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Relative Compactness of All Feasible Trajectories of a Second Order Control Problem

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Abstract

We study the relative compactness of all viable trajectories of a second order control problem with nonlinear boundary conditions by the lower and upper solutions method. To do that, we will need the Schauder's fixed point theorem for the existence result and the Arzelà-Ascoli's theorem for the relative compactness.

1 Introduction

Let $a, b \in \mathbb{R}$ such that $a < b$, $I = [a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$, $\overset{\circ}{I} =]a, b[= \{x \in \mathbb{R},\ a < x < b\}$, V a non-empty set of $\mathbb{R}$, $\alpha, \beta : I \to \mathbb{R}$ two applications of $C^2(I)$ such that $\forall\ t \in I,\ \alpha(t) \leq \beta(t)$. We give ourselves a dynamic system described by the equation

$$u''(t) = f(t, u(t), v(t)), \quad \forall t \in I, \tag{1.1}$$

with the boundary conditions

$$g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)), \tag{1.2}$$

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where $f : I\times\mathbb{R}\times V \to \mathbb{R}$ is a continuous map, $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ two continuous maps with some monotonies conditions and $v(t) \in V, \quad \forall\, t \in I$. Here, the control problem is of second order and the boundary conditions (1.2) generalize the classical boundary conditions of Cauchy, Dirichlet and Neumann.

The question is whether we can find a measurable map $v{:}I \to V$ for which problem (1.1)-(1.2) has a solution $u \in C^2(I)$ satisfying the following viability condition:

$$\alpha(t)\leq u(t)\leq \beta(t), \quad \forall\, t \in I. \tag{1.3}$$

That is to say, we seek what action $v(t)$ to apply to our dynamic system at each instant $t\in I$, so that the system follows a desired path.

We will suppose that the maps g_1 and g_2 verify :

(H1) : g_1 is decreasing according to the second argument,
(H2) : g_2 is increasing according to the second argument.

The lower and upper solutions method was initiated by Scorza Dragoni [9] in 1931 for a Dirichlet problem. Since then, a large number of contributions have enriched the theory and recent results have been found by J. Mawhin and K. Schmitt [11], A. Adjé [1;2], K. R. Ahoulou and A. Adjé [3], Coster and P. Habets [7;8] and Frigon [10], N. A. Asif, I. Talib and C. Tunc [5,6].

Our contribution in this paper is to complete the work done in [4] for using the lower and upper solutions method to show that the set of the viable trajectories of the control problem (1.1) with the nonlinear boundary conditions (1.2) is non-empty and relatively compact. To do that, we will need the Schauder's fixed point theorem and the Arzelà-Ascoli's theorem.

2 Trajectories with constant control

Definition 1. *$u \in C^2(I)$ is a constant control trajectory of the problem (1.1)-(1.2) if there exists $v \in V$ such that $u''(t) = f(t, u(t), v), \quad \forall\, t \in I$ with*

$$g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)).$$

For all $v \in V$, the application $f_v : I\times\mathbb{R} \to \mathbb{R}$ defined by $f_v(t, x) = f(t, x, v)$

being continuous, we consider the problem :

$$(Pv) \qquad \begin{cases} u''(t) = f_v(t, u(t)), & \forall\, t \in I, \\ g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)). \end{cases}$$

Definition 2. *α is a lower solution of (Pv) if :*
(H3) $\forall t \in \overset{o}{I}$, $\alpha''(t) \geq f_v(t, \alpha(t))$, $g_1(\alpha(a), \alpha'(a)) \leq 0$ and $g_2(\alpha(b), \alpha'(b)) \leq 0$;
and β is upper solution of (Pv) if :
(H4) $\forall t \in \overset{o}{I}$, $\beta''(t) \leq f_v(t, \beta(t))$, $g_1(\beta(a), \beta'(a)) \geq 0$ and $g_2(\beta(b), \beta'(b)) \geq 0$.

Proposition 1. *Let $v \in V$ and suppose the hypotheses (H1), (H2), (H3) and (H4) are satisfied. Then the problem (Pv) admits at least one solution $u_v \in C^2(I)$ such that:*

$$\alpha(t) \leq u_v(t) \leq \beta(t), \quad \forall\, t \in I.$$

Proof. This proof is based on the study of the modified problem :

$$\begin{cases} u''(t) = f_v(t, \gamma(t, u(t))) + u(t) - \gamma(t, u(t)), \quad \forall\, t \in I, \\ u(a) = \gamma\left(a\ ,\ u(a) + g_1(\gamma(a, u(a)), u'(a))\right), \\ u(b) = \gamma\left(b\ ,\ u(b) + g_2(\gamma(b, u(b)), u'(b))\right), \end{cases} \tag{2.4}$$

where γ is the continuous function from $I \times \mathbb{R}$ into $\mathbb{R}$ defined by :

$$\gamma(t, x) = \max[\alpha(t), \min(x, \beta(t))] = \begin{cases} \alpha(t) & if & x < \alpha(t), \\ x & if & \alpha(t) \leq x \leq \beta(t), \\ \beta(t) & if & x > \beta(t). \end{cases}$$

The proof will be done in two steps. First we are going to show that all solution u of the problem (2.4) verifies the inequality

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \forall\, t \in I,$$

and is therefore a solution of problem (Pv). Then we are going to show that (2.4) admits at least one solution.

Step 1 : All solution of problem (2.4) is wedged between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

Let u be a solution of (2.4). We will show that $\alpha(t) \leq u(t), \quad \forall\, t \in I$.
Suppose that there exists $t_0 \in I$ such that

$$\min_{t \in I}(u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0\ .$$

Then $\gamma(t_0, u(t_0)) = \alpha(t_0)$.

- If $t_0 \in \overset{\circ}{I}$, then $u'(t_0) - \alpha'(t_0) = 0$ *and* $u''(t_0) - \alpha''(t_0) \geq 0$.

According to the hypothesis (H3), $\alpha''(t_0) \geq f_v(t_0, \alpha(t_0))$.
So

$$u''(t_0) = f_v(t_0, \alpha(t_0)) + u(t_0) - \alpha(t_0) \leq \alpha''(t_0) + u(t_0) - \alpha(t_0) \quad ,$$

and we have the contradiction

$$0 \leq u''(t_0) - \alpha''(t_0) \leq u(t_0) - \alpha(t_0) < 0 \quad .$$

- If $t_0 = a$, that is to say

$$\min_{t \in I}(u(t) - \alpha(t)) = u(a) - \alpha(a) < 0 \quad ,$$

we have

$$u'(a) - \alpha'(a) \geq 0 \quad .$$

From

$$u(a) = \gamma(a, u(a) + g_1(\alpha(a), u'(a)))$$

and from (H1)

$$g_1(\alpha(a), u'(a)) \leq g_1(\alpha(a), \alpha'(a)) \leq 0 \quad ,$$

we get that

$$u(a) + g_1(\alpha(a), u'(a)) \leq u(a) < \alpha(a) \quad ;$$

which leads to the contradiction

$$\alpha(a) > u(a) = \gamma(a,\ u(a) + g_1(\alpha(a), u'(a))) = \alpha(a) \quad .$$

- If $t_0 = b$, that is to say

$$\min_{t \in I}(u(t) - \alpha(t)) = u(b) - \alpha(b) < 0 \quad ,$$

we have

$$u'(b) - \alpha'(b) \leq 0 \quad .$$

Using (H2) and the fact that $u(b) = \gamma(b\ ,\ u(b) + g_2(\alpha(b), u'(b)))$, we obtain the contradiction

$$\alpha(b) > u(b) = \gamma(b,\ u(b) + g_2(\alpha(b), u'(b))) = \alpha(b).$$

Then $\forall t \in I,\ \ \alpha(t) \leq u(t)$. In the same way, we prove that $u(t) \leq \beta(t),\ \ \forall\, t \in I$.

Step 2 : Existence of solution for the problem (2.4)

We are now going to show, via Schauder's fixed point theorem [9,p60], that (2.4) admits at least one solution. Let set down $X = C^2(I),\ \ Z = C(I) \times \mathbb{R}^2$ and consider the operator $L : X \to Z$ defined by:

$$Lu = (u'' - u, u(a), u(b))$$

and the function $N : X \to Z$ defined by :

$$Nu(t) = (f_v(t, \gamma(t, u(t))) - \gamma(t, u(t)) \quad ,$$

$$\gamma\,(a,\ u(a) + g_1(\gamma(a, u(a)), u'(a)))\,,$$

$$\gamma\,(b,\ u(b) + g_2(\gamma(b, u(b)), u'(b))) \quad .$$

Thus, the problem (2.4) can be written by:

$$Lu = Nu\ .$$

L is linear and bijective (hence is a Fredholm's operator of index zero) and L^{-1} is compact.
Indeed, it is easy to see that $KerL = L^{-1}(0_Z) = 0_X$, $ImL = L(X) = Z$ and moreover if $G : I \times I \to \mathbb{R}$ is the Green operator of the problem

$$\begin{cases} u''(t) - u(t) = h(t), & \forall\ t \in I, \\ u(a) = 0 = u(b), \end{cases}$$

then $\forall (h, A, B) \in Z$, the only solution of the problem

$$\begin{cases} u''(t) - u(t) = h(t), & \forall\ t \in I, \\ u(a) = A,\ u(b) = B, \end{cases}$$

is given by:

$$L^{-1}(h, A, B)(t) = \omega(t) + \int_a^b G(t, x)h(x)dx,$$

where

$$\omega(t) = Ae^{-(t-a)} + (B - Ae^{-(b-a)})(e^{b-a} - e^{-(b-a)})^{-1}(e^{t-a} - e^{-(t-a)}).$$

So L is bijective.
Then, the problem (2.4) can be written now by:

$$u = L^{-1}Nu\ .$$

As $\omega : I \to \mathbb{R}$ and $G : I \times I \to \mathbb{R}$ are continuous on compact sets, we show easily that, for all bounded subset K of Z, $L^{-1}K$ is bounded and equicontinuous [1,p167]. Therefore L^{-1} is compact.
We know that the interval I is compact and all the functions f_v, γ, g_1 and g_2 we use to define the function N are continuous. And according to the definition of γ, we have :

$$\gamma(t, u(t)) \in [\alpha(t), \beta(t)],$$

$$\gamma(a,\ u(a) + g_1(\gamma(a, u(a)), u'(a)) \in [\alpha(a), \beta(a)],$$
$$\gamma(b,\ u(b) + g_2(\gamma(b, u(b)), u'(b)) \in [\alpha(b), \beta(b)].$$

So, the function f_v being continuous on the compact

$$K = I \times [\min_{t\in I} \alpha(t), \max_{t\in I} \beta(t)]\quad ,$$

is bounded there.
Then the function N is continuous and bounded in $C(I)$.
Therefore, $L^{-1}N$ is compact and as X is convex in $C(I)$, by the Schauder's fixed point theorem, $L^{-1}N : X \to X$ has a fixed point which is solution of (2.4). □

3 Implicit formulation of the control problem

Suppose now that:

(H5) : the set V is compact.

As we look for solutions that evolve between α and β, we will consider the set value map $F : I \times [\min_{t\in I} \alpha(t), \max_{t\in I} \beta(t)] \to \mathbb{R}$ defined by :

$$F(t,x) = f(t,x,V) = \underset{v\in V}{\cup}\{f(t,x,v)\}$$

that is continuous and with compact values. Consider the differential inclusion problem:

$$\begin{cases} u''(t)\in F(t,u(t)) & \forall\ t\in I, \\ g_1\left(u\left(a\right), u'\left(a\right)\right) = 0 = g_2\left(u\left(b\right), u'\left(b\right)\right) \end{cases} \tag{3.5}$$

which is an implicit formulation of the control problem (1.1)-(1.2). The interest of the implicit formulation of control problems lies in the fact that it allows us to find constant control trajectories on I.

Moreover, since

$$Im\left(F\right) = \left\{F\left(t,x\right); \left(t,x\right) \in I \times \left[\min_{t\in I} \alpha(t), \max_{t\in I} \beta(t)\right]\right\}$$

is compact, the following definition can be given.

Definition 3. *α is a lower solution of the problem (3.5) if:*
(H6): $\forall t \in \overset{o}{I}$, $\alpha''(t) \geq y \quad \forall y \in F(t,\alpha(t))$, $\quad g_1(\alpha(a),\alpha'(a)) \leq 0$ and $g_2(\alpha(b),\alpha'(b)) \leq 0$;
and β is upper solution of the problem (3.5) if:
(H7): $\forall t \in \overset{o}{I}$, $\beta''(t) \leq y \quad \forall y \in F(t,\beta(t))$, $\quad g_1(\beta(a),\beta'(a)) \geq 0$ and $g_2(\beta(b),\beta'(b)) \geq 0$.

Proposition 2. *Under the assumptions (H1), (H2), (H5), (H6) and (H7), for all $v \in V$, the problem (3.5) admits a at least one solution $u_v \in C^2(I)$ such that:*

$$\alpha(t) \leq u_v(t) \leq \beta(t), \quad \forall\ t \in I.$$

Proof. Let $v \in V$. Then $f_v(t,x) \in F(t,x)$, $\forall (t,x) \in I\times[\min_{t\in I} \alpha(t), \max_{t\in I} \beta(t)]$. So,

$$f_v(t,\alpha(t)) \in F(t,\alpha(t))$$

and

$$f_v(t,\beta(t)) \in F(t,\beta(t))\ .$$

Thus by the hypotheses (H6) and (H7), $\forall\ t \in \overset{o}{I}$, we have:
$\alpha''(t) \geq f_v(t,\alpha(t))$ with $\quad g_1(\alpha(a),\alpha'(a)) \leq 0$ *and* $g_2(\alpha(b),\alpha'(b)) \leq 0$;

and
$\beta''(t) \leq f_v(t, \beta(t))$ with $g_1(\beta(a), \beta'(a)) \geq 0$ *and* $g_2(\beta(b), \beta'(b)) \geq 0$.

So α is a lower solution and β an upper solution of the problem (Pv). So according to proposition 1, the problem (Pv) admits a solution u_v included between α and β.
Finally, it is enough to note that u_v solution of the problem (Pv) is also solution of the problem(3.5). □

4 Compactness of all viable trajectories in $C^2(I)$

The relation (1.3) indicates that the set of viable trajectories of the problem (1.1)-(1.2) is bounded in $C(I)$. But, since any viable trajectory is in $C^2(I)$, it would be interesting to study the boundedness and the compactness of all viable trajectories in $C^2(I)$. For that, we suppose the following complementary hypothesis:

(H8): $\alpha(a) = \beta(a)$, $\quad -\infty < \alpha'(a) \quad$ and $\quad +\infty > \beta'(a)$.

Theorem 1. *Under the assumptions (H1), (H2), (H5), (H6), (H7) and (H8), the set of all viable trajectories of problem (1.1)-(1.2) is non-empty and relatively compact in $C^2(I)$.*

Proof. The proof will be done in two steps. First we will show the existence of viable trajectories solutions for the problem (1.1)-(1.2) and in the second part, we will show the relative compactness of the set of all the viable trajectories of the problem (1.1)-(1.2).

Step1: Existence of viable trajectories.

let $v \in V$, the constant function that we still to note v which at all every $t \in I$ associates $v(t) = v$ is measurable and according the proposition 2, there is a trajectory u_v verifying (1.1)-(1.2) such that $\alpha(t) \leq u_v(t) \leq \beta(t)$, $\forall\, t \in I$.

Step 2: Relative compactness of the set of all the viable trajectories of the problem (1.1)-(1.2).

Let $u \in C^2(I)$ a viable trajectory of the problem (1.1)-(1.2). Then, there is a measurable function $v : I \to V$ such that:

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \forall\, t \in I,$$

$$u''(t) = f(t, u(t), v(t)), \quad \forall\, t \in I,$$

and

$$g_1(u(a), u'(a)) = 0 = g_2(u(b), u'(b)).$$

The functions α and β being continuous on the compact interval I, they are bounded and reach there their extrema. Therefore

$$\min_{t\in I} \alpha\,(t) \;\leq u\,(t) \leq \max_{t\in I} \beta\,(t)\;, \quad \forall\, t \in I$$

and then there is $M_1 > 0$ such that $\|u\|_C < M_1$.
The function f being continuous, it is bounded on the compact

$$I \times \left[\min_{t\in I} \alpha\,(t)\;; \max_{t\in I} \beta\,(t)\;\right] \times V$$

and it reaches there its extrema.

So there is $M_2 > 0$ such that $\|u''\|_C < M_2$.

We must now show that u' is bounded on I.

We know that $u''\,(t) = f\,(t, u\,(t)\,, v\,(t))\,, \quad \forall\, t \in I$. Then,

$$u'\,(t) = u'\,(a) + \int_a^t f(s, u\,(s)\,, v\,(s)\,ds.$$

Therefore,

$$u'\,(a) - (b-a)\,M_2 \leq u'\,(t) \leq u'\,(a) + (b-a)M_2.$$

According the assumption (H8),

$$\alpha\,(a) = u\,(a) = \beta\,(a)\,.$$

And since

$$\alpha\,(t) \leq u\,(t) \leq \beta\,(t) \qquad \forall t \in I,$$

we have

$$-\infty < \alpha'\,(a) \leq u'\,(a) \leq \beta'\,(a) < +\infty.$$

So,

$$\alpha'(a) - (b-a)\, M_2 \leq u'(t) \leq \beta'(a) + (b-a)M_2.$$

And then there is $M_3 > 0$ such that $\|u'\|_C < M_3$.

Finally,

$$\|u\|_{C^2} = \max\{\|u\|_C \ ;\ \|u'\|_C \ ;\ \|u''\|_C \} \ \leq \max\{M_1 \ ;\ M_2 \ ;\ M_3\}\ .$$

We have just seen that the set of all viable trajectories of the problem (1.1)-(1.2) is bounded. It remains for us to show that it is equicontinuous to conclude that it is relatively compact.

Indeed, we know that $\|u'\|_C < M_3$ for all viable trajectory u of the problem (1.1)-(1.2).

Then by the finite-increment theorem, we have

$$\forall t_1,\ t_2 \in I, \quad |u(t_1) - u(t_2)| \leq M_3 |t_1 - t_2|\,.$$

So $\forall \varepsilon > 0,\ \ \exists \eta > 0$ ($\eta = \frac{\varepsilon}{M_3}$) such that for all viable trajectory u of (1.1)-(1.2),

$$|t_1 - t_2| < \eta \Longrightarrow |u(t_1) - u(t_2)| \leq \varepsilon.$$

Finally, we have shown that the set of viable trajectories of (1.1)-(1.2) is bounded and equicontinuous, and is therefore relatively compact according to the Arzelà-Ascoli's theorem. □

5 Example

Let $A \geq 1$, $I = [0;2]$, $V = [1;A]$, $v : I \to V$ a measurable function, $\alpha(t) = t^2 - 2t$ and $\beta(t) = -t^2 + 2t,\ \ \forall\, t \in I$. Consider the problem

$$\begin{cases} u''(t) = \frac{t}{v(t)} \sin(u(t)), \ \ \forall\, t \in I, \\ \\ -|u(0)|(u'(0))^3 = 0 = u(2) + |u(2)|u'(2). \end{cases} \tag{5.6}$$

Here, $f(t,u,v) = \frac{t}{v}\sin(u(t))$, $g_1(x,y) = -|x|y^3$ and $g_2(x,y) = x + |x|y$.
We have $g_1(x,y) = -|x|y^3$ is decreasing according to the second argument and $g_2(x,y) = x + |x|y$ is increasing according to the second argument.
It is easy to show that α is a lower solution and β is a upper solution of the

problem (6) such that $\alpha(t) \leq \beta(t), \quad \forall\, t \in I$, $\alpha(0) = \beta(0)$, $\alpha'(0) > -\infty$ and $\beta'(0) < +\infty$. Hence by Theorem 1, the set of all the solutions $u \in C^2(I)$ of the problem (5.6) such that

$$\alpha(t) \leq u(t) \leq \beta(t)$$

is non-empty and relatively compact in $C^2(I)$.

6 Conclusion

In this article, We establish the relative compactness of all viable trajectories of a second order control problem with nonlinear boundary conditions by the lower and upper solutions method. We give a example but more examples and applications can be given.

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