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Favorite mathematics topics from the 12^{th} Century to the 21^{st} Century

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Abstract

In this article we decided to begin with the 12^{th} Century because this century witnessed huge mathematical interest in Western Europe stimulated by Arabic original books as well as translated ones being translated into Latin (which became the venue of intellectual and scientific domains in Western Europe and remained in this academic function until the 18^{th} century) at Spain translation centers.
Despite the long war between the kingdoms of England and France from 1337 to 1453 resulting in famine and plague in the 14^{th} and the first half of the 15^{th} century, the year 1450 witnessed the advent of the printing press which had an enormous impact on printing arithmetic books for the purpose of teaching business people computational methods for their commercial needs.
A more detailed treatment is done on later centuries.

1 12^{th} Century

This century witnessed a breakthrough in Western Europe stimulated by Arabic original mathematics books as well as translated ones being translated

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into Latin at Spain translation centers mainly by three pioneers Gerard of Cremona, Robert of Chester, and Adelard of Bath (who all learned Arabic in Toledo, Spain). One of the translated works was for **Muhammad Ibn Musa Al-Khawrizmi** who lived in the 9^{th} century. From his family name, rendered in Latin, as "Algoritmi" the term "Algorithm" originated. From the title of his Arabic book on "al-jabr" the term "algebra" originated. In this book, Al-Khawrizmi gave a complete solution to the Quadratic Equation (Quadratic Formula). The problems which he included illustrate the application of the Quadratic Equation to numerous situations including inheritance questions which at times could be complicated under Muslim Law.
Another translated work in this epoch was Euclid's "Elements" in Arabic translated into Latin in 1142.

2 13^{th} Century

Italian merchants trading with Arabs in this century put **Leonard of Pisa (Fibonacci) (1175 - 1250)** in a unique position to disseminate the ideas of Arab Mathematicians in Western Europe. He was brought up in the region of North Africa known today as Algeria. His father was a Customs Officer at Bougie, the most important port on the North African Coast. Fibonacci traveled a lot to Egypt, Syria, and Greece. His best known work "Liber Abaci" (published in 1202 and then in a revised edition in 1228) showed the first use of "Abaci" (abacus) in a wider sense to refer to mathematics computation rather that the later meaning as a counting board.
However, despite Fibonacci push to adopt the Arabic numerals, there was a huge resistance to safeguard against possible forged alterations to 6 or 9 of a numeral like 0 (something not easy to do with Roman numerals, for example $X = 10$). Consequently, a money order was recorded also in words, a practice we see even nowadays in writing checks, say.
Despite Fibonacci wide contributions, he is most famous [3] for his sequence:

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+2} = u_n + u_{n+1}.$$

Here we give the explicit formula for a Fibonacci sequence, proved after Fibonacci's death, by Binet:
Let $x_n = u_{n+1}/u_n$ for $n \in \mathbb{N}$. First we prove $x_1 < x_3 < x_5 < ... < x_6 < x_4 < x_2$: Notice that $x_2 = 2$, $x_3 = \frac{3}{2}$, $x_4 = \frac{5}{3}$. We have $x_1 < x_3 < x_4 < x_2$. More generally, by induction

$$x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}. \qquad (*)_n$$

Indeed, assume $(*)_{k-1}$ holds; that is, $x_{2k-3} < x_{2k-1} < x_{2k} < x_{2k-2}$. Now $x_n = \frac{u_{n-1}+u_n}{u_n} = \frac{1}{x_{n-1}} + 1$. Therefore, $x_{2k-1} < x_{2k+1} < x_{2k} < x_{2k-2}$. By the same process $x_{2k-1} < x_{2k+1} < x_{2k+2} < x_{2k}$, which is $(*)_k$.
Next we prove that: $\lim_{n\to\infty}(x_{2n} - x_{2n-1}) = 0.$.

$$x_{2n+2} - x_{2n+1} = \frac{1}{x_{2n+1}} - \frac{1}{x_{2n}} = \frac{1}{\frac{1}{x_{2n}}+1} - \frac{1}{\frac{1}{x_{2n-1}}+1}$$

$$= \frac{x_{2n}}{x_{2n}+1} - \frac{x_{2n-1}}{x_{2n-1}+1} < \frac{x_{2n} - x_{2n-1}}{x_{2n}+1} < \frac{1}{2}(x_{2n} - x_{2n-1})$$

(since $x_{2n-1} < x_{2n} \Rightarrow -\frac{1}{x_{2n-1}+1} < -\frac{1}{x_{2n}+1}$ and $x_{2n} = \frac{1}{x_{2n-1}} + 1 > 1$).
Hence $0 < x_{2n} - x_{2n-1} < \left(\frac{1}{2}\right)^{n-1} \to 0$ as $n \to \infty$.
Now applying the Nested Sequence Principle which states: Let $\{I_n : n \in \mathbb{N}$ be a collection of closed intervals of $\mathbb{R}$ such that:
(i) $I_{n+1} \subseteq I_n$ for each $n \in \mathbb{N}$, and
(ii) For $\epsilon > 0$, the length of $I_n < \epsilon$ for some n.
Then $\cap_1^\infty I_n = \{x\}$ for some $x \in \mathbb{R}$.
We take $I_n = [x_{2n-1}, x_{2n}]$. Then using $(*)_n$ we see that x exists. Next we claim that $\lim_{n\to\infty} x_n = x$. To see this, let $\epsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $x_{2n_0} - x_{2n_0-1} < \epsilon$. Then

$$n \geq n_0 \Rightarrow 0 \leq x_{2n} - x \leq x_{2n} - x_{2n-1} \leq x_{2n_0} - x_{2n_0-1} < \epsilon.$$

Similarly,

$$n \geq n_0 \Rightarrow 0 \leq x - x_{2n-1} < \epsilon.$$

Thus $k \geq 2n_0 \Rightarrow |x_k - x| < \epsilon$ (Consider $k = 2n$ and $k = 2n-1$).
Next since $x_1 = 1 < x < x_2 = 2$, x satisfies the quadratic equation $x^2 - x - 1 = 0$ and so $x = \frac{1}{2} + \frac{\sqrt{5}}{2} := \alpha$ is accepted (Notice that the other root is $\beta := \frac{1}{2} - \frac{\sqrt{5}}{2}$ is rejected).
Let α and β be the roots of $x^2 = x + 1$. If $w_n = a\alpha^n + b\beta^n$, then we prove that the sequence $(w_n)_{n=0}^\infty$ satisfies $w_{n+2} = w_{n+1} + w_n$ for all $n \geq 0$.
If $x^2 - x - 1 = (x-\alpha)(x-\beta)$ and for $n \in \mathbb{N} \cup \{0\}$, $w_n = a\alpha^n + b\beta^n$, then $w_{n+2} = w_{n+1} + w_n$:
Indeed,

$$w_{n+2} = a\alpha^2\alpha^n + b\beta^2\beta^n = a(\alpha+1)\alpha^n + b(\beta+1)\beta^n = a\alpha^{n+1} + b\beta^{n+1} + a\alpha^n + b\beta^n = w_{n+1} + w_n.$$

Using the theory of double recurrence sequences, we can set $a + b = u_0 = 0$ and $a\alpha + b\beta = u_1 = 1$. It follows that $u_n = w_n$ and the system in matrix form is

$$\begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\alpha - \beta} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$

$$a = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}} = -b, \quad u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n).$$

Now

$$x_n = \frac{u_{n+1}}{u_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \frac{\alpha^{n+1} - (-\frac{1}{\alpha})^{n+1}}{\alpha^n - (-\frac{1}{\alpha})^n}$$

$$= \frac{\alpha\left(1 - (-1)^{n+1}\frac{1}{\alpha^{2n+2}}\right)}{1 - (-1)^n \frac{1}{\alpha^{2n}}} \to \alpha.$$

Thus we obtain Binet's formula:

$$u_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

3 14^{th} Century

Nicole Oresme (1320-1382) was a French who wrote five mathematics books on graphing functions and investigating infinite series (he had more interests besides mathematics–like philosophy and theology, when in 1369 he translated Aristotle works from Latin to French and added commentaries). Most of his mathematics work centered around infinite series. He gave the following proof that $\sum_1^\infty \frac{1}{n}$ diverges (which was remarkable especially because several mathematicians in the $14th$ Century thought that this series was convergent):

$$\sum_1^\infty \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+...$$

$$= 1+\frac{1}{2}+\{\frac{1}{3}+\frac{1}{4}\}+\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\}+\{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\}+...$$

$$\geq 1+\{\frac{1}{2}\}+\{\frac{1}{4}+\frac{1}{4}\}+\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\}+\{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\}+...$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$$

which diverges.
He discussed convergence and divergence conditions of geometric series and, whenever possible, evaluated sums of several geometric series.
In addition, he used a geometric argument to see that $\sum_1^\infty \frac{n}{2^n} = 2$.
Towards the end of this century algebraic symbolism was developed in Italy in which letters, like x, were used for an unknown. Other letters started appearing for constants.

4 15th Century

Nicolas Chuquet (1445-1488) was a French who has a Bachelor Degree in Medicine (which suggests that he didn't obtain the complete requirements to practice as a doctor) but was more interested in Mathematics. He used zero and negative numbers as exponents. His 1484 Algebra text "Triparty en la science des nombres" (Three parts in the science of numbers) contained not only algebra but also arithmetic including the words and phrases: million, billion, trillion, fourth quadrillion, fifth quyillion, sixth sixlion, seventh septyillion, eight ottyllion, ninth nonyllion and so on.
Chuquet was described as advanced for his time [15] that his contemporaries were unable to understand him and thus neglected him.
Chuquet is believed to be one of the inventors of logarithms and treated irrational numbers.

5 16th Century

The Italian mathematician **Niccolo Fontana (1499–1557)** discovered a general algebraic formula for solving cubic equations. The idea is to first use the substitution $t = x + \frac{a}{3}$ to reduce $x^3 + ax^2 + bx + c = 0$ to $t^3 + pt + q = 0$ (with $p = b - \frac{a^2}{3}, q = \frac{2a^3}{27} - \frac{ab}{3} + c$) whose solutions are:

$$t_1, t_2, t_3 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}$$

from which x_1, x_2, and x_3 are obtained easily.
The General Quartic formula, due to **Lodovico Ferrari (1522-1565),** (published in 1545), can be written out in full using the solution of one cubic equation and two quadratic ones:
Thus consider

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

As was done in the case of the general Cubic Equation, we similarly use the substitution $t = x + \frac{a}{4}$ to reduce the given equation to

$$t^4 + pt^2 + qt + r = 0.$$

Rewrite as

$$\left(t^2 + \frac{p}{2}\right)^2 = -qt - r + \frac{p^2}{4}.$$

Introduce a parameter m and expand to get

$$(t^2 + \frac{p}{2} + m)^2 = (t^2 + \frac{p}{2})^2 + 2m(t^2 + \frac{p}{2}) + m^2.$$

Thus

$$\left(t^2 + \frac{p}{2} + m\right)^2 = 2mt^2 - qt + m^2 + mp + \frac{p^2}{4} - r.$$

Choose m so that a perfect square results on the right-hand side. To do that the discriminant in t of this quadratic equation must be zero; that is, m is a solution of the equation

$$(-q)^2 - 4(2m)\left(m^2 + pm + \frac{p^2}{4} - r\right) = 0,$$

which may be rewritten as

$$8m^3 + 8pm^2 + (2p^2 - 8r)m - q^2 = 0.$$

The value of m may thus be obtained from Fontana's Method for the Cubic Equation. We consider two cases:
Case 1. $m = 0$: Here $q = 0$ which leads to an easy solution.
Case 2. $m \neq 0$: Here we can write

$$\frac{q^2}{8m} = m^2 + pm + \frac{p^2}{4} - r$$

and so

$$\left(\sqrt{2m}t - \frac{q}{2\sqrt{2m}}\right)^2 = 2mt^2 - tq + \frac{q^2}{8m} = (t^2 + \frac{p}{2} + m)^2$$

from which

$$\left(t^2 + \frac{p}{2} + m + \sqrt{2m}t - \frac{q}{2\sqrt{2m}}\right)\left(t^2 + \frac{p}{2} + m - \sqrt{2m}t + \frac{q}{2\sqrt{2m}}\right) = 0.$$

Consequently, we get the four solutions using the obvious quadratic formulas. In the context of what is discussed here it is important to mention that at age 19, Abel (1802–1829) was able to prove a 300-year old unsolved problem by proving that the fifth-degree equation

$$ax^5 + bx^4 + cx^3 + dx^2 + fx + t = 0,$$

has no solutions in terms of radicals (this was later generalized by his contemporary Galois for all $n > 4$, for a proof see [8], pp. 499–503), as it is the case for, say, quadratic equations.

6 17th Century

John Wallis (1616-1703) was an English mathematician who is given partial credit for the development of infinitesimal calculus. Between 1643 and 1689 he served as chief cryptographer for Parliament and, later, the royal court. He is credited with introducing the symbol for infinity.
However, Wallis most important result is the following formula carrying his name

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}...$$

I am going to obtain this formula as a pleasant application of a function due to Bernhard Riemann who lived in the 19$^t h$ Century and whose most important additional results we will discuss in a later section. For now let $z = \sigma + it$. For $\sigma > 1$, the **Riemann zeta function** ζ is defined by $\zeta(z) = \sum\limits_{n=1}^{\infty} \frac{1}{n^z}$.

We have $|\frac{1}{n^z}| = \frac{1}{|e^{z \log n}|} = \frac{1}{|e^{\sigma \log n}|} = \frac{1}{n^\sigma}$. By Weierstrass test, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly in the half-plane $\sigma > 1$ and hence on every compact subset of this half-plane. Thus ζ is analytic in the half-plane $\sigma > 1$ being the sum function of a uniformly convergent series of analytic functions. With some work, this function can be continued analytically to all complex $z \neq 1$. As a result, the zeta function is analytic everywhere except for a simple pole at $z = 1$ with residue 1. It is well-known that the only real zeros of the zeta function are on the negative even integers and are called **the trivial zeros.** When talking about the zeta function, it would be a miss not to mention the following famous conjecture:
The Riemann Hypothesis [16]. ALL NON-TRIVIAL ZEROS of the zeta function have real part equal to $\frac{1}{2}$.

The **completed zeta function** (or generically the xi-function), originally defined by Riemann [16], is

$$\xi(z)=\frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$$

The success of our proof (generalization) hinges on first finding a product representation of the completed zeta function which then is used in finding a product representation of the zeta function. To have a relatively self-contained paper we mention a few definitions:

Let $f(z)$ be an entire function. The **maximum modulus function**, denoted by $M(r)$, is defined by $M(r)=\max\{|f(z)|:|z|=r\}$.

Let $f(z)$ be a non-constant entire function. The **order** ρ of $f(z)$ is defined by

$$\rho=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}.$$

The order of any constant function is 0, by convention.

An entire function $f(z)$ of positive order ρ is said to be of **type** τ if

$$\tau=\limsup_{r\to\infty}\frac{\log M(r)}{r^{\rho}}.$$

The following [10] are some important properties of the completed zeta function:

1. $\xi(z)=\xi(1-z)$. This Functional Equation shows that the function $\xi(z)$ is symmetric about the critical line $Re(z)=\frac{1}{2}$.
2. The function $\xi(z)$ is entire.
3. The function $\xi(z)$ is of order one and infinite type.
4. The function $\xi(z)$ has infinitely many zeros.

It is clear now that the completed zeta function ξ is more convenient to use instead of the zeta function ζ since using the definition of ξ removes the simple pole of ζ at $z=1$ and as a result the theory of entire functions can be applied if needed, to ξ. In addition, since none of the factors of ξ except

ζ has a zero in $\mathbb{C} - \{0, 1\}$, no information is lost about the non-trivial zeros. The Riemann Hypothesis can therefore be stated as:

The Riemann Hypothesis using the completed zeta function. ALL ZEROS of $\xi(z)$ are on the critical line $Re(z) = \frac{1}{2}$.

We begin by finding a canonical representation of $\xi(z)$:

$\xi(z)$ is an entire function of order one and infinite type. Since the zeta function $\zeta(z)$ has a simple pole with residue 1 at $z = 1, \xi(1) = \frac{1}{2}$. Now, using the functional equation $\xi(z) = \xi(1 - z), \xi(0) = \frac{1}{2}$ Then, by ([11] p. 47), we have

$$\xi(z) = e^A e^{Bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \frac{z}{z_n},$$

where again $\{z_n\}_1^\infty$ are the non-zero zeros of $\xi(z)$ and hence, using the functional equation and the definition of the completed zeta function, are the non-trivial zeros of $\zeta(z)$ which are indeed in the critical strip and A and B are complex constants.

Now $\xi(0) = \frac{1}{2}$ implies that $e^A = \frac{1}{2}$ and so we can write

$$\xi(z) = \frac{1}{2} e^{Bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \frac{z}{z_n}.$$

The effort to find B using $\xi(1) = \frac{1}{2}$ leads to

$$1 = e^b \prod_{n=1}^{\infty} (1 - \frac{1}{z_n}) e^{\frac{1}{z_n}}.$$

Consider the product

$$p = \prod_{n=1}^{\infty} (1 - \frac{1}{z_n}) e^{\frac{1}{z_n}}.$$

Then

$$p^z = \prod_{n=1}^{\infty} (1 - \frac{1}{z_n})^z e^{\frac{z}{z_n}}.$$

Therefore,

$$\xi(z) = \frac{1}{2} e^{Bz} p^z \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Now $\xi(1) = \frac{1}{2}$ implies that $e^B p = 1$ and our identity reduces to the following representation of $\xi(z)$:

$$\xi(z) = \frac{1}{2} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Using the above product representation of $\xi(z)$ and that for the reciprocal of the Gamma function $\frac{1}{\Gamma(z)} = z\prod_{n=1}^{\infty}(1+\frac{z}{n})(1+\frac{1}{n})^{-z}$ ([11], p. 49) with the definition of $\xi(z)$ give the following explicit representation for $\zeta(z)$($\{z_n\}$ denotes the sequence of non-trivial zeros of $\zeta(z)$) :

$$\zeta(z) = \underbrace{\frac{1}{z-1}}_{\text{for singularity}} \pi^{\frac{z}{2}} \prod_{n=1}^{\infty} \underbrace{(1+\frac{z}{2n})}_{\text{for trivial zeros}} \underbrace{(1-\frac{z}{z_n})}_{\text{for non-trivial zeros}} (1+\frac{1}{n})^{-\frac{z}{2}}(1-\frac{1}{z_n})^{-z}.$$

Our discovered explicit representation above of the zeta function serves as a generalization of **Wallis Formula,**

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}...,$$

To see this, for $z \in \mathbb{C} - \{0,1\}$, we can rewrite the representation above as:

$$(z-1)\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\frac{z}{2}} \underbrace{\prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1-\frac{1}{z_n})^{-z}}_{2\xi(z) \text{ entire hence continuous}} \underbrace{\frac{z}{2}\prod_{n=1}^{\infty}(1+\frac{z}{2n})(1+\frac{1}{n})^{-\frac{z}{2}}}_{\frac{1}{\Gamma(\frac{z}{2})} \text{ entire hence continuous}}.$$

The result now follows using $\lim_{z\to 1}(z-1)\zeta(z) = 1$.

7 18^{th} Century

The Swiss mathematician **Leonard Euler (1707-1783)** was one of the most famous mathematicians of all times who had major contributions to Number Theory, Algebra, Combinatorics, among others. He read mathematics textbooks on his own and entered the university at age 14 and got his master's degree at age 17. The mathematician Johann Bernoulli (his father Nicholas and his brother Jacob were also mathematicians), discovered Euler's great potential for mathematics. The French physicist Arago stated that

"Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind."

In Euler's words:

" ... I soon found an opportunity to be introduced to a famous professor Johann Bernoulli... True, he was very busy and so refused flatly to give me private lessons; but he gave me much more valuable advice to start reading more difficult mathematical books on my own and to study them as diligently as I could; if I came across some obstacle or difficulty, I was given permission to visit him freely every Sunday afternoon and he kindly explained to me everything I could not understand ..."

The early result that made Euler famous was the exact value for $\zeta(2) = \sum_1^\infty \frac{1}{n^2}$ and he showed in 1735 that the value is $\frac{\pi^2}{6}$ and then went on to find $\zeta(4), \zeta(6), \zeta(8), \cdots$ He connected the symbols e, π, and i by the interesting relation $e^{\pi i} = -1$ and which is a special case of his formula $e^{i\theta} = \cos\theta + i\sin\theta$ which itself connects the basic trigonometric functions with the exponential function. He introduced his famous constant which appears in many parts of mathematics notably in Number Theory and Complex functions

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n),$$

and whose nature (transcendental or not) is still unknown. Euler describes the haphazardness of prime numbers by saying:

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

In 1727 he accepted a position at the Academy of Sciences in St. Petersburg, Russia and became a professor six years later. In 1734 Euler got married and afterwards he and his wife had 13 children of which, unfortunately, only 5 survived. Euler claimed that he made some of his important mathematical discoveries while holding a baby. Euler won numerous awards and in a letter to a friend he wrote:

"I can do just what I wish [in my research]... The king calls me his professor, and I think I am the happiest man in the world."

In 1766 Euler became blind following an illness but did not stop his research. Indeed, almost half of his research was done while he was blind dictating his results to a secretary. More than 500 books and articles were published in his life time. After Euler's death in 1783, the St. Petersburg Academy continued to publish his unpublished works over a period span of 50 years which shows the enormous quality mathematical work that Euler has done and which amounts to another 400 books and articles. Nowadays, the ongoing **Euler Project** is targeted to translating Euler's work into English and it is estimated that over 1000 volumes will be required for its completion.

8 19^{th} Century

Bernhard Riemann (1826-1866) was taught by his father until he was 10. He then attended a school where at one time he showed an interest in mathematics and the director of the school lend Riemann *the theory of numbers* by Legendre and Riemann read the 900-page book in 6 days. In 1846 Riemann attended the University of Göttingen in Germany and then moved to the University of Berlin where he worked on his general theory of complex variables. In 1849 he returned to the University of Göttingen and got his Ph.D. in 1851. In 1859 Riemann was elected to the Berlin Academy of Sciences and he presented one of his masterpieces "On the number of primes less than a given magnitude" which changed the direction of mathematical research in which he examined the Zeta function as a complex function (unlike Euler who had consider it only as a real function) $\zeta(z) = \sum_1^\infty \frac{1}{n^z}$ which as we have seen was represented by Euler as $\prod_{primes\ p}(1-p^{-z})^{-1}$. In this article, Riemann stated his famous **Riemann Hypothesis** that we mentioned in a previous section.

Even though Bernhard Riemann's 1859 condensed 8-page paper [16] was his only work spanning Number Theory since his preoccupation was developing the theory of complex functions (he emphasized the geometric aspects of the theory in contrast to the purely analytic approach taken by another co-founder, Cauchy (1789-1857)), it had a deep impact on Mathematics and in particular on Analytic Number Theory of which we mention $\prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^z}} = \sum_{n=1}^\infty \frac{1}{n^z}$ previously discovered by Euler but for real z. In other words, the Riemann zeta function is not only important as a function of a complex variable but also contains information about prime numbers and their distribution.

Riemann's defined zeta function can now be related to Bernoulli numbers $B_k, k = 0, 1, 2, 3...$ defined by $\frac{z}{e^z-1} = \sum_0^\infty B_k \frac{z^k}{k!}$ as in Euler famous identity [11], p. 125:

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}B_{2k}\pi^{2k}}{(2k)!}$$

and [14], p. 22: if k is a positive integer, then $\zeta(-k) = -\frac{B_{k+1}}{k+1}$. In particular, the latter case implies that $\zeta(-2k) = 0$. The negative even integers are called **trivial zeros** of the zeta function. The other zeros (there are plenty as we shall see later) are called the **non-trivial zeros** and we'll show that they are confined INSIDE what is known as the **Critical Strip** $\{z : 0 \leq Rez \leq 1\}$. To see this, first note that the Euler Product Formula implies that there are no zeros of $\zeta(z)$ with real part > 1 since convergent infinite products never vanish. Next, using Riemann Functional Equation $\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma(\frac{1-z}{2})\zeta(1-z)$ and the fact that Γ has no zeros in $\mathbb{C}$, it follows that there are no zeros of $\zeta(z)$ with real part < 0 apart from $..., -6, -4, -2$. Finally, using $\zeta(1+it) \neq 0, \forall t \in \mathbb{R}$ (More details about this coming up) and the Functional Equation again, the result follows.

In the same paper Riemann conjectured that ALL non-trivial zeros of his zeta function have real part equal to $\frac{1}{2}$ (Recent computer calculations have shown that the first discovered 10 trillion non-trivial zeros, ordered by increasing positive imaginary part, lie on the critical line $\frac{1}{2} + it$, where t is a real number; the approximate values of t for the first 6 zeros are $14.13472, 21.02203, 25.01085, 30.42487, 32.93506, \text{and } 37.58617$). This has been known as the **Riemann Hypothesis**.

Now, we elaborate on $\zeta(1+it) \neq 0$ which is essential for an analytic proof of a deep theorem known as the **Prime Number Theorem**:

On the basis of counting primes, one may be led to suspect that the number of primes less than or equal to a positive number x, denoted by $\pi(x)$, increases somehow like $\frac{x}{\log x}$. As a matter of fact, in 1791 at the age of 14, Gauss conjectured that $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$. In 1850, trying to settle the Gauss conjecture, Tchebycheff showed that there exist positive constants c and C such that

$$c\frac{x}{\log x} < \pi(x) < C\frac{x}{\log x}$$

for $x \geq 2$ with $c = .92$ and $C = 1.11$

In 1859, while attempting to find a formula for $\pi(x)$, Riemann (a student of Gauss) discovered analytic properties of his zeta function.

Throughout the last decade of the nineteenth century, J. Hadamard became interested in Gauss conjecture and the result was his theory of entire functions. It was not until 1896 that the Gauss conjecture was settled by Hadamard and, simultaneously by, de la Vallée Poussin and from then on it has been known as the **Prime Number Theorem.** Both Hadamard and de la Vallée Poussin proved that $\zeta(1+it) \neq 0$ from which they deduced the Prime Number Theorem.

Hadamard and de la Vallée Poussin also showed the converse to be true and for a while it appeared that the Prime Number Theorem was impossible to prove without using $\zeta(1+it) \neq 0$. However, in 1949, Erdös and Selberg proved the Prime Number Theorem by "elementary" methods meaning without using functions of a complex variable. Below we state Hadamard and de la Vallée Poussin key result:
$\zeta(1+it) \neq 0, \forall t \in \mathbb{R}$. That is, no zeros of the zeta function could lie on the line $x = 1$.

9 20^{th} Century

9.1 Viggo Brun (1885-1978)

Brun was born in Norway. In 1910, he studied at the University of Göttingen. In 1915, he introduced a number theory method, known as **Brun's sieve**, which showed some relevance in the Goldbach and the twin prime conjectures. In particular, he showed that:

There are infinitely many integers n such that n and $n+2$ have at most nine prime factors

and

All large even integers are the sum of two integers having at most nine prime factors.

The author naturally feels that Brun came across his theorem that the series of reciprocals of all twin primes:

$$\frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$$

is convergent [20] while trying to solve the twin prime conjecture where things went the opposite way (in comparison with the series of reciprocals of all primes is divergent). During $1919 - 1920$, Brun made another contribution to number theory with his algorithm on multi-dimensional continued fraction and its application to music. In 1923, Brun became a professor at the Technical University in Norway. In 1946, he moved to the University of Oslo and stayed there until his retirement in 1955.

9.2 Roger Apéry (1916–1994)

In 1979, Apéry [1] interestingly proved that $\zeta(3) = \sum_1^\infty \frac{1}{n^3}$ is an irrational number using the formula

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}},$$

which was discovered by Hjortnaes in 1953 [27]. However, finding the exact value that $\sum_1^\infty \frac{1}{n^3}$ converges to is one of the most notorious problems that did not even yield to Euler. An idea occurred to me and that was to consider the less difficult problem of finding a representation of $\zeta(3)$ in terms of $\sum_1^\infty \frac{1}{n^3\binom{2n}{n}}$) only which would be a beautiful result similar to Euler's $\zeta(2) = 3\sum_1^\infty \frac{1}{n^2\binom{2n}{n}}$. In [9] I have obtained the following representation

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi\sum_1^\infty \frac{1}{(3n-2)^2} - \frac{3}{4}\sum_1^\infty \frac{1}{n^3\binom{2n}{n}}$$

The simply-looking series $\sum_1^\infty \frac{1}{(3n-2)^2}$, however, is hard to evaluate. The idea is to use the LLL algorithm in the Computer Algebra System Maple to find the right integer relation which I'll then prove mathematically thus obtaining an Euler-type representation for $\zeta(3)$.

One should not hope much to find the exact value of $\zeta(3)$ as $\sum_1^\infty \frac{1}{n^3\binom{2n}{n}}$ is unknown even though, [11], we have

$$\sum_1^\infty \frac{1}{\binom{2n}{n}} = \frac{2\pi\sqrt{3}+9}{27},$$

$$\sum_1^\infty \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9},$$

$$\sum_1^\infty \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18},$$

$$\sum_1^\infty \frac{1}{n^4\binom{2n}{n}} = \frac{17\pi^4}{3240},$$

while there are no known values for

$$\sum_1^\infty \frac{1}{n^k\binom{2n}{n}}$$

for $k > 4$.

9.3 Paul Erdös (1913-1996)

Erdös was born to two Hungarian mathematics teachers. At age three he could do mental math involving three-digit numbers. At age four he tried to look for patterns to prime numbers. From then on he was hooked on mathematics and got his Ph.D. in mathematics at age 21 in 1934 and was a top notch mathematician. He traveled to England on a fellowship. During world war the British government restricted his movements for it feared that his mathematical correspondence with Hua, a colleague in Communist China, was code.
With about 1500 published papers, Erdös had a large number of collaborators; about 500 persons who had written articles with him. Those people were amused to define an Erdös number: everyone who published a co-authored paper with Erdö is assigned an **Erdös number equal to** 1. Those who published an article jointly with a person of Erdös number equal to 1 is assigned an **Erdös number equal to** 2, and so on. Albert Einstein has an Erdös number equal to 2. No wonder with so many collaborators, that Erdös traveled continuously throughout his life especially to the United States accompanied by two suitcases which contained all his belonging, going from a university to another, staying at a hotel or at a friend's house. Erdös was mainly a problem solver in the following fields of mathematics: Number Theory, Combinatorics, and Graph Theory. He did mathematics for an average of 19 hours a day and was fond of saying:

"A mathematician is a machine for turning coffee into theorems"

At the end of the 19-th century, Hadamard and de La Valle Poussin had demonstrated the Prime Number Theorem. In 1949, Atle Selberg found an important inequality leading to an elementary demonstration of the theorem and discussed it with Erdös and this collaboration led, in 1949, to an elementary proof of the Prime Number Theorem with Selberg. However, there was a regrettable story that followed that. Prior to the beginning of the e-mail days, the fastest courier of mathematical news was the frequent traveler Paul Erdös himself and he mentioned that with Selberg he had obtained an elementary proof of the prime number theorem. Now at that time mathematicians knew of Erdös, while few had heard of the young Norwegian Selberg. At a mathematics conference, there was talk about this fresh proof before it is being prepared by Selberg and Erdös who agreed to publish it back-to-back in the same journal, explaining each person's work. A mathe-

matician who was publicizing this work at the conference mentioned in front of other mathematicians that "Erdös and someone else" got an elementary proof of the Prime Number Theorem. When these exact words reached Selberg, it appeared to him that Erdös had claimed all the credit for himself. Selberg was not pleased and he rushed and published his proof and published first. As a result of this monumental proof, Selberg won the Fields Medal for this work. However, Erdös was not too concerned about the whole episode. Later, Erdös received the Wolf award in 1984.
Erdös published a new demonstration of Bertrand's Postulate, that there exists a prime number between n and $2n$, for each n. Tchebychev had already given a proof of this result during the 19-th century, but Erds proof was the most simple.

Erdös was not married. Despite more than 1500 papers that he authored or co-authored in his life, Erdös did not know how to drive a car depending mainly on friends to do the driving. He invented special vocabulary that mainly his collaborators understood. For instance, he used the word "preaching" for someone who is lecturing in mathematics, "died" for someone who stopped publishing, "left" for someone who died, "captured" for someone who is married, "liberated" for someone who is divorced, "recaptured" for someone who is married again, "epsilon" for a small child. Erdös had a strange personality and at many times when he was on the east coast in the United States, he would call at 7 a.m local time one of his collaborates in the west coast and wakes him therefore at 5 a.m. When asked about this once in an interview, he jokingly replied: "Good. That means he is home." He was mathematically active even at old age. Erdös was fond of saying:

"The first sign of senility is when a man forgets his theorems. The second sign is when he forgets to zip up. The third sign is when he forgets to zip down."

As Oliver Sacks described him:

"A mathematical genius of the first order. Paul Erdös was totally obsessed with his subject–he thought and wrote mathematics for nineteen hours a day until he died. He traveled constantly, living of a plastic bag, and had no interest in food, sex, companionship, art–all that is usually indispensable to a human life."

Near the end of his life, Erdös did not stop his mathematical activity. In his philosophy, to die means to stop doing mathematics. He left (died) on September $20, 1996$ while in a mathematics conference at Warsaw, Poland.

10 21^{st} Century

10.1 Andrew Wiles (1953–)

In 1995, and after spending 7 years on it and an additional year fixing it with his former research student Richard Taylor after a flaw was discovered in the proof, Wiles fulfilled his 10-year childhood dream by solving the 300-year old famous Fermat's Last Theorem that there are no positive integers x, y, and z satisfying the equation $x^n + y^n = z^n$ for all integers $n \geq 2$ (unlike the well-known Pythagoras Theorem):
Fermat(1601-1665) showed that there are no positive integers satisfying $x^4 + y^4 = z^4$. and Euler showed that there are no positive integer triples $x^3 + y^3 = z^3$. Later in the 18th and 19th century, mathematicians showed that there are no such positive integers for $n \leq 4000$. supposedly trying to disprove Fermat's Last "Theorem" and discrediting his unusual statement: "I have discovered a truly marvelous proof that it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. This margin is too narrow to contain it."
It is interesting to observe, in view of Fermat's remarks, that Wiles proof [17] is a 109-page proof. Another interesting remark was made by one of my Lebanese students, after I mentioned that Nada Canaan, Andrew Wiles wife, was Lebanese. He mentioned that this proof could not have been achieved without eating the famous Taboleh (Vegetable) dish.

10.2 Harald Helfgott (1977–)

For a long time it was believed that Analytic Number Theory Open Problems are easy to state and hard to solve. So it was a big surprise to the mathematics community to see progress in the same summer of 2013 on two of the most mysterious problems, we will discuss one in this section and the other in the next section:
In a letter to Euler in 1742, Christian Goldbach conjectured that: Every natural number $n > 5$ is a sum of three primes.

Euler restated the conjecture as: Every EVEN number ≥ 4 is the sum of two primes.

In 1937, Vinogradov [18] proved that every sufficiently large ODD number $n > n_o$ is the sum of three primes.
However, an elaborate study of the proof by Borodzkin [19] (Vinogradov student) [19] revealed that "sufficiently large" involves $n_o = 3^{3^{15}}$ which is approximately $10^{7000000}$. Even though the above estimate was refined by Chen and Wang [22] to $10^{43000} \sim e^{99012}$ and later by Liu and Wang [23] to e^{3100}, it was still too large for cases to be checked by computer.

On the other hand, Harald Helfgott [24] (May 13, 2013) proved that every ODD number > 5 is the sum of 3 primes.
Indeed, this result was proven, by Helfgott, for odd numbers $> (8.875)10^{30}$ but the result has been checked by Platt [25] by computer for odd numbers $< (8.875)10^{30}$.
On a related note, in 1966 Jingrun Chen [7] proved that every sufficiently large EVEN integer is the sum of a prime and the product of at most two primes.
In 1938, Estermann [26] and Tchudakov [28] proved that almost all EVEN integers are sums of two primes which is strengthened in 2013 by a result of Oliveira e Silva, Herzog, and Pardi [29] who have verified using computers that every even integer up to $(4)10^{18}$ is the sum of two primes, one suggestion is to strengthen Chen's theorem to show that every large EVEN integer is the sum of two primes.
Using the previous observations, another suggestion is to explicitly find a positive integer n_0 such that every EVEN number $n > n_0$ is the sum of two primes (presumably $n_0 > (4)10^{18}$) and then check that the desired result holds for all even numbers m with $(4)10^{18} < m \leq n_0$ using a (super)computer or many (super)computers working in parallel.

10.3 Yitang Zhang (1955–)

Prime numbers differing by 2 are called twin primes. The smallest pair is $(3, 5)$ and a large pair is

$$3756801695685.2^{666669} - 1, 3756801695685.2^{666669} + 1$$

in which each component has 200700 digits. Whether twin primes are finite or infinite in number is one of the most famous problems in number theory. The twin prime conjecture states that the number of twin primes is infinite. Many attempts to prove or disprove this 2300-year old conjecture have failed including Brun's result [20] that $\sum_{p,p+2primes} \frac{1}{p}$ is finite. A method to show the twin prime conjecture is to show that $\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$. In 2005, Goldston, Yildirim, and Pintz [30] obtained a major and deep result [30] that, with p_n denoting the n^{th} prime number, $\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0$; in other words, for every $\epsilon > 0$, there exist infinitely many pairs of consecutive primes p_n and p_{n+1} such that $p_{n+1} - p_n < \epsilon \log p_n$. Using Goldston, Yildirim, and Pintz result, Zhang [21] succeeded in showing the spectacular result:

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < 70000000$$

(One way to say that is that the gap between successive prime numbers remains finite).
This is a spectacular result because the finite upper bound 70 million is not important and as a matter of fact, the Polymath 8 project, started by Fields medalist Terence Tao at UCLA, has reduced this upper bound on the separation to 246 in relation to 2 for which the Prime Number Conjecture would be true.

References

[1] Roger Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Journées arithmétiques de Luminy, Astérisque, **61,** 1979, 11-13.

[2] Viggo Brun, La série $\frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$ est convergent ou finie, Bull. des Sciences Math., (2), **43,** 1919, 100-104, 124-128.

[3] David M. Burton, The History of Mathematics, Sixth ed., McGraw Hill, 2007.

[4] D. Shanks, J. W. Wrench, Brun's constant, Math. Comp., **28,** 1974, 293–299.

[5] R. P. Brent, Tables concerning irregularities in the distribution of primes and twin primes to 10^{11}, Math. Comp., **30,** 1976, 379.

[6] E. C. Titchmarsh, The theory of functions, Second ed., Oxford University Press, London, 1939.

[7] J. R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes I and II, Sci. Sinica, **16,** 1973, 157-176; **21,** 1978, 421-430.

[8] Michael I. Rosen, Niels Hendrik Abel and Equations of the Fifth Degree, Amer. Math. Monthly, **102,** no. 6, 1995, 495-505.

[9] Badih Ghusayni, Exploring new identities with Maple as a tool, WSEAS Trans. on Information Science and Applications, **1,** no. 5, 2004, 1151-1157.

[10] Badih Ghusayni, Results Connected to the Riemann Hypothesis, Int. J. Math. Analysis, **6**, 2012, 1235-1250.

[11] Badih Ghusayni, Number Theory from an Analytic Point of View, 2003.

[12] Badih Ghusayni, A Generalization of Wallis Formula, Int. J. Math. Comput. Sci., **10,** no. 1, 2015, 51-54.

[13] Badih Ghusayni, Half a dozen famous unsolved problems in mathematics with a dozen suggestions on how to try to solve them, **11,** no. 2, 2016, 257–273.

[14] Badih Ghusayni, The Value of the Zeta Function at an Odd Argument, Int. J. Math. Comput. Sci., **4**, no. 1, 2009, 21-30.

[15] Graham Flegg, Cynthia Hay, Barbara Moss, eds., Nicolas Chuquet, Renaissance Mathematician, 1985.

[16] Bernhard Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsberichte der Berliner Akademie, 1859 (translated as "On the number of prime numbers less than a given quantity"). English Translation by David Wilkins is available online via http://www.claymath.org/millenium-problems/riemann-hypothesis with the original hand-written manuscript available via http://www.claymath.org/sites/default/files/riemann1859.pdf

[17] Andrew Wiles, Modular Elliptic Curves and Fermat's Last Theorem, Annals Math., **142,** (1995), 443–551.

[18] Ivan Matveevich Vinogradov, Representation of an odd number as a sum of three primes (Russian), Dokl. Akad. Nauk. SSSR, **15,** (1937), 191-294; English translation in Goldbach Conjecture, Yuan Wang, Editor, World Scientific, 1983, 61–71.

[19] K. G. Borodzkin, On I. M. Vinogradov's constant, Proc. 3rd All-Union Math. Conf., **1**, Izdat. Akad. Nauk SSSR, (Russian), 1956.

[20] Viggo Brun, La Série $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+...$ est convergent ou finie, Bull. des Science Math., **43,** (1919), 100-104, 124-128.

[21] Yitang Zhang, Bounded Gaps between primes, Annals of Math., **179,** no. 3, (1994,) 1121-1174.

[22] Jingrun Chen, Tianze Wang, On the odd Goldbach problem, Acta Math. Sinica, **32,** (1989), 702-718.

[23] M. Liu, T. Wang, On the Vinogradov bound in the three primes Goldbach conjecture, Acta Arith., **105**, (2002), 133-175.

[24] Harald A. Helfgott, Major Arcs for Goldbach's Problem, available on arxiv.org

[25] Harald A. Helfgott, David J. Platt, Numerical Verification of the Ternary Goldbach Conjecture up to $(8.875)10^{30}$, Experimental Math., **22,** no. 4, (December 9, 2013), 406–409.

[26] T. Estermann, Proof that almost all even positive integers are sums of two primes, Proc. London Math. Soc., **44,** (1938), 307–314.

[27] M. M. Hjortnaes, Overϕring av rekken $\sum_{k=1}^{\infty}\frac{1}{k^3}$ til el bestemt integral, Proc. 12^{th} Cong. Scand. Math. (Lund 1953), Lund 1954.

[28] N. G. Tchudakov, On the density of the set of even integers which are representable as a sum of two odd primes (Russian), Akad. Nauk SSSR, Ser. Math., **1,** (1938), 25-40.

[29] Tomás Oliveira e Silva, S. Herzog, S. Pardi, Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $(4)10^{18}$, Math. Computation, **83,** no. 288, (2014), 2033-2060.

[30] Daniel A. Goldston, János Pintz, Cem Yildirim, Primes in Tuple I, Annals Math., **170,** (2009), 819–862.