \neq t$ and suppose that $x \neq y$. Then $x < y$ or $x > y$. If $x < y$, then $t = y \cdot x = x$, which is a contradiction. If $x > y$, then $t = x \cdot y = y$, which is a contradiction. Hence, $x = y$. Thus (UP-4) holds.

Hence, $(X, \cdot, t)$ is a UP-algebra. □

Example 2.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a totally ordered set with the natural order and choose $t = 2$. By Theorem 2.3, we have $(X, \cdot, 2)$ is a UP-algebra with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	2	0	1	3	4	5
2	2	0	1	3	4	5
0	2	2	2	2	2	2
1	2	0	2	2	2	2
3	2	0	1	2	2	2
4	2	0	1	3	2	2
5	2	0	1	3	4	2

Theorem 2.5. *Let X be a nonempty totally ordered set and let arbitrary $t \in X$. Define a binary operation $\cdot$ on X by: for all $x, y \in X$,*

$$x \cdot y = \begin{cases} y & \text{if } x < y \text{ or } x = t, \\ t & \text{otherwise.} \end{cases}$$

Then $(X, \cdot, t)$ is a UP-algebra.

Proof. For any $x \in X$, we have $t \cdot x = x$ and $x \cdot t = t$. Thus (UP-2) and (UP-3) hold.

Let $x, y, z \in X$.

Case 1: $x = y = z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot (t \cdot t) = t \cdot t = t.$$

Case 2: $x = y$ but $y \neq z$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (t \cdot (y \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t.$$

Case 3: $x = z$ but $z \neq y$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot ((x \cdot y) \cdot t) = (y \cdot z) \cdot t = t.$$

Case 4: $y = z$ but $z \neq x$. Then

$$(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = t \cdot ((x \cdot y) \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = t.$$

Case 5: $x \neq y, x \neq z$ and $y \neq z$.

Case 5.1: $x > y > z$. If $x = t$, then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$. If $x \neq t$, then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (t \cdot t) = (y \cdot z) \cdot t = t$.

Case 5.2: $x > z > y$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot ((x \cdot y) \cdot (x \cdot z))$. If $x = t$, then $z \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot z) = z \cdot z = t$. If $x \neq t$, then $z \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (t \cdot t) = z \cdot t = t$.

Case 5.3: $y > x > z$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot (x \cdot z))$. If $x = t$, then $(y \cdot z) \cdot (y \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$. If $x \neq t$, then $(y \cdot z) \cdot (y \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot t) = (y \cdot z) \cdot t = t$.

Case 5.4: $y > z > x$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (y \cdot z) = t$.

Case 5.5: $z > x > y$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot ((x \cdot y) \cdot z)$ If $x = t$, then $z \cdot ((x \cdot y) \cdot z) = z \cdot (y \cdot z) = z \cdot z = t$. If $x \neq t$, then $z \cdot ((x \cdot y) \cdot z) = z \cdot (t \cdot z) = z \cdot z = t$.

Case 5.6: $z > y > x$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = z \cdot (y \cdot z) = z \cdot z = t$.

Thus (UP-1) holds.

Let $x, y \in X$ be such that $x \cdot y = t$ and $y \cdot x = t$. We see that if $x = t$ or $y = t$, then $x = t = y$. Let $x \neq t$ and $y \neq t$ and suppose that $x \neq y$. Then $x < y$ or $x > y$. If $x < y$, then $t = x \cdot y = y$, which is a contradiction. If $x > y$, then $t = y \cdot x = x$, which is a contradiction. Hence, $x = y$. Thus (UP-4) holds.

Hence, $(X, \cdot, t)$ is a UP-algebra. □

Example 2.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a totally ordered set with the natural order and choose $t = 2$. By Theorem 2.5, we have $(X, \cdot, 2)$ is a UP-algebra with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	2	0	1	3	4	5
2	2	0	1	3	4	5
0	2	2	1	3	4	5
1	2	2	2	3	4	5
3	2	2	2	2	4	5
4	2	2	2	2	2	5
5	2	2	2	2	2	2

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