International Journal of Mathematics and
Computer Science, **13**(2018), no. 1, 45–50

On Two Integrability Methods of Improper Integrals

H. N. ÖZGEN

Mathematics Department
Faculty of Education
Mersin University, TR-33169
Mersin, Turkey

email: nogduk@gmail.com

(Received September 28, 2017, Accepted October 20, 2017)

Abstract

In this paper, we introduce a new concept of $\mid \bar{N}, p \mid_k, k \geq 1$ integrability of improper integrals. By using this definition we prove an analogous theorem due to Bor [H. Bor, On two summability methods, Math. Proc. Cambridge Philos. Soc., 97, 1985, 147–149] for improper integrals rather than infinite series.

1 Introduction

Throughout this paper we assume that f is a real valued function which is continuous on $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. By $\sigma(x)$, we denote the Cesàro mean of $s(x)$. The integral $\int_0^\infty f(t)dt$ is said to be integrable $\mid C, 1 \mid_k, k \geq 1$, in the sense of Flett [4], if

$$\int_0^\infty x^{k-1} \mid \sigma'(x) \mid^k = \int_0^\infty \frac{\mid v(x) \mid^k}{x} \tag{1}$$

is convergent. Here, $v(x) = \frac{1}{x} \int_0^x t f(t)dt$ is called the generator of the integral $\int_0^\infty f(t)dt$.

Key words and phrases: Cesàro and Riesz method, improper integrals, divergent integrals, inequalities for integrals, Hölder inequality.
AMS (MOS) Subject Classifications: 26D15, 40F05, 40A10, 35A23.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

Let p be a real valued, non-decreasing function on $[0,\infty)$ such that

$$P(x)=\int_0^x p(t)dt, p(x)\neq 0, p(0)=0.$$

The Riesz mean of $s(x)$ is defined by

$$\sigma_p(x)=\frac{1}{P(x)}\int_0^x p(t)s(t)dt.$$

We say that the integral $\int_0^\infty f(t)dt$ is integrable $\mid \bar{N},p\mid_k, k\geq 1$, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_p'(x)\mid^k dx \tag{2}$$

is convergent. In Particular, if we take $p(x)=1$ for all values of x, then $\mid \bar{N},p\mid_k$ integrability reduces to $\mid C,1\mid_k$ integrability of improper integrals. Given any functions f,g, it is customary to write $g(x)=O(f(x))$, if there exist η and N, for every $x>N, \mid \frac{g(x)}{f(x)}\mid\leq \eta$.
The difference between $s(x)$ and its nth weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

$$s(x)-\sigma_p(x)=v_p(x), \tag{3}$$

where

$$v_p(x)=\frac{1}{P(x)}\int_0^x P(u)f(u)du.$$

We note that if we take $p(x)=1$, for all values of x, then we have the following identity(see [3])

$$s(x)-\sigma(x)=v(x).$$

Since

$$\sigma_p'(x)=\frac{p(x)}{P(x)}v_p(x),$$

condition (3) can be rewritten as

$$s(x)=v_p(x)+\int_0^x \frac{p(u)}{P(u)}v_p(u)du. \tag{4}$$

In view of the identity (4), the function $v_p(x)$ is called the generator function of $s(x)$.
The absolute Nörlund summability of Fourier series and its allied series was studied by several workers. The corresponding problem of absolute summability of trigonometric integrals by the functional Nörlund methods up to the present was not been studied to the same degree. In fact, a paper of Boi͡cun [1] is the first one which takes up the problem in a special direction. Next, in 1974, Lal [5] generalized Boi͡cun's result.

Lal and Ram [6], [7], Lal and Singh [8], [9] investigated several conditions for the absolute Nörlund summability of integrals associated with the Fourier integral of a function. Recently, the author ([10], [11]) established several theorems dealing with the absolute Cesáro and Riesz integrability of improper integrals, respectively. In what follows, we give the main theorem.

2 Main result

The aim of this paper is to prove the theorem due to Bor [2] for improper integrals. Now we can give the following theorem.
Theorem 2.1. Let p be a real valued, non-decreasing functions on $[0, \infty)$ such that as $x \to \infty$ satisfying (2) and

$$xp(x) = O(P(x)), \tag{5}$$

$$P(x) = O(xp(x)). \tag{6}$$

If $\int_0^\infty f(t)dt$ is integrable $\mid C, 1 \mid_k$, then it is also integrable $\mid \bar{N}, p \mid_k, k \geq 1$.

3 Proof of the Theorem

Let $\sigma_p(x)$ denote the $(\bar{N}, p)$ means of the integral $\int_0^\infty f(t)dt$. Then we have

$$\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t)s(t)dt. \tag{7}$$

Since the integral $\int_0^\infty f(t)dt$ is integrable $\mid C, 1 \mid_k$, we can write

$$\int_0^\infty \frac{\mid v(x) \mid^k}{x} dx$$

is convergent, where $v(x)$ is the generator function of $s(x)$.

Differentiating the equation (7), we can write

$$\sigma_p'(x) = \frac{p(x)}{P^2(x)} \int_0^x P(t)f(t)dt = \frac{p(x)}{P^2(x)} \int_0^x \frac{P(t)}{t} tf(t)dt.$$

Integrating by parts of the second statement, we obtain

$$\begin{aligned}
\sigma_p'(x) &= \frac{p(x)}{P^2(x)}v(x) - \frac{p(x)}{P^2(x)} \int_0^x \left(\frac{P(t)}{t}\right)' tv(t)dt \\
&= \frac{p(x)}{P^2(x)}v(x) - \frac{p(x)}{P^2(x)} \int_0^x p(t)v(t)dt + \frac{p(x)}{P^2(x)} \int_0^x \frac{P(t)}{t}v(t)dt \\
&= \sigma_{p,1}(x) + \sigma_{p,2}(x) + \sigma_{p,3}(x), \ say.
\end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\int_0^m \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_{p,r}(x) \mid^k dx = O(1) \ as \ m \to \infty, \ for \ r = 1, 2, 3.$$

Using the condition (5), we have

$$\begin{aligned}
\int_0^m \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_{p,1}(x) \mid^k dx &= \int_0^m \frac{p(x)}{P(x)} \mid v(x) \mid^k dx \\
&= \int_0^m \frac{1}{x} \mid v(x) \mid^k dx \\
&= O(1) \ as \ m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem.

Applying Hölder's inequality with $k > 1$, we get

$$\begin{aligned}
\int_0^m \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_{p,2}(x) \mid^k dx &= O(1)\int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left(\int_0^x p(t) \mid v(t) \mid dt\right)^k \\
&= O(1)\int_0^m \frac{p(x)dx}{P(x)} \left(\int_0^x p(t) \mid v(t) \mid^k dt\right) \\
&\times \left(\frac{1}{P(x)}\int_0^x p(t)dt\right)^{k-1} \\
&= O(1)\int_0^m p(t) \mid v(t) \mid^k dt \int_t^m \frac{p(x)}{P^2(x)}dx \\
&= O(1)\int_0^m \frac{p(t)}{P(t)} \mid v(t) \mid^k dt \\
&= O(1)\int_0^m \frac{\mid v(t) \mid^k}{t}dt \\
&= O(1)\ as\ m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem.
Finally, as in $\sigma_{p,2}(x)$, by (6), we have that

$$\begin{aligned}
\int_0^m \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_{p,3}(x) \mid^k dx &= O(1)\int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left(\int_0^x \frac{P(t)}{t} \mid v(t) \mid dt\right)^k \\
&= O(1)\int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left(\int_0^x p(t) \mid v(t) \mid dt\right)^k \\
&= O(1)\ as\ m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem.
So, we get

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{k-1} \mid \sigma_p'(x) \mid^k dx$$

is convergent. This completes the proof of the theorem.
Therefore, a theorem of Bor [2] is obtained for the improper integrals.

Acknowledgement. The author acknowledges that the results in this paper were presented at the International Workshop, Mathematical Methods in Engineering, MME-2017, held in Cankaya University, Ankara, Turkey on April 27-29, 2017 and this work was supported by Mersin University Coordinatorship of Scientific Research Projects.

References

[1] L. G. Boicun, Absolute summability of conjugate Fourier integrals by the method of G. F. Voronoi, Izv. Vysš. Učebn. Zaved. Matematika, **61**(1967), no. 6, 11–21.

[2] H. Bor, On two summability methods, Math. Proc. Cambridge Philos. Soc. **97**(1985), 147–149.

[3] İ. Çanak, Ü. Totur, A Tauberian theorem for Cesàro summability factors of integrals, Appl. Math. Lett. **24**(2011), no. 3, 391–395.

[4] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc., **7**(1957), 113–141.

[5] S. N. Lal, On the absolute summability of the allied integral by a functional Nörlund method, Proc. Amer. Math. Soc., **42**(1974), no. 1, 113–120.

[6] S. N. Lal, S. Ram, On the absolute summability of a trigonometric integral by a functional Nörlund method, Indian J. Pure Appl. Math. **5**(1974), no. 10, 875–883.

[7] S. N. Lal, S. Ram, The absolute summability of infinite integrals by functional Nörlund methods, Indian J. Math., **17**(1975), no. 2, 101–105.

[8] S. N. Lal, A. K. Singh, Absolute Nörlund summability of Fourier integrals, Indian J. Math., **21**(1979), no. 2, 121–127.

[9] S. N. Lal, K. N. Singh, Allied integral and absolute functional Nörlund method of summability, Indian J. Pure Appl. Math., **11**(1980), no. 12, 1617–1625.

[10] H. N. Özgen, On $\mid C,1 \mid_k$ integrability of Improper integrals, Int. J. Anal. Appl., **11**(2016), no. 1, 19–22.

[11] H. N. Özgen, On equivalence of two integrability methods, Miskolc. Math. Notes, **18**(2017), no. 1, 391–396.