International Journal of Mathematics and
Computer Science, **13**(2018), no. 1, 29–36

The imaginary parts of the zeros of the Riemann zeta function

Bahattin Gunes
Istanbul Technical University
Istanbul, Turkey

email: gunbaha@yahoo.com

(Received August 3, 2017, Accepted October 26, 2017)

Abstract

Assuming the Riemann hypothesis, we provide a new formula to help in finding the imaginary parts of the non-trivial zeros of the Riemann zeta function. This formula may also be useful in studies involving prime numbers.

1 Introduction

The Riemann zeta function [2] is defined in the half-plane with the complex variable $s = \sigma + it$ as an absolutely convergent series on $\sigma > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots \qquad \text{for } \ 1 < R(s) \quad (1.1)$$

As shown by Riemann, $\zeta(s)$ extends to the whole complex plane $\mathbb{C}$ as a meromorphic function by analytic continuation and satisfies the functional equation:

$$\zeta(s)\,\Gamma\left(\frac{s}{2}\right) = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2} \qquad \text{for } \quad s \neq 1 \quad (1.2)$$

Also, if we consider the alternative functional equation ([5], p. 146),

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \qquad \text{for } \quad s \neq 1 \quad (1.3)$$

Key words and phrases: Non-trivial zeros, Zeta function, Imaginary parts.
AMS (MOS) Subject Classification: 11M26.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

we can see that the Riemann zeta function $\zeta(s)$ has zeros at the negative even integers -2, -4, -6, ... which are called as the trivial zeros. The Riemann hypothesis, proposed by Riemann in 1859, is concerned with the non-trivial zeros [2]. Let us denote such a zero by $s_{\mathbf{o}} = \sigma_{\mathbf{o}} + it_{\mathbf{o}}$.

The Riemann hypothesis: The real part of a non-trivial zero of the Riemann zeta function is 1/2. That means, $\zeta(s_{\mathbf{o}}) = 0 \quad \Rightarrow \quad \sigma_{\mathbf{o}} = 1/2$.

2 The partial sum with the Riemann zeta function

In 1914, Hardy [3] proved that $\zeta(1/2 + it)$ has infinitely many zeros.
Assuming the Riemann hypothesis, we can write the infinitely many non-trivial zeros of the Riemann zeta function as follows:

$$s_{\mathbf{o}} = 1/2 \mp it_{\mathbf{o}}$$

Now we study such zeros. Let us write with $s = \sigma + it$:

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\, dx \qquad \text{for} \quad 1 < \sigma$$

Then, we can write the following formula ([1], p. 1):

$$\zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s} \qquad \text{for} \quad 0 < \sigma \quad \text{and} \quad s \neq 1 \tag{2.1}$$

For a sufficiently large positive integer N, let

$$\zeta_1(s) = \sum_{n=1}^{N} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s} \qquad \text{for} \quad 0 < \sigma < 1 \tag{2.2}$$

We can now make the following rearrangement:

$$\zeta_1(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \left(\int_1^2 \frac{1}{x^s}\, dx + \int_2^3 \frac{1}{x^s}\, dx + \cdots \int_{N-1}^{N} \frac{1}{x^s}\, dx + \int_N^{N+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s}$$

$$\zeta_1(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N+1} \frac{1}{x^s}\, dx \qquad \text{on} \quad 0 < \sigma < 1$$

3 A new formula as $\zeta_N(s)$

We obtained the following function from equation (2.1):

$$\zeta_1(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N+1} \frac{1}{x^s}\, dx \qquad \text{for} \quad 0 < \sigma < 1 \tag{3.1}$$

Inspired by the above-formula, we can derive:

$$\zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s} \quad \Rightarrow \quad \zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_{n-1}^{n} \frac{1}{x^s}\, dx \right)$$

Continuing:

$$\zeta_2(s) = \sum_{n=1}^{N} \left(\frac{1}{n^s} - \int_{n-1}^{n} \frac{1}{x^s}\, dx \right) \qquad \text{for} \quad 0 < \sigma < 1$$

Rearranging:

$$\zeta_2(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N} \frac{1}{x^s}\, dx \qquad \text{for} \quad 0 < \sigma < 1 \tag{3.2}$$

Since we have obtained two partial sums (3.1) and (3.2) from the same source (2.1), we can expect: $\zeta_1(s)$ and $\zeta_2(s)$ bound $\zeta(s)$by both sides as lower and upper with the sufficiently N values. As N increases, we can observe that the gap between the three will decrease naturally.

Now here, let us write the optimal equation between $\zeta_1(s)$ and $\zeta_2(s)$. We now have a chance for a most perfect approach to the Riemann zeta function.

Taking the average of the N and $N+1$ values on the integrals, we have

$$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N+1/2} \frac{1}{x^s}\, dx \qquad \text{for} \quad 0 < \sigma < 1 \tag{3.3}$$

In Section 4, we will do some calculations to show how $\zeta_N(s)$ is precise when we compare the Riemann zeta function.

4 Two examples with numeric calculations

The first example: First, let us do a calculation with the Riemann zeta function:

$$\zeta(1/5 + 109i) = 8.27517... - \ i0.783862...$$

Then, let us do the same calculation with $\zeta_N(s)$, $\zeta_1(s)$ and $\zeta_2(s)$. To show the differences among them more clearly, let us take the N value a little smaller as 100000. For their calculations, let us use one of the online calculators on the web. For example, if we choose "the Wolfram Alpha Widget Limit Calculator" [6], the calculation of $\zeta_N(1/5 + 109i)$ can be done as follows:

$$\zeta_N(1/5 + 109i) = \lim_{t \to 109} \left(\sum_{1}^{100000} \frac{1}{n^{1/5+109i}} - \frac{(100000 + 1/2)^{4/5-it}}{4/5 - it} \right)$$

$$\zeta_N(1/5 + 109i) = 8.27517... - i0.783862...$$

Then, let us see the calculations with the partial sums of the Riemann zeta function:

$$\begin{aligned} \zeta_1(1/5 + 109i) &= 8.28296... - \ i0.83325... \\ \zeta_2(1/5 + 109i) &= 8.26734... - \ i0.734478... \end{aligned}$$

One might want to see the calculation with the following sum (which is the partial sum of the Riemann zeta function) with the same conditions:

$$\zeta_3(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}$$

$$\zeta_3(1/5 + 109i) = 8.24321... - \ i0.727209...$$

Thus, we have seen how $\zeta_N(s)$ is sensitive when we compare it with $\zeta_1(s)$, $\zeta_2(s)$ and $\zeta_3(s)$.

The second example: This time, let us perform a very sophisticated example to show how $\zeta_N(s)$ is extremely precise with a sufficiently large N value when we compare the Riemann zeta function.
Let us consider the following functions $\zeta_{N^-}(s)$ and $\zeta_{N^+}(s)$ by taking this 1/2 with the differences $\mp$ 0.0001; for $0 < \sigma < 1$:

$$\zeta_{N^-}(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N+1/2-0.0001} \frac{1}{x^s}\, dx$$

$$\zeta_{N^+}(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \int_0^{N+1/2+0.0001} \frac{1}{x^s}\, dx$$

Then let us start the example. We know by [4]:
$\zeta(s_{\mathbf{o}}) = 0$ by $s_{\mathbf{o}} = 1/2 + i30.42487612585951321031189753058409132018156 00...$

Now with $s = 1/2 + 30.424876125859513210311897530584091320181 56i$ and the 1/2 with the differences **if we take the N integer as $\mathbf{10^{11}}$**:

$$\zeta_{N^-}(s) = \sum_{n=1}^{10^{11}} \frac{1}{n^s} - \int_0^{10^{11}+1/2-0.0001} \frac{1}{x^s}\, dx = -2.15346*10^{-10} + 2.85953*10^{-10} i$$

$$\zeta_{N^+}(s) = \sum_{n=1}^{10^{11}} \frac{1}{n^s} - \int_0^{10^{11}+1/2+0.0001} \frac{1}{x^s}\, dx = 1.91577*10^{-10} - 2.53271*10^{-10} i$$

Thus, we see the above-calculations bound the zero of $\zeta(s)$ from both sides as:

The real part of them $: -2.15346 * 10^{-10} < 0 < 1.91577 * 10^{-10}$

The imaginary part of them $: -2.53271 * 10^{-10} < 0 < 2.85953 * 10^{-10}$

You see how these small differences ∓ 0.0001 on $\zeta_N(s)$ can bound the zero of the Riemann zeta function so sensitively after 100 billions sums. Even, we have just taken the imaginary part of the zeros of the Riemann zeta function with **41 decimal digits** on the functions $\zeta_{N^-}(s)$ and $\zeta_{N^+}(s)$.

5 Calculations resulting from a specific value

Now, with the function $\zeta_N(1/2+it)$ in the region $0 \leqslant t \leqslant t_a$, let us see how all calculations as sensitive compared to the Riemann zeta function.
For these calculations, our goal is to catch **100 decimal digits** with a very precise computing program. We can not use the above calculator [6] for this purpose.
Denoting this decimal digits as c, one can consider the c decimal digits as large as needed $(500, 1000$, or larger$)$.

- Now, let us take the N value as 10^{12}.

- Take the t value as t_a which is the largest value in the region $0 \leqslant t \leqslant t_a$.

- With $\zeta_N(1/2+it_a)$, suppose that we have made a calculation which has 100 decimal digits.

Then, let us ask ourselves the following critical question:
How can we be sure that this calculation is definitely the same as $\zeta(1/2+it_a)$ which has also the 100 decimal digits?
Let us do a second calculation with $N = 10^{13}$.

- If both 100 decimal digits are the same, we can say that our first calculation is equivalent to $\zeta(1/2+it_a)$.

- If not, we can check it with $N = 10^{13}$ & 10^{14} this time.

- If still not, we go on in the same way with $N = 10^{14}$ & 10^{15} as ditto ...

- Once we have found this specific N value, we can talk about these equivalents in all the region $0 \leqslant t \leqslant t_a$ by this specific N value.

6 The imaginary parts of the zeros of Riemann zeta function

Let us consider the specific N value here which provided catching the calculations with the c decimal digits accuracy in the region $0 \leqslant t \leqslant t_a$ when compared with the Riemann zeta function.
Denoting this specific N value as N_c, let us change the appearance of the function $\zeta_N(1/2+it)$ with it as follows:

$$\zeta_{N_c}(1/2+it) = \sum_{n=1}^{N_c} \frac{1}{n^{1/2+it}} - \int_0^{N_c+1/2} \frac{1}{x^{1/2+it}}\, dx$$

So now, with the c decimal digits accuracy provided by the N_c value, we can say again that:

All the calculations of $\zeta_{N_c}(1/2+it)$ will be the same as the ones of $\zeta(1/2+it)$.

Thus, we can find the zeros of the Riemann zeta function precisely.
In the region $0 \leqslant t \leqslant t_a$, if we intend to find the zeros of the Riemann zeta function with the c decimal digits, we can find them using the following:

$$\zeta_{N_c}(1/2+it) = 0$$

$$\sum_{n=1}^{N_c} \frac{1}{n^{1/2+it}} - \int_0^{N_c+1/2} \frac{1}{x^{1/2+it}}\, dx = 0 \qquad (6.1)$$

Going one step further by applying Euler's formula to equation 6.1, we have:

The terms of the real parts:

$$\sum_{n=1}^{N_c} \frac{\cos(t\ln n)}{n^{1/2}} - \int_0^{N_c+1/2} \frac{\cos(t\ln x)}{x^{1/2}} dx = 0$$

The terms of the imaginary parts:

$$\sum_{n=1}^{N_c} \frac{\sin(t\ln n)}{n^{1/2}} - \int_0^{N_c+1/2} \frac{\sin(t\ln x)}{x^{1/2}} dx = 0$$

Even though the real terms as free from the complex ones, both are zero at the same points.

References

[1] N. D. Elkies, Introduction to Analytic Number Theory, http://www.math.harvard.edu/~elkies/M259.02/zeta1.pdf

[2] B. Enrico, The Riemann hypothesis, http://www.claymath.org

[3] G. H. Hardy, Sur les zéros de la fonction ζ de Riemann, C. R. Acad. Sci. Paris 158: 1012–1014, JFM 45.0716.04 reprinted in (Borwein et al., 2008).

[4] A. Odlyzko, Tables of zeros of the Riemann zeta function, http://www.dtc.umn.edu/~odlyZko/zeta_tables/zeros2

[5] F. Saidak, P. Zvengrowski, On the modulus of the Riemann zeta function in the critical strip, Math. Slovaca, 2003.

[6] Wolfram Alpha Widget Limit Calculator, Math Help Boards on the web.