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A note on absolute summability method involving almost increasing and δ-quasi-monotone sequences

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Abstract

In this paper, a known result dealing with $\left|\bar{N}, p_n\right|_k$ summability of infinite series has been generalized to the $\varphi-\mid \bar{N}, p_n; \beta \mid_k$ summability of infinite series by using almost increasing and δ-quasi-monotone sequences.

1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leq b_n \leq Lc_n$ (see [1]).
A sequence (d_n) is said to be δ-quasi-monotone, if $d_n \to 0$, $d_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.1}$$

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The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (z_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [9]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \mid z_n - z_{n-1} \mid^k < \infty. \tag{1.3}$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid \bar{N}, p_n; \beta \mid_k$, $k \geq 1$ and $\beta \geq 0$, if (see [10])

$$\sum_{n=1}^{\infty} \varphi_n^{\beta k + k - 1} \mid z_n - z_{n-1} \mid^k < \infty. \tag{1.4}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \mid \bar{N}, p_n; \beta \mid_k$ summability is the same as $\mid \bar{N}, p_n; \beta \mid_k$ summability (see [4]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 0$, then we get $|\bar{N}, p_n|_k$ summability.

2 Known Result

The following theorem is known dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

Theorem 2.1 ([7, 8]). Let (Y_n) be an almost increasing sequence such that $\mid \Delta Y_n \mid = O(Y_n/n)$ and $\lambda_n \to 0 \quad as \quad n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n Y_n \delta_n < \infty$, $\sum A_n Y_n$ is convergent and $|\Delta \lambda_n| \leq |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{1}{n} \mid \lambda_n \mid = O(1) \quad as \quad m \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n \mid^k = O(Y_m) \quad as \quad m \to \infty \tag{2.6}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \mid t_n \mid^k = O(Y_m) \quad as \quad m \to \infty, \tag{2.7}$$

where (t_n) is the n-th $(C, 1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

3 Main Result

Some works dealing with generalized absolute summability methods of infinite series have been done (see [5]-[6], [11]-[12]). The purpose of this paper is to generalize Theorem 2.1 to $\varphi - |\bar{N}, p_n; \beta|_k$ summability, in the following form.

Theorem 3.1. Let (φ_n) be a sequence of positive real numbers such that

$$\varphi_n p_n = O(P_n), \tag{3.8}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} = O({\varphi_v}^{\beta k} \frac{1}{P_v}) \quad as \quad m \to \infty. \tag{3.9}$$

If all conditions of Theorem 2.1 with conditions (2.6) and (2.7) are replaced by

$$\sum_{n=1}^{m} \varphi_n^{\beta k} \frac{\mid t_n \mid^k}{n} = O(Y_m) \quad as \quad m \to \infty \tag{3.10}$$

and

$$\sum_{n=1}^{m} \varphi_n^{\beta k-1} \mid t_n \mid^k = O(Y_m) \quad as \quad m \to \infty \tag{3.11}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |\bar{N}, p_n; \beta|_k$, $k \geq 1$ and $0 \leq \beta < \frac{1}{k}$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2 ([7]). Let (Y_n) be an almost increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. If (A_n) is δ-quasi-monotone with $\sum A_n Y_n$ is convergent and $|\Delta \lambda_n| \leq |A_n|$ for all n, then we have

$$|\lambda_n| Y_n = O(1) \quad as \quad n \to \infty. \tag{3.12}$$

Lemma 3.3 ([8]). Let (Y_n) be an almost increasing sequence such that $n|\Delta Y_n| = O(Y_n)$. If (A_n) is δ-quasi-monotone with $\sum nY_n\delta_n < \infty$, $\sum A_nY_n$ is convergent, then

$$nA_nY_n = O(1) \quad as \quad n \to \infty, \tag{3.13}$$

$$\sum_{n=1}^{\infty} n|\Delta A_n|Y_n < \infty. \tag{3.14}$$

4 Proof of Theorem 3.1

Let (J_n) indicate $(\bar{N}, p_n)$ means of the series $\sum a_n\lambda_n$. Then, for $n \geq 1$, we obtain

$$\bar{\Delta} J_n = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v.$$

Applying Abel's formula, we get

$$\begin{aligned}\bar{\Delta} J_n &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{1}{v}P_vt_v\lambda_{v+1} - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{v+1}{v}p_v\lambda_vt_v \\ &\quad + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{v+1}{v}P_vt_v\Delta\lambda_v + \frac{n+1}{nP_n}p_n\lambda_nt_n \\ &= J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4}.\end{aligned}$$

For the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta k+k-1} \mid J_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{4.15}$$

By using Hölder's inequality and Abel's formula, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid J_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} (\frac{p_n}{P_n P_{n-1}})^k \left(\sum_{v=1}^{n-1} P_v \, |t_v| \, |\lambda_{v+1}| \, \frac{1}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \, |t_v| \, |\lambda_{v+1}| \, \frac{1}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v \, |t_v|^k \, |\lambda_{v+1}| \, \frac{1}{v} \right) \\
&\quad \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \, |\lambda_{v+1}| \, \frac{1}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v \, |t_v|^k \, |\lambda_{v+1}| \, \frac{1}{v} \right) \\
&= O(1) \sum_{v=1}^{m} P_v \, |t_v|^k \, |\lambda_{v+1}| \, \frac{1}{v} \sum_{n=v+1}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta k} \, |t_v|^k \, |\lambda_{v+1}| \, \frac{1}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_{v+1}| \sum_{r=1}^{v} \varphi_r^{\beta k} \frac{|t_r|^k}{r} + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \varphi_v^{\beta k} \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} |A_{v+1}| Y_{v+1} + O(1) |\lambda_{m+1}| Y_{m+1} \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of (2.5), (3.8), (3.9), (3.10) and (3.12). Again, using Hölder's inequality and Abel's formula, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid J_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} (\frac{p_n}{P_n P_{n-1}})^k \left(\sum_{v=1}^{n-1} p_v \, |\lambda_v| \, |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v \, |\lambda_v| \, |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} p_v \, |\lambda_v|^k \, |t_v|^k \right) \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} p_v \, |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta k-1} |\lambda_v| |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{r=1}^{v} \varphi_r^{\beta k-1} |t_r|^k + O(1) |\lambda_m| \sum_{v=1}^{m} \varphi_v^{\beta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |A_v| Y_v + O(1) |\lambda_m| Y_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of (3.8), (3.9), (3.11) and (3.12).
By (3.8), (3.9), (3.10), (3.13) and (3.14), we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid J_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} (\frac{p_n}{P_n P_{n-1}})^k \left(\sum_{v=1}^{n-1} P_v \, |t_v| \, |\Delta\lambda_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \, |t_v| \, |\Delta\lambda_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v \, |t_v|^k \, |\Delta\lambda_v| \right) \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \, |t_v|^k \, |A_v| \\
&= O(1) \sum_{v=1}^{m} P_v |t_v|^k |A_v| \sum_{n=v+1}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} = O(1) \sum_{v=1}^{m} \varphi_v^{\beta k} |t_v|^k v |A_v| \frac{1}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|A_v|) \sum_{r=1}^{v} \varphi_r^{\beta k} \frac{|t_r|^k}{r} + O(1) m |A_m| \sum_{v=1}^{m} \varphi_v^{\beta k} \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|A_v|) Y_v + O(1) m |A_m| Y_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta A_v| Y_v + O(1) \sum_{v=1}^{m-1} |A_{v+1}| Y_{v+1} + O(1) m |A_m| Y_m \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Finally, as in $J_{n,2}$, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta k+k-1} \mid J_{n,4} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\beta k+k-1} (\frac{p_n}{P_n})^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta k-1} |\lambda_n| |t_n|^k \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of (3.8), (3.11) and (3.12).
Thus the proof of Theorem 3.1 is completed.

5 Conclusion

If we take $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 0$ in Theorem 3.1, then we get Theorem 2.1. In this case, conditions (3.10) and (3.11) reduce to conditions (2.6) and (2.7), respectively. Also, the conditions (3.8) and (3.9) are automatically satisfied.

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