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A recurrence equation for b-digital sequences and its solutions

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Abstract

The b-parts of real numbers were introduced and studied by M. H. Hooshmand. Those have some applications to b-digital sequences, radix or b-adic, and also unique finite b-representation of real numbers. Applying them for radix representation of the sum of two real numbers x and y, we obtain a recurrence equation in generalized form (for every two-sided number sequences x_n and y_n):

$$M_n = [\frac{x_n + y_n + M_{n-1}}{b}] \;\; ; \;\; n \in \mathbb{Z},$$

where $b \neq 0$ is a fixed real number. In the way, we show that this equation has infinitely many solutions and, in particular, that some of them have many more properties regarding to their series and b-parts. Moreover, we prove a uniqueness theorem for the equation whenever $b > 2$ is an integer number and x_n, y_n are b-digital sequences.

1 Introduction and preliminaries

The b-parts of real numbers [3] were introduced in 2001. Then, some of their properties and applications were considered in [2,4]. Suppose $b \neq 0$ is a fixed

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real number. For any real number a denote by $[a]$ the largest integer not exceeding a and put $(a) = a - [a]$ (the decimal or fractional part of a). The b-parts are defined as follows

$$[a]_b = b[\frac{a}{b}] \ , \ (a)_b = b(\frac{a}{b}).$$

We call $[a]_b$ *b-integer part of* a and $(a)_b$ *b-decimal part of* a. Since $(a)_1 = (a)$ to avoid any confusion between decimal and parentheses notation, sometimes we use the symbol $(a)_1$ instead of (a).

The b-parts have so many algebraic, analytic, elementary properties and number theoretic explanations. If b is a positive integer, then $[a]_b$ is the same unique integer of the residue class $\{[a]-b+1, \cdots, [a]\}$ (mode b) that is divisible by b (because $[a]-b+1 \leq [a]_b \leq [a]$). Also, if $b > 0$, then $[a]_b$ is the largest element of $b\mathbb{Z}$ not exceeding a and if $b < 0$, then $[a]_b$ is the smallest element of $b\mathbb{Z}$ not less than a. Also, we have $[a+\beta]_b = [a]_b + \beta \ , \ (a+\beta)_b = (a)_b$, for all $\beta \in b\mathbb{Z}$. Now, let a , b be positive integers. By the division algorithm we have $a = bq + r$, where q , r are integers with $0 \leq r < b$. So $(a)_b = (bq+r)_b = (r)_b = r$. That is, $(a)_b$ is the same remainder of the division of a by b. This important fact leads us to the generalized division algorithm for real numbers and their unique finite b-representation (see [4]).

We call a function $a : \mathbb{Z} \to S$ (where $S \neq \emptyset$ is an arbitrary set) a "two sided sequence " and denote it by $\{a_n\}_{\infty}^{-\infty}$.

Let $b > 1$ be a fixed positive integer. A *b-digital sequence* (to the base b) is a two-sided sequence $\{a_n\}_{\infty}^{-\infty}$ of integers which satisfy the following conditions

i) $0 \leq a_n < b \quad : \forall n \in \mathbb{Z}$,

ii) There exists an integer N such that $a_n = 0$, for all $n > N$,

iii) For every integer m, there exists an integer $n \leq m$ such that $a_n \neq b-1$.

A nonzero b-digital sequence $\{a_n\}_{\infty}^{-\infty}$ will be denoted by $\{a_n\}_{\infty,b}^{-\infty}$ or $\{a_n\}_{N,b}$, where N is the largest integer such that $a_N \neq 0$, if a_n is non-zero sequence (we set $N = 0$ for the zero b-digital sequence). The following theorem is a well-known result about b-adic digits that has a new proof by using b-parts.

Theorem A(Fundamental theorem of b-digital sequences). Let $b > 1$ be a fixed positive integer. A two-sided sequence $\{a_n\}_{\infty}^{-\infty}$ of integers is a b-digital sequence if and only if there exists a non-negative real number a such that

$$a_n = ([ab^{-n}])_b \quad : \forall n \in \mathbb{Z}.$$

Moreover

$$a_n = [(ab^{-n})_b] = (ab^{-n})_b - (ab^{-n}) = (ab^{-n})_b - ((ab^{-n})_b)$$

$$= [ab^{-n}] - [[ab^{-n}]]_b = [ab^{-n}] - [ab^{-n}]_b,$$

for all $n \in \mathbb{Z}$, and we have $N = [\log_b a]$,

$$a = \sum_{\infty}^{-\infty} a_n b^n = \sum_{N}^{-\infty} a_n b^n = \sum_{N}^{-\infty} ([ab^{-n}])_b b^n.$$

Proof. See [4] for a proof by using the b-parts.

Therefore, there is a one-to-one correspondence between the set of all b-digital sequences and non-negative real numbers. Regarding this unique correspondence, the b-digital sequence a_n is called a b-radix representation or b-adic expansion of a (see [1,5]) and is written as

$$a = a_N a_{N-1} \cdots a_0 \centerdot a_{-1} a_{-2} \cdots_b = \cdots a_{n+1} a_n a_{n-1} \cdots_b \tag{1.1}$$

Also, we call a_n the n-th digit of a and denote it by $\mathrm{dgt}_{n,b}(a)$. Therefore, $\mathrm{dgt}_{n,b}(a) = ([b^{-n}a])_b$, for all integers n, and

$$a = \sum_{\infty}^{-\infty} \mathrm{dgt}_{n,b}(a) b^n = \sum_{N}^{-\infty} \mathrm{dgt}_{n,b}(a) b^n$$

For example $\mathrm{dgt}_{1,10}(\pi) = ([10^{-1}\pi])_{10} = 0$, $\mathrm{dgt}_{0,10}(\pi) = ([\pi])_{10} = 3$, $\mathrm{dgt}_{-1,10}(\pi) = ([10\pi])_{10} = 1$, $\mathrm{dgt}_{-2,10}(\pi) = ([10^2\pi])_{10} = 4$.

2 Intermediate sequence of addition of b-digital sequences

Let x_n , y_n be b-digital sequences and x, y the corresponding (positive) real numbers, respectively. Therefore $x_n = \mathrm{dgt}_{n,b}(x)$, $y_n = \mathrm{dgt}_{n,b}(y)$ and

$$\begin{aligned} x &= \textstyle\sum_{\infty}^{-\infty} x_n b^n = \cdots x_{n+1} x_n x_{n-1} \cdots_b \\ y &= \textstyle\sum_{\infty}^{-\infty} y_n b^n = \cdots y_{n+1} y_n y_{n-1} \cdots_b \end{aligned} \tag{2.1}$$

Now, put $z = x+y$ and let z_n be the unique b-digital sequence corresponding to z (so $z_n = \mathrm{dgt}_{n,b}(z)$ and thus $z_n = ([xb^{-n}+yb^{-n}])_b$, $z = \sum_{\infty}^{-\infty} z_n b^n$). Then, a natural question to raise is to find a relation between z_n and x_n, y_n.

Considering the usual algorithm of addition of two real numbers, by using their b-adic representation:

$$\begin{array}{cccccc} \cdots & 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \text{ or } 1 & \cdots & \\ \cdots & x_{n+1} & x_n & x_{n-1} & \cdots & \\ \cdots & y_{n+1} & y_n & y_{n-1} & \cdots & + \\ -- & -- & -- & -- & -- & \\ \cdots & z_{n+1} & z_n & z_{n-1} & \cdots & \end{array} \tag{2.2}$$

we conclude that there exists a two-sided binary sequence $\mu_n = \mu_n(x, y)$ such that

$$z_n = (x_n + y_n + \mu_{n-1})_b \Rightarrow \mu_n = [\frac{x_n + y_n + \mu_{n-1}}{b}] \quad : \quad \forall n \in \mathbb{Z}$$

Here, we consider its generalized form (without the mentioned conditions on μ) and prove existence of infinitely many solutions and some other properties, by using the b-parts.

Definition 2.1. Suppose $b \neq 0$ is a fixed real number and x_n, y_n are two given two-sided sequences. We call the following recursive equation "*b-intermediate equation of x_n and y_n*"

$$M_n = [\frac{x_n + y_n + M_{n-1}}{b}] \;\; ; \;\; n \in \mathbb{Z} \tag{2.3}$$

If b is an integer > 1 and x_n, y_n are b-digital sequences, then we call (2.3) "*b-intermediate b-digital equation of x_n and y_n*" .

Lemma 2.2. *Assume x_n, y_n are two-sided sequences. The b-intermediate equation of x_n and y_n has infinitely many solutions and if M_n is one solution, then*

(a) M_n is a two-sided sequence of integers.

(b) $(x_n + y_n + M_{n-1})_b = x_n + y_n + M_{n-1} - bM_n$ for all integers n (in fact, this is a necessary and sufficient condition for M_n to be a solution of (2.3)).

(c) If $b > 1$ and there exists an integer N such that $M_N \geq 0$ and $x_n + y_n = 0$ for every integer $n > N$, then there exists an integer $N' \geq N$ such that $M_n = 0$ for all $n > N'$.

(d) If $b > 1$, M_n is bounded and there exists an integer N such that $M_N \geq 0$ and $x_n + y_n = 0$, for every integer $n > N$, then $\sum_{\infty}^{-\infty} M_n b^n$ is convergent and

$$\sum_{\infty}^{-\infty} (x_n + y_n + M_{n-1})_b b^n = \sum_{\infty}^{-\infty} (x_n + y_n) b^n$$

(note that both are convergent or divergent).
(e) Suppose $b > 1$ is an integer and x_n, y_n are b-digital sequences (i.e. (2.3) is b-intermediate b-digital equation of x_n and y_n) and x, y their corresponding real numbers. If M_n is bounded and there exists an integer N such that $M_N \geq 0$, then

$$\sum_{\infty}^{-\infty}(x_n + y_n + M_{n-1})_b b^n = x + y$$

Proof. Put $M_0 = t$, $M_{-1} = bt - (x_0 + y_0)$, ... ,

$$M_{-k} = b^k t - ((x_0 + y_0)b^{k-1} + (x_{-1} + y_{-1})b^{k-2} + \cdots + (x_{-k+1} + y_{-k+1}))$$

for every positive integer k. Also, set $M_1 = [\frac{x_1+y_1+t}{b}]$ and define M_k inductively by $M_k = [\frac{x_k+y_k+M_{k-1}}{b}]$, for every positive integer k. By letting t run over $\mathbb{Z}$, we obtain infinitely many solutions $\{M_n\}_{\infty}^{-\infty}$ for the equation.
(a) Obvious.
(b) A simple calculation shows that

$$M_n = [\frac{x_n + y_n + M_{n-1}}{b}] \Leftrightarrow (x_n + y_n + M_{n-1})_b = x_n + y_n + M_{n-1} - bM_n.$$

(c) We have $M_n = [\frac{M_{n-1}}{b}] \leq \frac{M_{n-1}}{b}$ for every $n \geq N + 1$. Since $b > 1$ and $M_N \geq 0$, $M_{N+1} \geq 0$, $M_{N+2} \geq 0$, ... , and so $M_n < M_{n-1}$ for every $n \geq N+1$ such that $M_{n-1} > 0$. Thus there exists an integer $k_0 \geq 1$ such that $M_{N+k_0} = 0$ and so (c) holds.
(d) Part (c) implies $M_n = 0$ for all $n > N'$. In addition, the series $M_0 + M_{-1}b^{-1} + M_{-2}b^{-2} + \cdots$ is convergent (because $b > 1$ and M_n is bounded). So $\sum_{\infty}^{-\infty} M_n b^n$ is convergent and therefore $\sum_{\infty}^{-\infty}(M_{n-1} - bM_n)b^n = 0$. Now applying (b) We conclude that

$$\sum_{\infty}^{-\infty}(x_n + y_n + M_{n-1})_b b^n = \sum_{\infty}^{-\infty}(x_n + y_n)b^n$$

(e) Since x_n, y_n are b-digital sequences and $b > 1$, there exists an integer N_0 such that $M_{N_0} \geq 0$ and $x_n + y_n = 0$, for every integer $n > N_0$. Now part (d) implies

$$\sum_{\infty}^{-\infty}(x_n + y_n + M_{n-1})_b b^n = \sum_{\infty}^{-\infty}(x_n + y_n)b^n = \sum_{\infty}^{-\infty} x_n b^n + \sum_{\infty}^{-\infty} y_n b^n = x + y.$$

□

Now we are ready to prove the existence of the unique binary solution for the b-intermediate b-digital equation.

Theorem 2.3. *If $b > 2$, then the b-intermediate b-digital equation of x_n and y_n has a unique binary b-digital solution $M_n = \mu_n$ such that $(x_n + y_n + \mu_{n-1})_b$ is also b-digital.*

Proof. Put

$$\mu_n = \mu_{n,b}(x, y) = \begin{cases} 0\ ; & x_n + y_n < b - 1 \\ 1\ ; & (x_n + y_n > b - 1) \text{ or } (x_m + y_m = b - 1\ ;\ \text{for every } m \le n) \\ 0\ ; & x_n + y_n = b - 1\ ,\ x_m + y_m \le b - 1\ ,\ x_{m_0} + y_{m_0} \ne b - 1\ ;\ \text{for} \\ & \text{every } m \le n \text{ and some } m_0 \le n \\ 1\ ; & x_n + y_n = b - 1\ ,\ x_{m_0} + y_{m_0} \ge b - 1\ ,\ x_k + y_k = b - 1\ ;\ \text{for} \\ & \text{some } m_0 \le n \text{ and every } m_0 < k \le n \\ 0\ ; & \text{otherwise} \end{cases}$$

Fix an integer n. Since $0 \le x_n, y_n < b$ and $0 \le \mu_n \le 1$, $0 \le \frac{x_n+y_n+\mu_{n-1}}{b} \le \frac{b-1}{b} \le 1$ if $x_n + y_n < b - 1$, so $\mu_n = 0 = [\frac{x_n+y_n+\mu_{n-1}}{b}]$. Also, if $x_n + y_n > b - 1$, then

$$1 \le \frac{x_n + y_n + \mu_{n-1}}{b} \le \frac{x_n + y_n + 1}{b} \le \frac{2b - 1}{b} \le 2$$

thus $\mu_n = 1 = [\frac{x_n+y_n+\mu_{n-1}}{b}]$.
Now let $x_n + y_n = b - 1$. Consider the following cases:
(i) $x_m + y_m = b - 1$ for every $m \le n$.
In this case $\mu_n = \mu_{n-1}$ and so μ_n satisfies the equation clearly.
(ii) $x_m + y_m \le b - 1$ for every $m \le n$ and there exists $m = m_0$ such that $x_{m_0} + y_{m_0} \ne b - 1$.
If $m_0 = n - 1$, then $x_{n-1} + y_{n-1} < b - 1$ so $\mu_{n-1} = 0$ and μ_n satisfies the equation. Also, if $m_0 < n - 1$, then $\mu_{m_0} = 0$ thus $\mu_{m_0+1} = 0$ (because $x_{m_0+1} + y_{m_0+1} + \mu_{m_0} \ne b - 1$) and so $\mu_{m_0} = \mu_{m_0+1} = \cdots = \mu_n = 0$ and (2.3) is satisfied.
(iii) $x_{m_0} + y_{m_0} > b - 1$ for some $m_0 \le n$ and $x_k + y_k = b - 1$ for every $m_0 < k \le n$.
In this case $x_{m_0} + y_{m_0} > b - 1$, $x_{m_0+1} + y_{m_0+1} = \cdots = x_n + y_n = b - 1$ so $\mu_{m_0} = \mu_{m_0+1} = \cdots = \mu_n = 1$ and so μ_n satisfies the equation.
(iv) None of the above cases occur.
Here there exists an integer $m_0 < n - 1$ such that $x_{m_0} + y_{m_0} > b - 1$ and $x_k + y_k \le b - 1$ for every $m_0 < k < n$ and there exists $k = k_0$ such that

$x_{k_0} + y_{k_0} < b - 1$. Thus $\mu_{k_0} = \mu_{k_0+1} = \cdots = \mu_n = 0$ and (2.3) is satisfied. Therefore, we have proved that μ_n is a binary solution of the equation. Also, Lemma 2.2 implies μ_n is b-digital sequence (note that $0 \leq \mu_n \leq 1 < b$).
On the other hand, clearly $(x_n + y_n + \mu_{n-1})_b$ satisfies the first and second conditions of b-digital sequences. Now, if there exists an integer m such that $(x_n+y_n+\mu_{n-1})_b = b-1$ for every $n \leq m$, then $x_n+y_n = b(\mu_n+1)-(\mu_{n-1}+1)$. If $\mu_m = 1$, then $\mu_{m-1} = 1$ (because $0 \leq x_m + y_m \leq 2b-2$, $x_m + y_m = 2b - 1 - (\mu_{m-1}))$ so $x_n + y_n = 2b - 2$ and thus $x_n = y_n = b - 1$ for every $n \leq m$ which is a contradiction. A similar argument shows that $\mu_{m-1} \neq 1$, $\mu_{m-2} \neq 1$, and so on. Hence, $\mu_n = 0$ for all $n \leq m$ so $x_n + y_n = b - 1$ and so $\mu_n = 1$ for all $n \leq m$, which is a contradiction. Therefore, $(x_n + y_n + \mu_{n-1})_b$ is b-digital sequence.
Finally, if M_n is another distinct binary b-digital solution of the equation such that $(x_n + y_n + M_{n-1})_b$ is b-digital, then

$$\sum_{\infty}^{-\infty}(x_n + y_n + M_{n-1})_b b^n = x + y = \sum_{\infty}^{-\infty}(x_n + y_n + \mu_{n-1})_b b^n$$

so Theorem A implies

$$(x_n + y_n + M_{n-1})_b = (x_n + y_n + \mu_{n-1})_b \;\; ; \;\; \forall n \in \mathbb{Z}$$

thus $|M_{n-1} - \mu_{n-1}| = b|M_n - \mu_n|$ so b divides $|M_{n-1} - \mu_{n-1}| \leq 2$, for every integer n, which is a contradiction (because $b > 2$). Therefore, the proof is complete.

□

Corollary 2.4. *Let $b > 2$ be a fixed integer. If x and y are two non-negative real numbers, then there exists a unique binary b-digital sequence $\mu_{n,b}(x, y)$ such that*

$$dgt_{n,b}(x + y) = (dgt_{n,b}(x) + dgt_{n,b}(y) + \mu_{n-1,b}(x, y))_b.$$

Proof. Put $x_n = \mathrm{dgt}_{n,b}(x)$, $y_n = \mathrm{dgt}_{n,b}(y)$ and let μ_n be the binary digital sequence in Theorem 2.3. Therefore, $\mathrm{dgt}_{n,b}(x + y) = (x_n + y_n + \mu_{n-1})_b$, by Lemma 2.2 (e) and Theorem 2.3. Since here $(x_n + y_n + \mu_{n-1})_b$ is a b-digital sequence, μ_n is unique, by the last part of the proof of Theorem 2.3. □

Note. As a result of the study, Theorem 2.3 and Corollary 2.4 imply that the carry over of addition of real numbers is unique if $b > 2$. But for $b = 2$ we have a question.
Question. For the case $b = 2$, can one obtain a unique solution ($M_n = \mu_n$ or another one) for the recurrence equation (2.3)?

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